# Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

## 2021 Lecture 4

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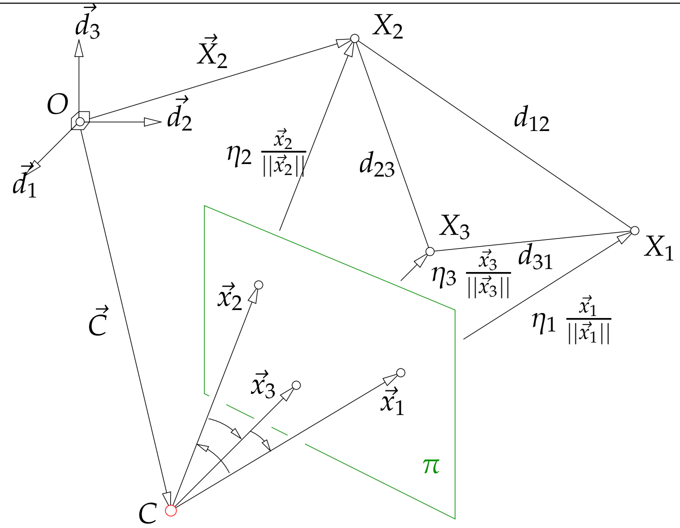
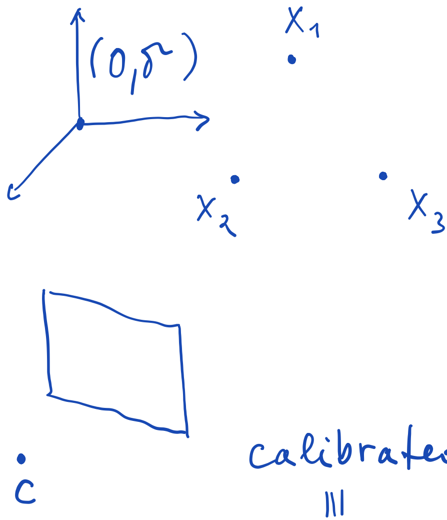



Figure 6.4: A calibrated camera pose can be computed from projections of three known points.




calibrated

$\equiv$

K is known

$\Downarrow$

get camera pose

$\vec{C}_\delta$, $R$

$P = [KR \mid -KR\vec{C}_\delta]$

## 6.3 Calibrated camera pose computation

We have seen how to find (uncalibrated) perspective camera pose from projections of known six points. In fact, we have recovered the calibration of the camera. Next we shall show that when the calibration is known, we are able to find the pose of the camera from projections of three points. This is a very classical problem which has been known since [14].

Figure 6.4 shows a camera with center $C$, which projects three points $X_1$, $X_2$ and $X_3$, represented by vectors $\vec{X}_{1\delta}$, $\vec{X}_{2\delta}$ and $\vec{X}_{3\delta}$ in $(O, \delta)$, into image points represented by $\vec{x}_{1\beta}$, $\vec{x}_{2\beta}$ and $\vec{x}_{3\beta}$.

## § 1 Classical formulation of the calibrated camera pose computation

We introduce distances between pairs of points as

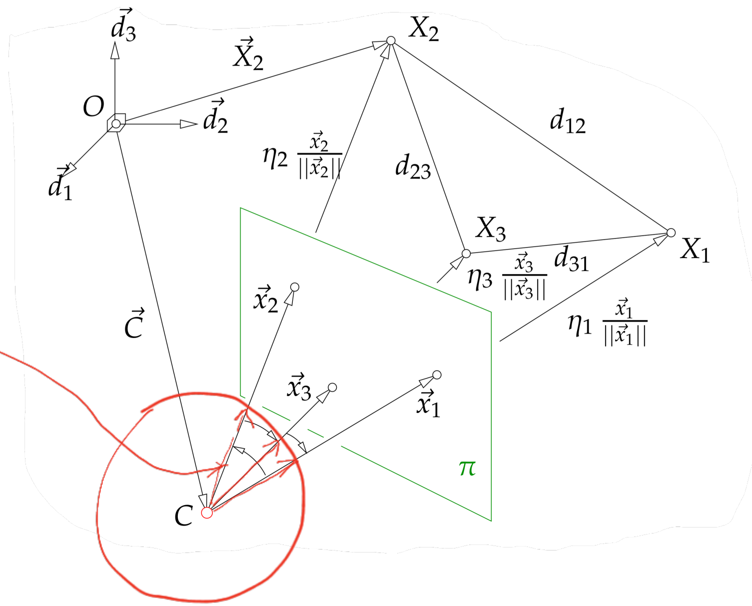
$$d_{12} = ||\vec{X}_{2\delta} - \vec{X}_{1\delta}||, \quad d_{23} = ||\vec{X}_{3\delta} - \vec{X}_{2\delta}||, \quad d_{31} = ||\vec{X}_{1\delta} - \vec{X}_{3\delta}|| \qquad (6.57)$$

Since we see three different points, we know that all distances are positive.

Points $X_1$, $X_2$ and $X_3$ are in $(C, \gamma)$ represented by vectors

$$\eta_i \boxed{\frac{\vec{x}_{i\gamma}}{||\vec{x}_{i\gamma}||}} = \eta_i \frac{\mathtt{K}^{-1}\vec{x}_{i\beta}}{||\mathtt{K}^{-1}\vec{x}_{i\beta}||}, \quad i = 1,2,3 \qquad (6.58)$$

with $\eta_i$ representing the distance from $C$ to $X_i$. Distances $\eta_i$ are positive since otherwise we could not see the points.




## § 2 Computing distances to the camera center

Calibrated perspective camera measures angles between projection rays

$$c_{ij} = \cos\angle(\vec{x}_i, \vec{x}_j) = \frac{\vec{x}_{i\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{j\beta}}{\|\mathtt{K}^{-1}\vec{x}_{i\beta}\|\|\mathtt{K}^{-1}\vec{x}_{j\beta}\|}, \quad i = 1,2,3, \; j = (i-1) \bmod 3 + 1 \qquad (6.59)$$

Hence we have all quantities $\eta_i$, $\cos\angle(\vec{x}_i, \vec{x}_j)$ and $d_{ij}$, which we need to construct a set of equations using the rule of cosines $d_{ij}^2 = \eta_i^2 + \eta_j^2 - 2\,\eta_i\,\eta_j \cos\angle(\vec{x}_i, \vec{x}_j)$, i.e.

$$\boxed{\begin{aligned} d_{12}^2 &= \eta_1^2 + \eta_2^2 - 2\,\eta_1\,\eta_2\,c_{12} & (6.60)\\ d_{23}^2 &= \eta_2^2 + \eta_3^2 - 2\,\eta_2\,\eta_3\,c_{23} & (6.61)\\ d_{31}^2 &= \eta_3^2 + \eta_1^2 - 2\,\eta_3\,\eta_1\,c_{31} & (6.62)\end{aligned}}$$

Solve ← 3 eqns deg 2

with $c_{ij} = \cos\angle(\vec{x}_i, \vec{x}_j)$.

