Probabilistic decisions

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(Re-)introduction uncertainty/probability

- Markov Decision Processes (MDP)/RL – uncertainty about outcome of actions
  - *Sequential* decisions (robot/agent goes from s_0 to s_G)
  - $\pi : \mathcal{S} \rightarrow \mathcal{A}$
  - *Policy (Strategy):* knowing what to do for all possible states.
- Now: uncertainty associated with states
  - Different states may have different prior probabilities.
  - The states $s \in \mathcal{S}$ are not directly observable.
  - They need to be inferred from features $x \in \mathcal{X}$.
  - *Single (repeated)* decision $\delta : \mathcal{X} \rightarrow \mathcal{S}$ ($\delta : \mathcal{X} \rightarrow \mathcal{D}$);
  - *Strategy:* knowing how to decide for all possible measurements.
- Decision example, crossing street:
  - x = camera image; $\mathcal{X}$ is the space of all possible images
  - $\mathcal{S} = \{\text{car, bus, bicycle, truck}\}$ approaching
  - I decide to: $\mathcal{D} = \{\text{go, wait}\}$

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Decision example: Insure or not? (from late 1980s) [5]

Known about HIV testing: HIV test falsely positive only in 1 case out of 1000.
A doctor calls: "Your HIV test is positive, 999/1000 you will die in 10 years. I'm sorry …".
Insurance company does not want to insure a married couple.

- Was the doctor right?
- Was the insurance company rational?

$\mathcal{S} = \{\text{healthy}, \text{infected}\}$, $\mathcal{X} = \{\text{positive\_test}, \text{negative\_test}\}$

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What is the probability the man is infected?

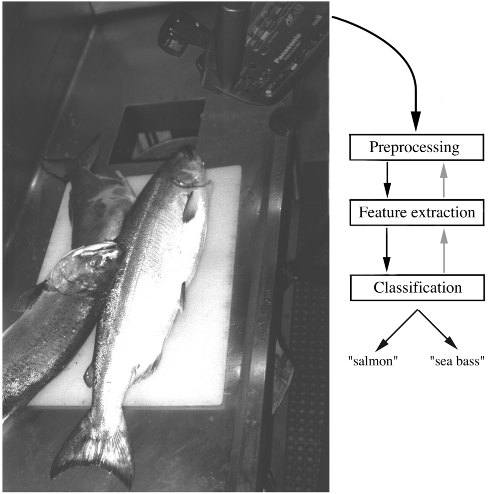
A: $\frac{1}{1000}$

B: $\frac{999}{1000}$

C: Don't know yet, more info needed, but less than $\frac{1}{2}$

D: Don't know yet, more info needed, but more than $\frac{1}{2}$

Classification example: What's the fish?



- Factory for fish processing
- 2 classes $s_{1,2}$:
  - salmon
  - sea bass
- Features $\vec{x}$: length, width, lightness etc. from a camera

Fish – classification using probability

$$posterior = \frac{likelihood \times prior}{evidence}$$

- Notation for classification problem
  - Classes $s_j \in \mathcal{S}$ (e.g., salmon, sea bass)
  - Features $x_i \in \mathcal{X}$ or feature vectors ($\vec{x}_i$) (also called attributes)
- Optimal classification of $\vec{x}$:

$$\delta^*(\vec{x}) = \arg\max_j P(s_j|\vec{x})$$

- We thus choose the most probable class for a given feature vector .
- Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j|\vec{x}) = \frac{P(\vec{x}|s_j)P(s_j)}{P(\vec{x})}$$

- Can we do (classify) better?

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Decision making under uncertainty

- An important feature of intelligent systems
  - make the best possible decision
  - in uncertain conditions
- **Example**: Take a tram OR subway from A to B?
  - Tram: timetables imply a quicker route, but adherence uncertain.
  - Subway: longer route, but adherence almost certain.
- **Example**: where to route a letter with this ZIP?
  - 15700? 15706? 15200? 15206?
- What is the optimal decision ?
- What is the cost of the decision? What is the associated loss ?
- What is the relation between loss and utility ?

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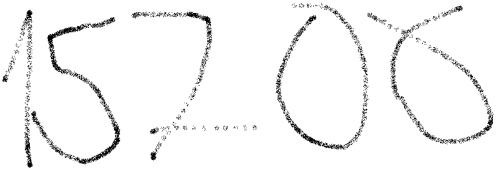

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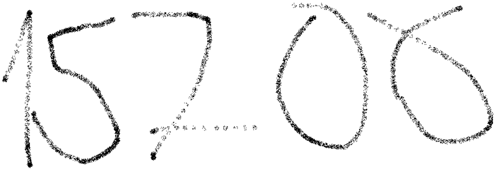

