Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 1

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1 Linear algebra

We rely on linear algebra [1, 2, 3, 4, 5, 6]. We recommend excellent text books [4, 1] for acquiring basic as well as more advanced elements of the topic. Monograph [2] provides a number of examples and applications and provides a link to numerical and computational aspects of linear algebra. We will next review the most crucial topics needed in this text.

1.1 Change of coordinates induced by the change of basis

Let us discuss the relationship between the coordinates of a vector in a linear space, which is induced by passing from one basis to another. We shall derive the relationship between the coordinates in a three-dimensional linear space over real numbers, which is the most important when modeling the geometry around us. The formulas for all other n-dimensional spaces are obtained by passing from 3 to n.

§1 Coordinates Let us consider an ordered basis $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ of a three-dimensional vector space V^3 over scalars $\mathbb{R}$. A vector $\vec{v} \in V^3$ is uniquely expressed as a linear combination of basic vectors of V^3 by its *coordinates* $x, y, z \in \mathbb{R}$, i.e. $\vec{v} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$, and can be represented as an ordered triple of coordinates, i.e. as $\boxed{\vec{v}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top.}$

We see that an ordered triple of scalars can be understood as a triple of coordinates of a vector in V^3 w.r.t. a basis of V^3. However, at the same

Basic concepts

- Linear space
- Basis
- Coordinates
- Linear mappings

- 3D Affine space $\equiv \mathbb{R}^3$
- 3D Projective space $\equiv \mathbb{R}^4 / \vec{x} \equiv \alpha\vec{x},\ \alpha \neq 0$

$\left.\begin{array}{l}\bullet\ \text{Points}\\ \bullet\ \text{Lines}\\ \bullet\ \text{Planes}\end{array}\right\} \equiv$ Subspaces of $\mathbb{R}^4$

$\vec{x}$ ... geometrical vector
(arrow $\equiv$ an ordered pair of points)

$\vec{x}_\beta$ ... coordinates of $\vec{x}$ in β

time, the set of ordered triples $\begin{bmatrix} x & y & z \end{bmatrix}^\top$ is also a three-dimensional *coordinate linear space* $\mathbb{R}^3$ over $\mathbb{R}$ with $\begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^\top + \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^\top = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 & z_1 + z_2 \end{bmatrix}^\top$ and $s \begin{bmatrix} x & y & z \end{bmatrix}^\top = \begin{bmatrix} s\,x & s\,y & s\,z \end{bmatrix}^\top$ for $s \in \mathbb{R}$. Moreover, the ordered triple of the following three particular coordinate vectors

$\mathbb{R}^3$

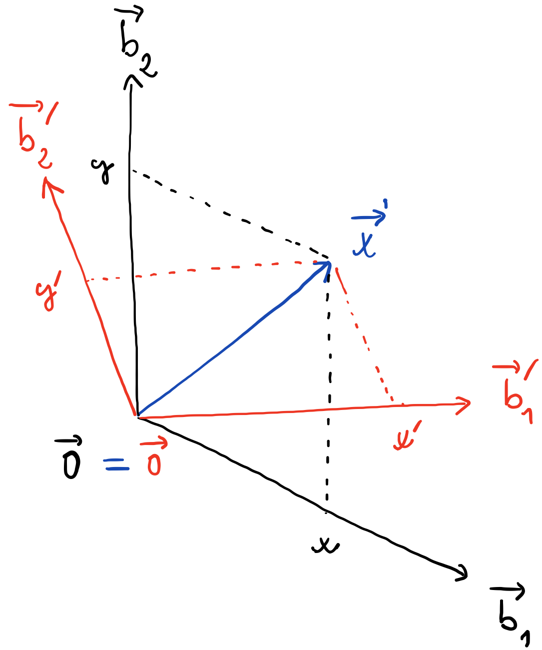
$$\sigma = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \tag{1.1}$$

The standard basis

forms an ordered basis of $\mathbb{R}^3$, the *standard basis*, and therefore a vector $\vec{v} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ is represented by $\vec{v}_\sigma = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ w.r.t. the standard basis in $\mathbb{R}^3$. It is noticeable that the vector $\vec{v}$ and the coordinate vector $\vec{v}_\sigma$ of its coordinates w.r.t. the standard basis of $\mathbb{R}^3$, are identical.