We have three quadratic equations in three variables. We shall solve this system by manipulating the three equations to generate one equation in one variable, solving it and then substituting back to get the remaining two variables.

Known : $d_{12}, d_{23}, d_{31}$
$c_{12}, c_{23}, c_{31}$

Unknowns : $\underbrace{\eta_1, \eta_2, \eta_3}_{\text{compute}}$

≡ Solve equations

**§ 3 A classical solution** Let us first get two equations in two variables. Let us generate new equations by multiplying the left hand side of (6.60) and (6.62) by the right hand side of (6.61) and right hand side of (6.60) and (6.62) by the left hand side of (6.61)

$$
\begin{aligned}
d_{12}^2\,(\eta_2^2+\eta_3^2-2\,\eta_2\,\eta_3\,c_{23}) &= d_{23}^2\,(\eta_1^2+\eta_2^2-2\,\eta_1\,\eta_2\,c_{12}) && (6.63)\\
d_{31}^2\,(\eta_2^2+\eta_3^2-2\,\eta_2\,\eta_3\,c_{23}) &= d_{23}^2\,(\eta_3^2+\eta_1^2-2\,\eta_3\,\eta_1\,c_{31}) && (6.64)
\end{aligned}
$$

3 equations in 3 unknowns

$$
\begin{cases}
d_{12}^2 = \eta_1^2+\eta_2^2-2\,\eta_1\,\eta_2\,c_{12}\\
d_{23}^2 = \eta_2^2+\eta_3^2-2\,\eta_2\,\eta_3\,c_{23}\\
d_{31}^2 = \eta_3^2+\eta_1^2-2\,\eta_3\,\eta_1\,c_{31}
\end{cases}
$$

We could have made three different choices which equation to use twice but since all $d_{ij}\neq 0$, and hence all sides of the equations are nonzero, all the choices are equally valid.

We have now two equations with three variables but since the equations are homogeneous, we will be able to reduce the number of variables to two by dividing equations by (e.g.) $\eta_1^2$ (which is non-zero) to get

$$
\begin{aligned}
d_{12}^2\,(\eta_{12}^2+\eta_{13}^2-2\,\eta_{12}\,\eta_{13}\,c_{23}) &= d_{23}^2\,(1+\eta_{12}^2-2\,\eta_{12}\,c_{12}) && (6.65)\\
d_{31}^2\,(\eta_{12}^2+\eta_{13}^2-2\,\eta_{12}\,\eta_{13}\,c_{23}) &= d_{23}^2\,(1+\eta_{13}^2-2\,\eta_{13}\,c_{31}) && (6.66)
\end{aligned}
$$

2 equations in 2 unknowns

$\eta_1 \neq 0$

with $\boxed{\eta_{12}=\frac{\eta_2}{\eta_1} \text{ and } \eta_{13}=\frac{\eta_3}{\eta_1}}$. Notice that we have a simpler situation than before with only two quadratic equations in two variables. Let us proceed further towards one equation in one variable.

We rearrange the terms to get a polynomials in $\eta_{13}$ on the left and the rest on the right

$$
\begin{aligned}
d_{12}^2\,\eta_{13}^2+(-2\,d_{12}^2\,\eta_{12}\,c_{23})\,\eta_{13} &= d_{23}^2\,(1+\eta_{12}^2-2\,\eta_{12}\,c_{12})-d_{12}^2\eta_{12}^2\\
(d_{31}^2-d_{23}^2)\,\eta_{13}^2+(2\,d_{23}^2\,c_{31}-2\,d_{31}^2\,\eta_{12}\,c_{23})\,\eta_{13} &= d_{23}^2-d_{31}^2\,\eta_{12}^2 && (6.67)
\end{aligned}
$$

Rearrange

to get two quadratic equations

$$
\begin{aligned}
m_1\,\eta_{13}^2+p_1\,\eta_{13} &= q_1 && (6.68)\\
m_2\,\eta_{13}^2+p_2\,\eta_{13} &= q_2
\end{aligned}
$$

Hide $\eta_{12}$

in $\eta_{13}$ with

$$\begin{aligned}
m_1 &= d_{12}^2 && (6.69)\\
p_1 &= -2\, d_{12}^2\, \eta_{12}\, c_{23} && (6.70)\\
q_1 &= d_{23}^2\,(1+\eta_{12}^2 - 2\,\eta_{12}\, c_{12}) - d_{12}^2 \eta_{12}^2 && (6.71)\\
m_2 &= d_{31}^2 - d_{23}^2 && (6.72)\\
p_2 &= 2\, d_{23}^2\, c_{31} - 2\, d_{31}^2\, \eta_{12}\, c_{23} && (6.73)\\
q_2 &= d_{23}^2 - d_{31}^2\, \eta_{12}^2 && (6.74)
\end{aligned}$$

$$\begin{aligned}
m_1\,\eta_{13}^2 + p_1\,\eta_{13} &= q_1\\
m_2\,\eta_{13}^2 + p_2\,\eta_{13} &= q_2
\end{aligned}$$

We have "hidden" the variable $\eta_{12}$ in the new coefficients. We can now look upon Equations 6.68 as on a linear system

$$\begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix}\begin{bmatrix} \eta_{13}^2 \\ \eta_{13} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad (6.75)$$

The matrix of the system (6.75) either is or is not singular.

**§ 4 Case A** If it is not singular, we can solve the system by Cramer's rule [3, 4, 2]

Generic case

$$\left|\begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix}\right| \neq 0$$

$$\begin{aligned}
\eta_{13}^2 \left|\begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix}\right| &= \left|\begin{bmatrix} q_1 & p_1 \\ q_2 & p_2 \end{bmatrix}\right| && (6.76)\\
\eta_{13} \left|\begin{bmatrix} m_1 & p_1 \\ m_2 & p_2 \end{bmatrix}\right| &= \left|\begin{bmatrix} m_1 & q_1 \\ m_2 & q_2 \end{bmatrix}\right| && (6.77)
\end{aligned}$$

giving

$$\begin{aligned}
\eta_{13}^2\,(m_1\, p_2 - m_2\, p_1) &= q_1\, p_2 - q_2\, p_1 && (6.78)\\
\eta_{13}^2\,(m_1\, p_2 - m_2\, p_1)^2 &= (m_1\, q_2 - m_2\, q_1)^2 && (6.79)
\end{aligned}$$