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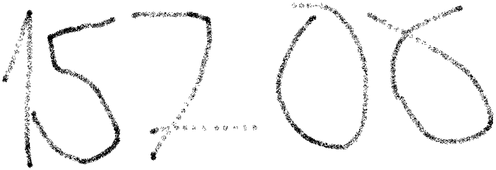

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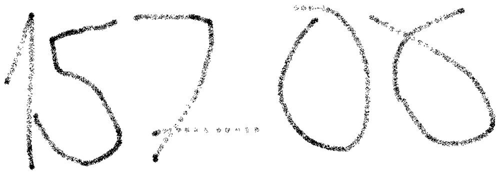

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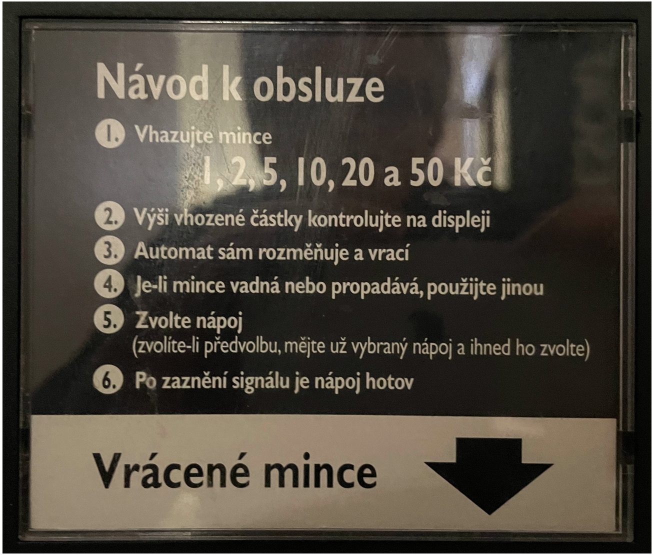
Introducing decision loss: Coin recognition



Recognizing/classifying coins: components

- $s \in \{1, 2, 5, 10, 20, 50\}$ – state - the true value
- $x \in \{0.0, 0.1, \cdots, 9.9\}[g]$ – measurement, observation
- $P(s, x)$ joint probability
- $d \in \{1, 2, 5, 10, 20, 50\}$– decision, result of the algorithm

Loss function $\ell(?)$
is a function of:

A s

B s, d

C s, x, d

D d

How many strategies?:

A 100

B 100^6

C 600

D 6^{100}

Strategy $d = \delta(?)$
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Introducing decision loss: What to cook for dinner [4]

- *Wife is coming back from work. Husband: what to cook for dinner?*
- 3 dishes (decisions) in his repertoire:
  - *nothing* … **don't bother cooking** ⇒ no work but makes wife upset
  - *pizza* … **microwave a frozen pizza** ⇒ not much work but won't impress
  - *g.T.c.* … **general Tso's chicken** ⇒ will make her day, but very laborious
- "Hassle" incurred by the individual options depends on wife's mood.
- For each of the 9 possible situations (3 possible decisions × 3 possible states), the cost is quantified by a loss function $\ell(d, s)$:

$\ell(s, d)$	$d =$ *nothing*	$d =$ *pizza*	$d =$ *g.T.c.*
$s =$ *good*	0	2	4
$s =$ *average*	5	3	5
$s =$ *bad*	10	9	6

The wife's state of mind is an uncertain state.

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Example (cont'd), State uncertain, symptoms, ...

- Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction.
- Anticipates 4 possible reactions:
  - *mild* ... all right, we keep our memories.
  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
  - *alarming* ... silence
- The reaction is a measurable attribute/symptom ("feature") of the mind state.
- From experience, the husband knows how probable individual reactions are in each state of mind; this is captured by the joint distribution $P(x, s)$.

$P(x, s)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$s = good$	0.35	0.28	0.07	0.00
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- i.e. function $d = \delta(x)$.
- Example of husband's possible strategies:

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$\delta_3(x) =$	*g.T.c.*	*g.T.c.*	*g.T.c.*	*g.T.c.*
$\delta_4(x) =$	*nothing*	*nothing*	*nothing*	*nothing*

- How many strategies?
- How to define which strategy is the best? How to sort them by quality?
- Define the risk of a strategy as a mean (expected) loss value .

$$r(\delta) = \sum_x \sum_s \ell(s, \delta(x)) P(x, s)$$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Do we need to evaluate all possible strategies? $P(x,s) = P(s|x)P(x)$

Calculating $r(\delta) = \sum_x \sum_s \ell(s, \delta(x)) P(x, s)$

$\ell(s,d)$	$d = nothing$	$d = pizza$	$d = g.T.c.$
$s = good$	0	2	4
$s = average$	5	3	5
$s = bad$	10	9	6

$P(x,s)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$s = good$	0.35	0.28	0.07	0.00
$s = average$	0.04	0.10	0.04	0.02
$s = bad$	0.00	0.02	0.05	0.03