In 2D for simplicity

§2 Two bases Having two ordered bases $\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$ and $\beta' = \begin{bmatrix} \vec{b}_1' & \vec{b}_2' & \vec{b}_3' \end{bmatrix}$ leads to expressing one vector $\vec{x}$ in two ways as $\vec{x} = x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3$ and $\vec{x} = x'\,\vec{b}_1' + y'\,\vec{b}_2' + z'\,\vec{b}_3'$. The vectors of the basis β can also be expressed in the basis β' using their coordinates. Let us introduce



$$\beta \qquad \begin{aligned} \vec{b}_1 &= a_{11}\,\vec{b}_1' + a_{21}\,\vec{b}_2' + a_{31}\,\vec{b}_3' \\ \vec{b}_2 &= a_{12}\,\vec{b}_1' + a_{22}\,\vec{b}_2' + a_{32}\,\vec{b}_3' \\ \vec{b}_3 &= a_{13}\,\vec{b}_1' + a_{23}\,\vec{b}_2' + a_{33}\,\vec{b}_3' \end{aligned} \qquad \beta' \tag{1.2}$$

§3 Change of coordinates We will next use the above equations to relate the coordinates of $\vec{x}$ w.r.t. the basis β to the coordinates of $\vec{x}$ w.r.t. the basis β'

$$\begin{aligned}
\vec{x} &= x\,\vec{b}_1 + y\,\vec{b}_2 + z\,\vec{b}_3 \\
&= x\,(a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3) + y\,(a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3) + z\,(a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3) \\
&= (a_{11}\,x + a_{12}\,y + a_{13}\,z)\,\vec{b}'_1 + (a_{21}\,x + a_{22}\,y + a_{23}\,z)\,\vec{b}'_2 + (a_{31}\,x + a_{32}\,y + a_{33}\,z)\,\vec{b}'_3 \\
&= x'\,\vec{b}'_1 + y'\,\vec{b}'_2 + z'\,\vec{b}'_3 \qquad (1.3)
\end{aligned}$$

Since coordinates are unique, we get

coordinates are unique

$$x' = a_{11}\,x + a_{12}\,y + a_{13}\,z \qquad (1.4)$$

$$y' = a_{21}\,x + a_{22}\,y + a_{23}\,z \qquad (1.5)$$

$$z' = a_{31}\,x + a_{32}\,y + a_{33}\,z \qquad (1.6)$$

Coordinate vectors $\vec{x}_\beta$ and $\vec{x}_{\beta'}$ are thus related by the following matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (1.7)$$

compare

$$\begin{aligned}
\vec{b}_1 &= a_{11}\,\vec{b}'_1 + a_{21}\,\vec{b}'_2 + a_{31}\,\vec{b}'_3 \\
\vec{b}_2 &= a_{12}\,\vec{b}'_1 + a_{22}\,\vec{b}'_2 + a_{32}\,\vec{b}'_3 \\
\vec{b}_3 &= a_{13}\,\vec{b}'_1 + a_{23}\,\vec{b}'_2 + a_{33}\,\vec{b}'_3
\end{aligned}$$

which we concisely write as

$$\boxed{\vec{x}_{\beta'} = \mathtt{A}\,\vec{x}_\beta} \qquad (1.8)$$

The columns of matrix $\mathtt{A}$ can be viewed as vectors of coordinates of basic vectors, $\vec{b}_1, \vec{b}_2, \vec{b}_3$ of β in the basis β'

$$\mathtt{A} = \begin{bmatrix} | & | & | \\ \vec{b}_{1_{\beta'}} & \vec{b}_{2_{\beta'}} & \vec{b}_{3_{\beta'}} \\ | & | & | \end{bmatrix} \qquad (1.9)$$

and the matrix multiplication can be interpreted as a linear combination of the columns of $\mathtt{A}$ by coordinates of $\vec{x}$ w.r.t. β

$$\vec{x}_{\beta'} = x\,\vec{b}_{1_{\beta'}} + y\,\vec{b}_{2_{\beta'}} + z\,\vec{b}_{3_{\beta'}} \qquad (1.10)$$