Eliminating $\eta_{13}$ (by squaring the second equation, multiplying the first one by $m_1\, p_2 - m_2\, p_1$, which is non-zero, and comparing the left hand sides) yields

$$\boxed{(m_1\, p_2 - m_2\, p_1)\,(q_1\, p_2 - q_2\, p_1) = (m_1\, q_2 - m_2\, q_1)^2} \qquad (6.80)$$

Substituting Formulas 6.69-6.74 into Equation 6.80 yields

$$0 \;=\; a_4\,\eta_{12}^4 + a_3\,\eta_{12}^3 + a_2\,\eta_{12}^2 + a_1\,\eta_{12} + a_0 \tag{6.81}$$

$$(m_1\,p_2 - m_2\,p_1)\,(q_1\,p_2 - q_2\,p_1) = (m_1\,q_2 - m_2\,q_1)^2$$

with coefficients polynomial deg 4

$$\begin{aligned}
a_4 \;=\;& -d_{23}^8 - d_{12}^4\,d_{23}^4 - d_{23}^4\,d_{31}^4 - 2\,d_{12}^2\,d_{23}^4\,d_{31}^2 + 2\,d_{23}^6\,d_{31}^2 + 2\,d_{12}^2\,d_{23}^6 && (6.82)\\
&+4\,d_{12}^2\,c_{23}^2\,d_{23}^4\,d_{31}^2\\
a_3 \;=\;& 4\,d_{12}^4\,d_{23}^4\,c_{31}\,c_{23} - 4\,d_{12}^2\,d_{23}^6\,c_{12} - 4\,d_{12}^2\,c_{23}\,d_{23}^6\,c_{31} + 4\,d_{23}^4\,c_{12}\,d_{31}^4 && (6.83)\\
&+4\,d_{23}^8\,c_{12} - 4\,d_{12}^2\,d_{23}^4\,c_{31}\,d_{31}^2\,c_{23} - 8\,d_{12}^2\,c_{23}^2\,d_{23}^4\,d_{31}^2\,c_{12} - 8\,d_{23}^6\,c_{12}\,d_{31}^2\\
&+4\,d_{12}^2\,d_{23}^4\,c_{12}\,d_{31}^2\\
a_2 \;=\;& 8\,d_{23}^6\,c_{12}^2\,d_{31}^2 + 4\,d_{23}^6\,d_{31}^2 - 2\,d_{23}^4\,d_{31}^4 + 2\,d_{12}^4\,d_{23}^4 - 4\,d_{12}^4\,d_{23}^4\,c_{31}^2 && (6.84)\\
&-4\,d_{23}^8\,c_{12}^2 - 4\,d_{12}^4\,c_{23}^2\,d_{23}^4 - 2\,d_{23}^8 + 8\,d_{12}^2\,c_{23}\,d_{23}^6\,c_{31}\,c_{12}\\
&+4\,d_{12}^2\,c_{23}^2\,d_{23}^4\,d_{31}^2 - 4\,d_{23}^4\,c_{12}^2\,d_{31}^4 + 4\,d_{12}^2\,d_{23}^6\,c_{31}^2 + 8\,d_{12}^2\,d_{23}^4\,c_{31}\,d_{31}^2\,c_{23}\,c_{12}\\
a_1 \;=\;& 4\,d_{23}^4\,c_{12}\,d_{31}^4 + 4\,d_{12}^2\,d_{23}^6\,c_{12} + 4\,d_{23}^8\,c_{12} - 4\,d_{12}^2\,c_{23}\,d_{23}^6\,c_{31} && (6.85)\\
&-8\,d_{12}^2\,d_{23}^6\,c_{31}^2\,c_{12} - 4\,d_{12}^2\,d_{23}^4\,c_{31}\,d_{31}^2\,c_{23} - 4\,d_{12}^2\,d_{23}^4\,c_{12}\,d_{31}^2\\
&+4\,d_{12}^4\,d_{23}^4\,c_{31}\,c_{23} - 8\,d_{23}^6\,c_{12}\,d_{31}^2\\
a_0 \;=\;& 2\,d_{23}^6\,d_{31}^2 + 2\,d_{12}^2\,d_{23}^4\,d_{31}^2 - d_{23}^4\,d_{31}^4 - d_{12}^4\,d_{23}^4 + 4\,d_{12}^2\,d_{23}^6\,c_{31}^2 && (6.86)\\
&-d_{23}^8 - 2\,d_{12}^2\,d_{23}^6
\end{aligned}$$

$$\begin{aligned}
m_1 &= d_{12}^2\\
p_1 &= -2\,d_{12}^2\,\eta_{12}\,c_{23}\\
q_1 &= d_{23}^2\,(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}) - d_{12}^2\eta_{12}^2\\
m_2 &= d_{31}^2 - d_{23}^2\\
p_2 &= 2\,d_{23}^2\,c_{31} - 2\,d_{31}^2\,\eta_{12}\,c_{23}\\
q_2 &= d_{23}^2 - d_{31}^2\,\eta_{12}^2
\end{aligned}$$

Solving $f(\eta_{12}) = 0$ (deg 4) for $\eta_{12} \in \mathbb{C}$

We will use eigenvalue computation to find a numerical solution to Equation 6.81. Construct the following *companion matrix*

$$\mathtt{C} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} 0 & 0 & 0 & -\frac{a_0}{a_4}\\ 1 & 0 & 0 & -\frac{a_1}{a_4}\\ 0 & 1 & 0 & -\frac{a_2}{a_4}\\ 0 & 0 & 1 & -\frac{a_3}{a_4}\end{bmatrix}\begin{array}{l} \} -a_0 \;\; 4\times 4\\ -a_1\\ -a_2\\ -a_3\end{array} \tag{6.87}$$

$f(\eta_{12}) = 0$

$\Updownarrow \quad a \neq 0$

$\frac{1}{a_4} f(\eta_{12}) = 0$ ... monic polynomial

and observe that

$$\boxed{|\eta_{12}\,\mathtt{I} - \mathtt{C}| \;=\; \eta_{12}^4 + \frac{a_3}{a_4}\,\eta_{12}^3 + \frac{a_2}{a_4}\,\eta_{12}^2 + \frac{a_1}{a_4}\,\eta_{12} + \frac{a_0}{a_4}} \tag{6.88}$$