$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$\delta_1(x) =$	$nothing$	$nothing$	$pizza$	$g.T.c.$
$\delta_2(x) =$	$nothing$	$pizza$	$g.T.c.$	$g.T.c.$
$\delta_3(x) =$	$g.T.c.$	$g.T.c.$	$g.T.c.$	$g.T.c.$
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Do we need to evaluate all possible strategies? $P(x,s) = P(s|x)P(x)$

Bayes optimal strategy

- The Bayes optimal strategy : one minimizing mean risk.

$$\delta^* = \arg\min_{\delta} r(\delta)$$

- From $P(x, s) = P(s|x)P(x)$ (Bayes rule), we have

$$r(\delta) = \sum_x \sum_s \ell(s, \delta(x))P(x, s) = \sum_s \sum_x \ell(s, \delta(x))P(s|x)P(x)$$

$$= \sum_x P(x) \underbrace{\sum_s \ell(s, \delta(x))P(s|x)}_{\text{Conditional risk}}$$

- The optimal strategy is obtained by minimizing the conditional risk *separately* for each x:

$$\delta^*(x) = \arg\min_{d} \sum_s \ell(s, d)P(s|x)$$

Optimal strategy: $\delta^*(x) = \arg\min_d \sum_s \ell(s,d)P(s|x)$

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$\delta^*(x) =$	??	??	??	??

Statistical decision making: wrapping up

- **Given**:
  - A set of possible states : $\mathcal{S}$
  - A set of possible decisions : $\mathcal{D}$
  - A loss function $\ell : \mathcal{D} \times \mathcal{S} \to \Re$
  - The range $\mathcal{X}$ of the attribute
  - Distribution $P(x, s)$, $x \in \mathcal{X}, s \in \mathcal{S}$.
- **Define**:
  - Strategy : function $\delta : \mathcal{X} \to \mathcal{D}$
  - Risk of strategy δ : $r(\delta) = \sum_x \sum_s \ell(s, \delta(x)) P(x, s)$
- **Bayes problem**:
  - Goal: find the optimal strategy $\delta^* = \arg\min_\delta r(\delta)$
  - Solution: $\delta^*(x) = \arg\min_d \sum_s \ell(s, d) P(s|x)$ (for each x)

A special case - Bayesian *classification*

- Bayesian classification is a special case of statistical decision theory:
  - Attribute vector $\vec{x} = (x_1, x_2, \dots)$: pixels 1, 2, ...
  - **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \dots 9\}$.**
  - State = actual class, Decision = recognized class
  - Loss function:

$$\ell(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{\ell(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d [1 - P(d|\vec{x})] = \arg\max_d P(d|\vec{x})$$

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References I

Further reading: Chapter 13 and 14 of [7] (Chapters 12 and 13 in [8]). Books [2] (for this lecture, read Chapter 1) and [3] are classical textbooks in the field of pattern recognition and machine learning. Interesting insights into how people think and interact with probabilities are presented in [5] (in Czech as [6]).

[1] People vs. Collins.
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[4] Zdeněk Kotek, Petr Vysoký, and Zdeněk Zdráhal.
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Additional material for thinking

Decision: guilty or not? (people of CA vs Collins, 1968) [5]

- Robbery, LA 1964, fuzzy evidence of the offenders:
  - female, around 65 kg
  - wearing something dark
  - hair of light color, between light and dark blond, in a ponytail
- At the same time, additional evidence close to the crime scene:
  - loud scream, yelling, looking at the this direction ...
  - a woman sitting into a yellow car
  - car starts immediately and passes close to the additional witness
  - a black man with beard and moustache was driving
- No more evidence
- Testimony of both the victim and the witness not unambiguous (didn't recognize suspects)
- Still, the suspects were sentenced to jail.

Decision: guilty or not? (people of CA vs Collins, 1968) [5]

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\begin{aligned}
P(\text{yellow car}) &= 1/10 \\
P(\text{man with moustache}) &= 1/4 \\
P(\text{black man with beard}) &= 1/10 \\
P(\text{woman with pony tail}) &= 1/10 \\
P(\text{woman blond hair}) &= 1/3 \\
P(\text{mix race pair in a car}) &= 1/1000
\end{aligned}
$$

Assume (wrong!) mutual indepedence:

$$P(?) = \frac{1}{12,000,000}$$

What probability?

A Convicted pair not guilty.

B A randomly selected pair matches characteristics.

C Some other.

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Judge needs:

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possible/existing pairs in California … N

$P(\text{pair will never appear in } N) = P(NA) = (1 - P_r)^N$

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people of CA vs Collins, 1968, [1]

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$P(MTO|ALO) = f(N)$; people of CA vs Collins, 1968