Columns of are basic vectors of β in β'

$$\vec{x}_\beta \xrightarrow{A} \vec{x}_{\beta'}$$

Coordinates $\vec{x}_\beta = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ must combine vectors of β

Matrix A plays such an important role here that it deserves its own name. Matrix A is very often called the *change of basis matrix from basis β to β'* or the *transition matrix from basis β to basis β'* [2, 7] since it can be used to pass from coordinates w.r.t. β to coordinates w.r.t. β' by Equation 1.8.

However, literature [3, 8] calls A the *change of basis matrix from basis β' to β*, i.e. it (seemingly illogically) swaps the bases. This choice is motivated by the fact that A relates vectors of β and vectors of β' by Equation 1.2 as

$$\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}a_{11}\vec{b}'_1 + a_{21}\vec{b}'_2 + a_{31}\vec{b}'_3 & a_{12}\vec{b}'_1 + a_{22}\vec{b}'_2 + a_{32}\vec{b}'_3 \\ & a_{13}\vec{b}'_1 + a_{23}\vec{b}'_2 + a_{33}\vec{b}'_3\end{bmatrix} \quad (1.11)$$

$$\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} \underbrace{\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}}_{A} \quad (1.12)$$

$$\begin{aligned}\vec{b}_1 &= a_{11}\vec{b}'_1 + a_{21}\vec{b}'_2 + a_{31}\vec{b}'_3\\ \vec{b}_2 &= a_{12}\vec{b}'_1 + a_{22}\vec{b}'_2 + a_{32}\vec{b}'_3\\ \vec{b}_3 &= a_{13}\vec{b}'_1 + a_{23}\vec{b}'_2 + a_{33}\vec{b}'_3\end{aligned}$$

and therefore giving

$$\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} \mathtt{A} \quad (1.13)$$

or equivalently

$$\begin{bmatrix}\vec{b}'_1 & \vec{b}'_2 & \vec{b}'_3\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} \mathtt{A}^{-1} \quad (1.14)$$

Let's get to

where the multiplication of a row of column vectors by a matrix from the right in Equation 1.13 has the meaning given by Equation 1.11 above. Yet another variation of the naming appeared in [5, 6] where $\mathtt{A}^{-1}$ was named the *change of basis matrix from basis β to β'*.

We have to conclude that the meaning associated with the *change of basis matrix* varies in the literature and hence we will avoid this confusing name and talk about A as about the *matrix transforming coordinates of a vector from basis β to basis β'*.

There is the following interesting variation of Equation 1.13

$$\boxed{\begin{bmatrix} \vec{b}_1' \\ \vec{b}_2' \\ \vec{b}_3' \end{bmatrix} = \mathtt{A}^{-\top} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}} \tag{1.15}$$

where the basic vectors of β and β' are understood as elements of column vectors. For instance, vector $\vec{b}_1'$ is obtained as

$$\vec{b}_1' = a_{11}^\star\, \vec{b}_1 + a_{12}^\star\, \vec{b}_2 + a_{13}^\star\, \vec{b}_3 \tag{1.16}$$

where $[a_{11}^\star, a_{12}^\star, a_{13}^\star]$ is the first row of $\mathtt{A}^{-\top}$.