Therefore, a numerical approximation of $\eta_{12}$ can be obtained by computing, e.g., `>>eig(C) in Matlab`. Complex solutions are artifacts of the method and should not be further considered. For every real solution, we can then substitute back to Equation 6.79 to obtain the corresponding

$$\eta_{13} (m_1 p_2 - m_2 p_1) = m_1 q_2 - m_2 q_1 \tag{6.79}$$

$$\eta_{13} = \frac{m_1 q_2 - m_2 q_1}{m_1 p_2 - m_2 p_1} \tag{6.89}$$

$$= \frac{d_{12}^2 (d_{23}^2 - d_{31}^2 \eta_{12}^2) + (d_{23}^2 - d_{31}^2)(d_{23}^2 (1 + \eta_{12}^2 - 2\eta_{12} c_{12}) - d_{12}^2 \eta_{12}^2)}{2 d_{12}^2 (d_{23}^2 c_{31} - d_{31}^2 c_{23} \eta_{12}) + 2 (d_{31}^2 - d_{23}^2) d_{12}^2 c_{23} \eta_{12}}$$

To get $\eta_1$, $\eta_2$ and $\eta_3$, we consider Equation 6.60, which can be rearranged as

$$d_{12}^2 = \eta_1^2 (1 + \eta_{12}^2 - 2\eta_{12} c_{12}) \tag{6.90}$$

and hence yields positive

$$\eta_1 = \frac{d_{12}}{\sqrt{1 + \eta_{12}^2 - 2\eta_{12} c_{12}}} \tag{6.91}$$

$$\eta_2 = \eta_1 \eta_{12} \tag{6.92}$$

$$\eta_3 = \eta_1 \eta_{13} \tag{6.93}$$

§5 **Case B** Let us now look at what happens when the matrix of the system (6.75) is singular. Then, after substituting $m_1$, $m_2$, $p_1$ and $p_2$ from Equations 6.69-6.74, we have

$$m_1 p_2 - m_2 p_1 = 0 \tag{6.94}$$

$$-2 d_{12}^2 d_{23}^2 (\eta_{12} c_{23} - c_{31}) = 0 \tag{6.95}$$

$$\eta_{12} c_{23} = c_{31} \tag{6.96}$$

We used the fact that neither $d_{12} \neq 0$ nor $d_{23} \neq 0$.

$$\mathrm{eig}(C) \rightarrow \begin{bmatrix} \eta_{12}^{(1)} \\ \eta_{12}^{(2)} \\ \eta_{12}^{(3)} \\ \eta_{12}^{(4)} \end{bmatrix} \text{ 4 solutions}$$

$\in \mathbb{R}$

$\cancel{\notin \mathbb{R}}$

Algebraic formulation of the absolute calibrated camera pose (P3P)

3 points in 3D & 2D

4 sols → 1 sol in reality

take additional pts → clear!

**§ 6 Case B1** When $c_{23} \neq 0$, then we get

$$\eta_{12} = \frac{c_{31}}{c_{23}} \tag{6.97}$$

Substituting it to Equations 6.65 we get

$$d_{12}^2 \left( (\frac{c_{31}}{c_{23}})^2 + \eta_{13}^2 - 2 \frac{c_{31}}{c_{23}} \eta_{13} c_{23} \right) = d_{23}^2 \left( 1 + (\frac{c_{31}}{c_{23}})^2 - 2 \frac{c_{31}}{c_{23}} c_{12} \right) \tag{6.98}$$

$$d_{12}^2 \left( c_{31}^2 + c_{23}^2 \eta_{13}^2 - 2 c_{31} c_{23}^2 \eta_{13} \right) = d_{23}^2 \left( c_{23}^2 + c_{31}^2 - 2 c_{31} c_{23} c_{12} \right) \tag{6.99}$$

and after some more manipulation obtain a quadratic equation

$$(d_{12}^2 c_{23}^2) \eta_{13}^2 + (-2 d_{12}^2 c_{23}^2 c_{31}) \eta_{13} + d_{12}^2 c_{31}^2 - d_{23}^2 c_{23}^2 - d_{23}^2 c_{31}^2 + 2 d_{23}^2 c_{12} c_{23} c_{31} = 0 \tag{6.100}$$

in $\eta_{13}$. We get $\eta_1$, $\eta_2$ and $\eta_3$ from Equations 6.91, 6.92, 6.93.

**§ 7 Case B2** When $c_{23} = 0$, then it follows from Equation 6.96 that $c_{31} = 0$ as well. Returning back to equations 6.65, 6.66 provides

$$d_{12}^2 (\eta_{12}^2 + \eta_{13}^2) = d_{23}^2 (1 + \eta_{12}^2 - 2 \eta_{12} c_{12}) \tag{6.101}$$

$$d_{31}^2 (\eta_{12}^2 + \eta_{13}^2) = d_{23}^2 (1 + \eta_{13}^2) \tag{6.102}$$

Expressing $\eta_{13}$ from Equation 6.102 gives

$$(d_{23}^2 - d_{31}^2) \eta_{13}^2 = d_{31}^2 \eta_{12}^2 - d_{23}^2 \tag{6.103}$$

**§ 8 Case B2.1** When $d_{23}^2 \neq d_{31}^2$, then we can write

$$\eta_{13}^2 = \frac{d_{31}^2 \eta_{12}^2 - d_{23}^2}{d_{23}^2 - d_{31}^2} \tag{6.104}$$

to substitute it into Equation 6.101

$$d_{12}^2 \left( \eta_{12}^2 + \frac{d_{31}^2\, \eta_{12}^2 - d_{23}^2}{d_{23}^2 - d_{31}^2} \right) = d_{23}^2\, (1 + \eta_{12}^2 - 2\, \eta_{12}\, c_{12}) \tag{6.105}$$

which we further manipulate to get a quadratic equation in $\eta_{12}$

$$(d_{12}^2 - d_{23}^2 + d_{31}^2)\ \eta_{12}^2 + 2\, c_{12}\, (d_{23}^2 - d_{31}^2)\, \eta_{12} + d_{31}^2 - d_{12}^2 - d_{23}^2 = 0 \tag{6.106}$$

We get $\eta_1$, $\eta_2$ and $\eta_3$ from Equations 6.91, 6.92, 6.93.

§ **9 Case B2.2** Finally, when $d_{23}^2 = d_{31}^2$, then we get from Equation 6.103

$$\eta_{12} = 1 \tag{6.107}$$

and from Equation 6.101

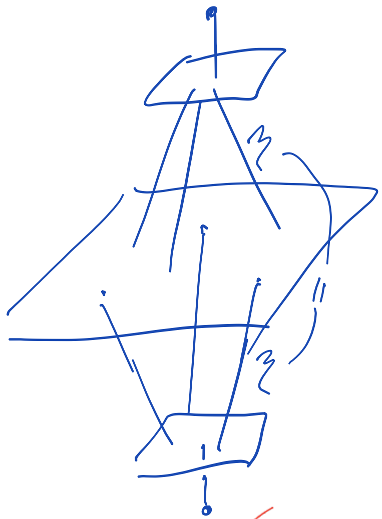
$$\eta_{13}^2 = \frac{d_{23}^2}{d_{12}^2}\, (2 - 2\, c_{12}) - 1 \tag{6.108}$$

and hence the positive

$$\eta_{13} = \sqrt{\frac{d_{23}^2}{d_{12}^2}\, (2 - 2\, c_{12}) - 1} \tag{6.109}$$

We get $\eta_1$, $\eta_2$ and $\eta_3$ from Equations 6.91, 6.92, 6.93.

§ **10 Selecting solutions** The above process of $\eta_i$ computation often delivers several solutions. It is important to notice that some of them may not satisfy the original Equations 6.62–6.60. For instance, we always obtain solutions for the case A as well as for some of the cases B but only one of the cases is actually valid. Hence, we need to select only the solutions that satisfy Equations 6.62–6.60 and are meaningful, i.e. are real and positive.



$\cancel{\notin \mathbb{R}}$

4 sols

$\mathbb{R} \longrightarrow$ pose

$\downarrow$

verify with other points

**§ 11 A modern (more elegant) solution** The classical solution is perfectly valid but it was quite tedious to derive it. Let us now present another, somewhat more elegant, solution, which exploits some of more recent results of algebraic geometry [15, 16].

$\eta_1, \eta_2, \eta_3$

Let us consider Equations 6.60, 6.61, 6.62 and proceed to Equations 6.65, 6.66, but, this time, using all three pairs to get three equations in $\eta_{12}$, $\eta_{13}$

$$f_1 = d_{12}^2\,(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}) - d_{23}^2\,(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}) = 0 \quad (6.110)$$
$$f_2 = d_{31}^2\,(\eta_{12}^2 + \eta_{13}^2 - 2\,\eta_{12}\,\eta_{13}\,c_{23}) - d_{23}^2\,(1 + \eta_{13}^2 - 2\,\eta_{13}\,c_{31}) = 0 \quad (6.111)$$
$$f_3 = d_{12}^2\,(1 + \eta_{13}^2 - 2\,\eta_{13}\,c_{31}) - d_{31}^2\,(1 + \eta_{12}^2 - 2\,\eta_{12}\,c_{12}) = 0 \quad (6.112)$$

3 eqns. in 2 unknowns

$\eta_{12}, \eta_{13}$

It is known [15, 16] that solutions to a set of $k$ algebraic equations

$$f_i(x_1, \ldots, x_n) = 0, \quad i = 1 \ldots, k \quad (6.113)$$

$k = 3$ equations (algebraic polynomial) in 2 unknowns

in $n$ variables, which have a fininte number of solutions, can always be obtained by deriving a polynomial $g(x_n) = 0$ in the last variable by the following procedure. If the system, does not have any solution, the procedure will generate polynomial $g_n = 1$, i.e. a non-zero constant, leading to the contradiction $1 = 0$.

$f_i(x_1, \ldots, x_n) \longrightarrow g(x_n)$

elimination procedure

The procedure is as follows. First generate new equations by multiplying all $f_i$ by all possible monomials up to degree $m$

$$x_1, \ldots, x_n, x_1^2, x_1\,x_2, \ldots, x_n^2, x_1^3, x_1^2\,x_2, \ldots, x_n^m \quad (6.114)$$

Monomial multiples of $f_i$

to get equations

$$f_1 = 0, \ldots, f_n = 0,\ x_1 f_1 = 0, \ldots, x_n f_n = 0,\ x_1^2 f_1 = 0,\ x_1\,x_2 f_1 = 0, \ldots, x_n^m f_n = 0 \quad (6.115)$$

$kn$

$x_1, \ldots, x_n, x_1^2, \ldots, x_n^?$

$x_1 x_2, x_1 x_3, \ldots$

The degree $m$ needs to be chosen such that the next step yields the desired result. It is always possible to choose such $m$ but it may sometimes be found only by using more and more monomials until the Gaussian elimination of the matrix of coefficients, which combine monomials, does not produce a row corresponding to an equation in $x_n$ only. Let us demonstrate this process by solving our problem.

We use the following four monomials of maximal degree two

$$\eta_{12},\ \eta_{13},\ \eta_{12}\,\eta_{13},\ \eta_{12}^2 \tag{6.116}$$

Notice that we did not include the second degree monomial $\eta_{13}^2$ since it turns out that equations generated by that monomial are not necessary. We obtain $15 = 3 + 4 \times 3$ equations

$n,\ d$

$\binom{n+d}{d}$

equations

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \eta_{12}\, f_1 \\ \eta_{12}\, f_2 \\ \eta_{12}\, f_3 \\ \eta_{13}\, f_1 \\ \eta_{13}\, f_2 \\ \eta_{13}\, f_3 \\ \eta_{12}\,\eta_{13}\, f_1 \\ \eta_{12}\,\eta_{13}\, f_2 \\ \eta_{12}\,\eta_{13}\, f_3 \\ \eta_{12}^2\, f_1 \\ \eta_{12}^2\, f_2 \\ \eta_{12}^2\, f_3 \end{bmatrix} = \mathtt{M} \begin{bmatrix} \eta_{12}\,\eta_{13}^3 \\ \eta_{13}^3 \\ \eta_{12}^2\,\eta_{13}^2 \\ \eta_{13}^2\,\eta_{12} \\ \eta_{13}^2 \\ \eta_{12}^3\,\eta_{13} \\ \eta_{13}\,\eta_{12}^2 \\ \eta_{13}\,\eta_{12} \\ \eta_{13} \\ \eta_{12}^4 \\ \eta_{12}^3 \\ \eta_{12}^2 \\ \eta_{12} \\ 1 \end{bmatrix} = \mathtt{M}\,\mathtt{m} = 0 \tag{6.117}$$

3

$n$

1

$f_1$

=

1

with

$$\mathtt{M} = \begin{bmatrix}
0 & 0 & 0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 \\
0 & 0 & 0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 \\
0 & 0 & 0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 \\
0 & 0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 & 0 \\
0 & 0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 & 0 \\
0 & 0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 & 0 \\
0 & m_1 & 0 & -m_7 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 & 0 & 0 & 0 \\
0 & m_5 & 0 & m_9 & -m_{10} & 0 & -m_3 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -m_1 & 0 & 0 & m_{11} & 0 & m_3 & -m_{12} & m_6 & 0 & 0 & 0 & 0 & 0 \\
m_1 & 0 & -m_7 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_5 & 0 & m_9 & -m_{10} & 0 & -m_3 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-m_1 & 0 & 0 & m_{11} & 0 & m_3 & -m_{12} & m_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_1 & 0 & 0 & -m_7 & 0 & 0 & 0 & m_4 & m_8 & -m_2 & 0 & 0 \\
0 & 0 & m_5 & 0 & 0 & m_9 & -m_{10} & 0 & 0 & -m_3 & 0 & m_2 & 0 & 0 \\
0 & 0 & -m_1 & 0 & 0 & 0 & m_{11} & 0 & 0 & m_3 & -m_{12} & m_6 & 0 & 0
\end{bmatrix} \tag{6.118}$$