§ 4 Example We demonstrate the relationship between vectors and bases on a concrete example. Consider two bases α and β represented by coordinate vectors, which we write into matrices

$$\alpha = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.17}$$

$$\beta = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{1.18}$$

and a vector $\vec{x}$ with coordinates w.r.t. the basis α

$$\vec{x}_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.19}$$

We see that basic vectors of α can be obtained as the following linear combinations of basic vectors of β

$$\vec{a}_1 = +1\,\vec{b}_1 + 0\,\vec{b}_2 + 0\,\vec{b}_3 \quad (1.20)$$
$$\vec{a}_2 = +1\,\vec{b}_1 - 1\,\vec{b}_2 + 1\,\vec{b}_3 \quad (1.21)$$
$$\vec{a}_3 = -1\,\vec{b}_1 + 0\,\vec{b}_2 + 1\,\vec{b}_3 \quad (1.22)$$

$$\alpha = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$
$$\beta = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 1 & 1\end{bmatrix},$$

$$\vec{x}_\alpha = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

or equivalently

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}\mathtt{A}$$

Coordinates of $\vec{x}$ w.r.t. β are hence obtained as

$$\vec{x}_\beta = \mathtt{A}\,\vec{x}_\alpha, \qquad \mathtt{A} = \begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 1 & 1\end{bmatrix} \quad (1.24)$$

$$\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} \quad (1.25)$$

$$\boxed{\begin{array}{l}\vec{x}_{\beta'} = A\,\vec{x}_\beta\\ \beta = \beta' A\end{array}}$$

Vectors

Bases

We see that

$$\alpha = \beta\,\mathtt{A} \quad (1.26)$$

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 1 & 1\end{bmatrix} \quad (1.27)$$

The following questions arises: When are the coordinates of a vector $\vec{x}$ (Equation 1.8) and the basic vectors themselves (Equation 1.15) transformed in the same way? In other words, when $\mathtt{A} = \mathtt{A}^{-\top}$. We shall give the answer to this question later in paragraph 1.4.

Quiz

4 Image coordinate system

Digital image Im is a matrix of pixels. We assume that Im is obtained by measuring intensity of light by sensors (pixels) arranged in a grid, Figure 4.1.

We will work with images in two ways. First, we will work with intensity values, which are stored in the memory as a three-dimensional array of bytes indexed by the row index `i`, the column index `j`, and the color index `k`, Figure 4(a). Color index attains three values 1, 2, 3, with 1 corresponding to red, 2 corresponding to green and 3 corresponding to blue colors.

In `Matlab`, image `Im` is accessed using the row index `i`, the column index `j` and the color index `k` as `>>Im(i,j,k)`. The most top left pixel has row as well as column index equal to 1. The red channel of the pixel with row index 2 and column index 3 is accessed as `>>Im(2,3,1)`.

§1 Image coordinate system For geometrical computation, we introduce an *image coordinate system* as in Figure 4(b). The origin of the image coordinate system is chosen to assign coordinates 1, 1 to the center of the most top left pixel. Horizontal axis $\vec{b}_1$ goes from left to right. The vertical axis $\vec{b}_2$ goes from top down. The pixel that is accessed as `>>Im(i,j,k)` is in the image coordinate system represented by the vector $\vec{u} = [\mathtt{j}, \mathtt{i}]^\top$. A digital image with H rows and W columns will be in indexed in `Matlab` as `>>Im(1:H,1:W,1:3)` and `>>size(Im)` will return [H W 3]. The center of the most bottom right pixel will have coordinates $[W, H]^\top$ in the image coordinate system.

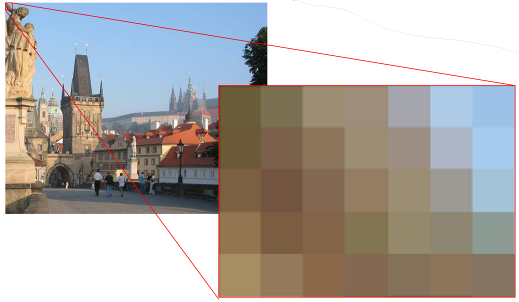
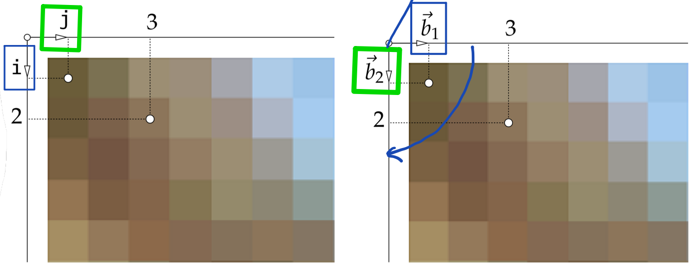