$15 \times 14$ Macaulay matrix

and

$$
\begin{array}{lllllllllll}
m_1 & = & d_{12}^2 & m_4 & = & d_{12}^2 - d_{23}^2 & m_7 & = & 2\,d_{12}^2\,c_{23} & m_{10} & = & 2\,d_{23}^2\,c_{31} \\
m_2 & = & d_{23}^2 & m_5 & = & d_{23}^2 - d_{31}^2 & m_8 & = & 2\,d_{23}^2\,c_{12} & m_{11} & = & 2\,d_{12}^2\,c_{31} \\
m_3 & = & d_{31}^2 & m_6 & = & d_{31}^2 - d_{12}^2 & m_9 & = & 2\,d_{31}^2\,c_{23} & m_{12} & = & 2\,d_{31}^2\,c_{12}
\end{array}
\tag{6.119}
$$

Matrix $\mathtt{M}$ contains coefficients and vector $\mathtt{m}$ contains the monomials.

Notice in Equation 6.117 that the last five monomials contain only on $\eta_{12}$. We have deliberately ordered monomials to achieve this. Next, we do Gaussian elimination (with pivoting) of matrix $\mathtt{M}$ and get a new matrix $\mathtt{M}'$.

One can verify that that the 10th row of $\mathtt{M}'$ has the first nine elements equal to zero. Therefore

$$
\mathtt{M}'_{10,:}\,\mathtt{m} = 0 \tag{6.120}
$$

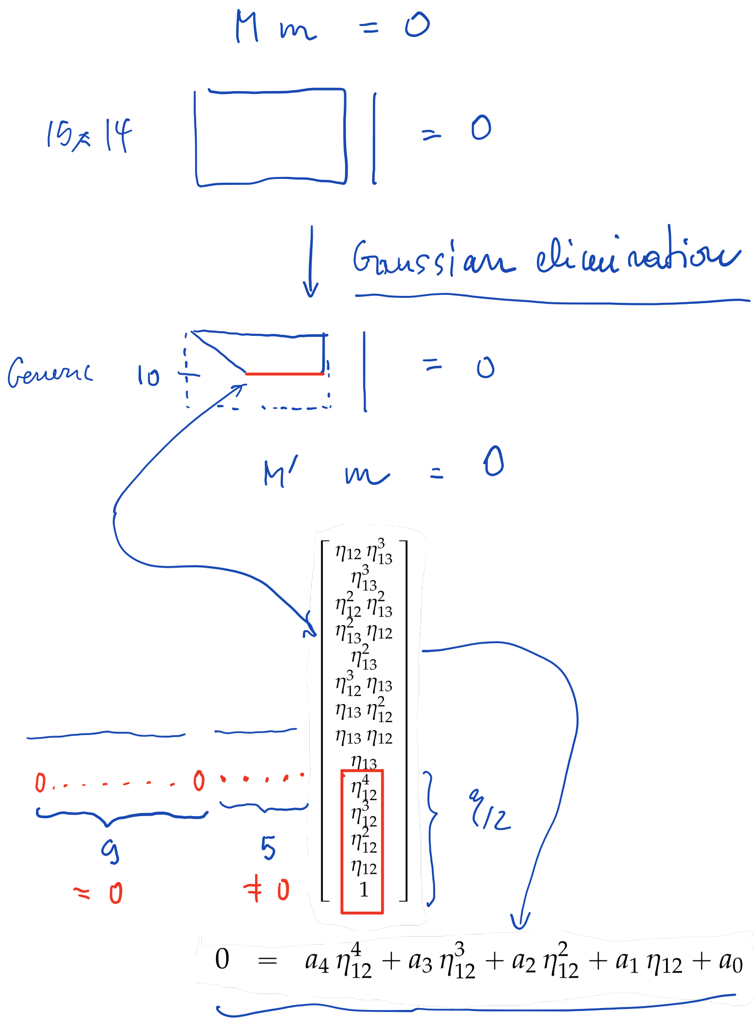
is a polynomial only in $\eta_{12}$. In fact, it is exactly a non-zero multiple of polynomials obtained in cases A, B1, B2.1 and B2.2 above.

Discussion of the cases happens in the Gaussian elimination with pivoting, which avoids dividing by elements close to zero. The resulting polynomial may be of degree four (case A) but will have lower degrees in other cases.

**§ 12 Computing camera orientation and camera center** Having quantities $\eta_1, \eta_2, \eta_3$, we shall compute camera projection center $\vec{C}_\delta$ and camera rotation $\mathtt{R}$ from Equation 6.24.

The three points $X_1$, $X_2$ and $X_3$ are represented in the world coordinate system $(O, \delta)$ by vectors $\vec{X}_{1\delta}$, $\vec{X}_{2\delta}$ and $\vec{X}_{3\delta}$. With known $\eta_1, \eta_2, \eta_3$, we can represent them also in the camera (orthonormal) coordinate system $(C, \epsilon)$ by vectors

$$
\vec{Y}_{i\epsilon} = \eta_i\,\vec{y}_{i\epsilon} = \eta_i\,\frac{\vec{x}_{i\epsilon}}{||\vec{x}_{i\epsilon}||} = \eta_i\,\frac{f\,\vec{x}_{i\gamma}}{||f\,\vec{x}_{i\gamma}||} = \eta_i\,\frac{\vec{x}_{i\gamma}}{||\vec{x}_{i\gamma}||}, \qquad i = 1,2,3 \tag{6.121}
$$

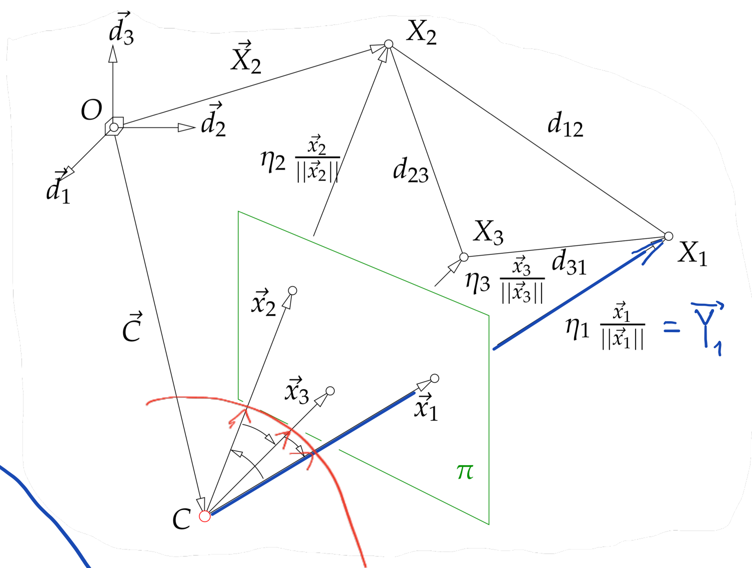


Coordinate vectors $\vec{X}_{i\delta}$ are related to coordinate vectors $\vec{Y}_{i\epsilon}$ as follows

$$\vec{Y}_{i\epsilon} = \eta_i\, \vec{y}_{i\epsilon} = \eta_i \frac{\vec{x}_{i\epsilon}}{||\vec{x}_{i\epsilon}||} = \eta_i \frac{f\, \vec{x}_{i\gamma}}{||f\, \vec{x}_{i\gamma}||} = \eta_i \frac{\vec{x}_{i\gamma}}{||\vec{x}_{i\gamma}||}, \qquad i = 1,2,3$$

$$\vec{Y}_{1\epsilon} = \mathtt{R}\,(\vec{X}_{1\delta} - \vec{C}_\delta) \tag{6.122}$$
$$\vec{Y}_{2\epsilon} = \mathtt{R}\,(\vec{X}_{2\delta} - \vec{C}_\delta) \tag{6.123}$$
$$\vec{Y}_{3\epsilon} = \mathtt{R}\,(\vec{X}_{3\delta} - \vec{C}_\delta) \tag{6.124}$$

Eliminate $\vec{C}_\delta$




There are three vector equations in $\mathbb{R}^3$, which is nine scalar equations, and 12 unknowns in $\mathtt{R}$ and $\vec{C}_\delta$. Additional seven equations are provided by the fact that $\mathtt{R}$ is an orthonormal matrix, i.e. $\boxed{\mathtt{R}^\top \mathtt{R} = \mathtt{I} \text{ and } |\mathtt{R}| = 1}$.

To compute $\mathtt{R}$, we shall next eliminate $\vec{C}_\delta$ from Equations 6.122-6.124

3x1 3x1

Unknown R : 2 eqns,

$$\vec{Y}_{2\epsilon} - \vec{Y}_{1\epsilon} = \mathtt{R}\,(\vec{X}_{2\delta} - \vec{X}_{1\delta}) \tag{6.125}$$
$$\vec{Y}_{3\epsilon} - \vec{Y}_{1\epsilon} = \mathtt{R}\,(\vec{X}_{3\delta} - \vec{X}_{1\delta}) \tag{6.126}$$

$(R^{-1})^{-1} = R$

and use the property (Equation 1.50 in Section 1.3)

+ 1 eqn

$$\boxed{\vec{X}_\epsilon \times \vec{Y}_\epsilon = \frac{\mathtt{R}^{-\top}}{|\mathtt{R}^{-\top}|}(\vec{X}_\delta \times \vec{Y}_\delta) = \mathtt{R}\,(\vec{X}_\delta \times \vec{Y}_\delta)} \tag{6.127}$$

of the vector product of any two vectors $\vec{X}$, $\vec{Y}$ in $\mathbb{R}^3$ and an orthonormal matrix $\mathtt{R}$ to write

$$(\vec{Y}_{2\epsilon} - \vec{Y}_{1\epsilon}) \times (\vec{Y}_{3\epsilon} - \vec{Y}_{1\epsilon}) = \left(\mathtt{R}\,(\vec{X}_{2\delta} - \vec{X}_{1\delta})\right) \times \left(\mathtt{R}\,(\vec{X}_{3\delta} - \vec{X}_{1\delta})\right) \tag{6.128}$$
$$= \mathtt{R}\left((\vec{X}_{2\delta} - \vec{X}_{1\delta}) \times (\vec{X}_{3\delta} - \vec{X}_{1\delta})\right) \tag{6.129}$$

General formula

See the following three pages

which provides a triplet of independent vectors expressed in the two bases

$$\vec{Z}_{2\epsilon} = \vec{Y}_{2\epsilon} - \vec{Y}_{1\epsilon}, \quad \vec{Z}_{2\delta} = \vec{X}_{2\delta} - \vec{X}_{1\delta} \tag{6.130}$$
$$\vec{Z}_{3\epsilon} = \vec{Y}_{3\epsilon} - \vec{Y}_{1\epsilon}, \quad \vec{Z}_{3\delta} = \vec{X}_{3\delta} - \vec{X}_{1\delta} \tag{6.131}$$
$$\vec{Z}_{1\epsilon} = \vec{Z}_{2\epsilon} \times \vec{Z}_{3\epsilon}, \quad \vec{Z}_{1\delta} = \vec{Z}_{2\delta} \times \vec{Z}_{3\delta} \tag{6.132}$$

Rotation R can then be recovered from

rank 3

$$\begin{bmatrix} \vec{Z}_{1\epsilon} & \vec{Z}_{2\epsilon} & \vec{Z}_{3\epsilon} \end{bmatrix} = \mathtt{R} \begin{bmatrix} \vec{Z}_{1\delta} & \vec{Z}_{2\delta} & \vec{Z}_{3\delta} \end{bmatrix} \tag{6.133}$$

rank 3

as

$$\mathtt{R} = \begin{bmatrix} \vec{Z}_{1\epsilon} & \vec{Z}_{2\epsilon} & \vec{Z}_{3\epsilon} \end{bmatrix} \begin{bmatrix} \vec{Z}_{1\delta} & \vec{Z}_{2\delta} & \vec{Z}_{3\delta} \end{bmatrix}^{-1} \tag{6.134}$$

With known R we get $\vec{C}_\delta$ as

$$\boxed{\vec{C}_\delta = \vec{X}_{i\delta} - \mathtt{R}^\top \vec{Y}_{i\epsilon}, \qquad i = 1,2,3} \tag{6.135}$$

§ 1 **Vector product** Assume two linearly independent coordinate vectors $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$ in $\mathbb{R}^3$. The following system of linear equations

rank 2

$$\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}}_{2\times 3} \vec{z} = 0 \quad \rightarrow \text{1D s of sols} \tag{1.41}$$

has a one-dimensional subspace $V$ of solutions in $\mathbb{R}^3$. The solutions can be written as multiples of one non-zero vector $\vec{w}$, the basis of $V$, i.e.

$$\boxed{\vec{z} = \lambda\, \vec{w}, \quad \lambda \in \mathbb{R}} \tag{1.42}$$

Let us see how we can construct $\vec{w}$ in a convenient way from vectors $\vec{x}$, $\vec{y}$.