Figure 4.1: Image is digitized by a rectangular array of pixels

$\vec{b}_3$



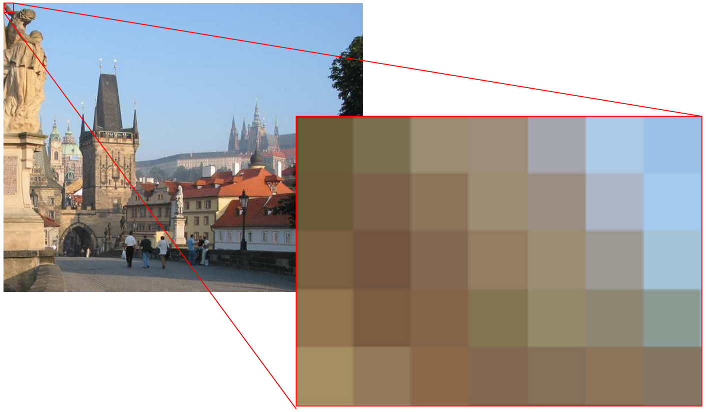
(a) Image `Im` is a matrix of pixels. In `Matlab`, it is accessed using the row index `i`, the column index `j` and color index `k` as >> Im(i,j,k). The most top left pixel has row as well as column index equal to 1. The red channel of the pixel with row index 2 and column index 3 is accessed as >> Im(2,3,1).

(b) The image coordinate system is defined with horizontal axis $\vec{b}_1$ and vertical axis $\vec{b}_2$. The origin of the coordinate system is chosen to assign coordinates 1, 1 to the most top left pixel. Notice that pixel, which is accessed as >> Im(2,3,1), is represented in the image coordinate system by the vector $\vec{u} = [3, 2]^\top$.

The image coordinate system coincides with the `Matlab` coordinate system image, i.e. commands

```
>> axis image
>> plot(j,i,'.b')
```

plot a blue dot on the pixel accessed as `>>Im(i,j,:)`;



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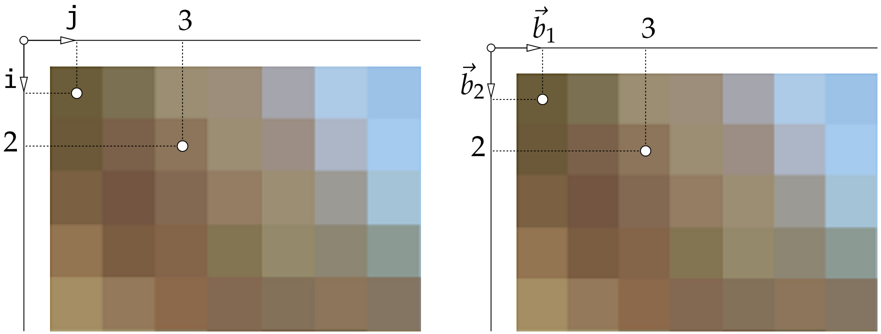
```
>> axis image
>> plot(j,i,'.b')
```

plot a blue dot on the pixel accessed as `>>Im(i,j,:);`

The image coordinate system is non-standard in two dimensions since it is a left-handed system. The reason for such a unnatural choice is that this system will be next augmented into a three-dimensional right-handed coordinate system in such a way that the $\vec{b}_3$ vector will be pointing towards the scene.



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(b) The image coordinate system is defined with horizontal axis $\vec{b}_1$ and vertical axis $\vec{b}_2$. The origin of the coordinate system is chosen to assign coordinates $1, 1$ to the most top left pixel. Notice that pixel, which is accessed as >> Im(2,3,1), is represented in the image coordinate system by the vector $\vec{u} = [3, 2]^\top$.

Figure 4.2: Image coordinate system.