Consider determinants of two matrices constructed from the matrix of the system (1.41) by adjoining its first, resp. second, row to the matrix of the system (1.41)

$$\left|\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}}_{3\times 3}\right| = 0 \qquad \left|\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 \end{bmatrix}}_{3\times 3}\right| = 0 \tag{1.43}$$

which gives

$$x_1\,(x_2\,y_3 - x_3\,y_2) + x_2\,(x_3\,y_1 - x_1\,y_3) + x_3\,(x_1\,y_2 - x_2\,y_1) = 0 \tag{1.44}$$

$$y_1\,\underline{(x_2\,y_3 - x_3\,y_2)} + y_2\,\underline{(x_3\,y_1 - x_1\,y_3)} + y_3\,\underline{(x_1\,y_2 - x_2\,y_1)} = 0 \tag{1.45}$$

and can be rewritten as

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \underline{\begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix}} = 0 \tag{1.46}$$

We see that vector

$$\vec{w} = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} \tag{1.47}$$

General formula

$$\boxed{\vec{X}_\epsilon \times \vec{Y}_\epsilon = \frac{\mathtt{R}^{-\top}}{|\mathtt{R}^{-\top}|}(\vec{X}_\delta \times \vec{Y}_\delta) = \mathtt{R}\,(\vec{X}_\delta \times \vec{Y}_\delta)}$$

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

solves Equation 1.41.

Notice that elements of $\vec{w}$ are the three two by two minors of the matrix of the system (1.41). The rank of the matrix is two, which means that at least one of the minors is non-zero, and hence $\vec{w}$ is also non-zero. We see that $\vec{w}$ is a basic vector of $V$. Formula 1.47 is known as the *vector product* in $\mathbb{R}^3$ and $\vec{w}$ is also often denoted by $\vec{x} \times \vec{y}$.

**§ 2 Vector product under the change of basis** Let us next study the behavior of the vector product under the change of basis in $\mathbb{R}^3$. Let us have two bases $\beta$, $\beta'$ in $\mathbb{R}^3$ and two vectors $\vec{x}$, $\vec{y}$ with coordinates $\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, $\vec{y}_\beta = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$ and $\vec{x}_{\beta'} = \begin{bmatrix} x'_1 & x'_2 & x'_3 \end{bmatrix}^\top$, $\vec{y}_{\beta'} = \begin{bmatrix} y'_1 & y'_2 & y'_3 \end{bmatrix}^\top$. We introduce

$$\vec{x}_\beta \times \vec{y}_\beta = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} \qquad \vec{x}_{\beta'} \times \vec{y}_{\beta'} = \begin{bmatrix} x'_2 y'_3 - x'_3 y'_2 \\ -x'_1 y'_3 + x'_3 y'_1 \\ x'_1 y'_2 - x'_2 y'_1 \end{bmatrix} \tag{1.48}$$

$$\begin{tikzcd} \vec{x}_\beta, \vec{y}_\beta \arrow[r] \arrow[d, "R\ A"] & \vec{x}_\beta \times \vec{y}_\beta \arrow[d, "R\ \frac{A^{-\top}}{|A^\top|} \neq A"] \\ \vec{x}_{\beta'}, \vec{y}_{\beta'} \arrow[r] & \vec{x}_{\beta'} \times \vec{y}_{\beta'} \end{tikzcd}$$

To find the relationship between $\vec{x}_\beta \times \vec{y}_\beta$ and $\vec{x}_{\beta'} \times \vec{y}_{\beta'}$, we will use the following fact. For every three vectors $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, $\vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^\top$, $\vec{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^\top$ in $\mathbb{R}^3$ there holds

$\vec{z}^\top (\vec{x} \times \vec{y}) =$

$$\vec{z}^\top (\vec{x} \times \vec{y}) = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} = \left| \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \right| = \left| \begin{bmatrix} \vec{x}^\top \\ \vec{y}^\top \\ \vec{z}^\top \end{bmatrix} \right| \tag{1.49}$$

A general fact

*

We can write

$$
\begin{aligned}
\vec{x}_{\beta'} \times \vec{y}_{\beta'} &= \begin{bmatrix} [1\,0\,0]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \\ [0\,1\,0]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \\ [0\,0\,1]\,(\vec{x}_{\beta'} \times \vec{y}_{\beta'}) \end{bmatrix} \overset{*}{=} \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 1\,0\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 0\,1\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta'}^\top \\ \vec{y}_{\beta'}^\top \\ 0\,0\,1 \end{bmatrix}\right| \end{bmatrix}^\top \\
&= \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_{\beta}^\top \mathtt{A}^\top \\ \vec{y}_{\beta}^\top \mathtt{A}^\top \\ 1\,0\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta}^\top \mathtt{A}^\top \\ \vec{y}_{\beta}^\top \mathtt{A}^\top \\ 0\,1\,0 \end{bmatrix}\right| & \left|\begin{bmatrix} \vec{x}_{\beta}^\top \mathtt{A}^\top \\ \vec{y}_{\beta}^\top \mathtt{A}^\top \\ 0\,0\,1 \end{bmatrix}\right| \end{bmatrix}^\top \\
&= \begin{bmatrix} \left|\begin{bmatrix} \vec{x}_{\beta}^\top \\ \vec{y}_{\beta}^\top \\ [1\,0\,0]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| & \left|\begin{bmatrix} \vec{x}_{\beta}^\top \\ \vec{y}_{\beta}^\top \\ [0\,1\,0]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| & \left|\begin{bmatrix} \vec{x}_{\beta}^\top \\ \vec{y}_{\beta}^\top \\ [0\,0\,1]\,\mathtt{A}^{-\top} \end{bmatrix} \mathtt{A}^\top\right| \end{bmatrix}^\top \\
&\overset{*?}{=} \begin{bmatrix} [1\,0\,0]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \\ [0\,1\,0]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \\ [0\,0\,1]\,\mathtt{A}^{-\top}(\vec{x}_\beta \times \vec{y}_\beta) \end{bmatrix} |\mathtt{A}^\top| \\
&= \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,(\vec{x}_\beta \times \vec{y}_\beta) \quad = A^{-\top}|A^\top|\,(\vec{x}_\beta \times \vec{y}_\beta)
\end{aligned}
\tag{1.50}
$$

$\vec{x}_{\beta'} = A\,\vec{x}_\beta$

$\vec{x}_{\beta'}^\top = \vec{x}_\beta^\top A^\top$

**§3 Vector product as a linear mapping** It is interesting to see that for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ there holds

$$
\vec{x} \times \vec{y} = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{1.51}
$$

and thus we can introduce matrix

$$
[\vec{x}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{1.52}
$$

and write

$$
\vec{x} \times \vec{y} = [\vec{x}]_\times\,\vec{y} \tag{1.53}
$$

# *Quiz*