

Sequential decisions under uncertainty

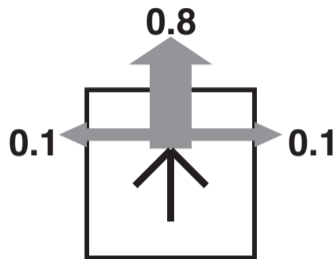
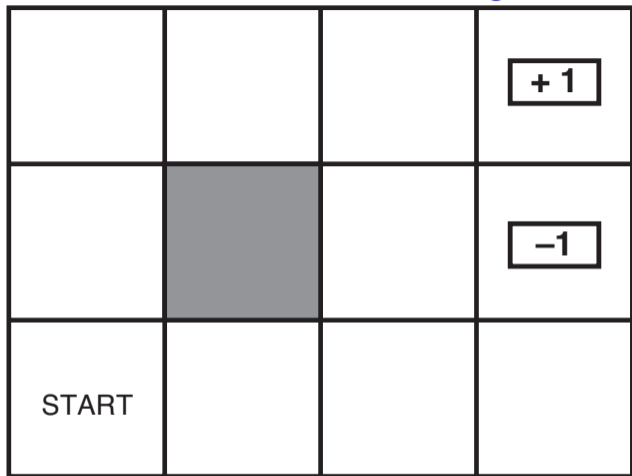
Markov Decision Processes (MDP)

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Vision for Robots and Autonomous Systems, Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering, Czech Technical University in Prague

March 21, 2024

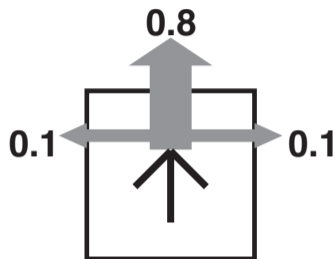
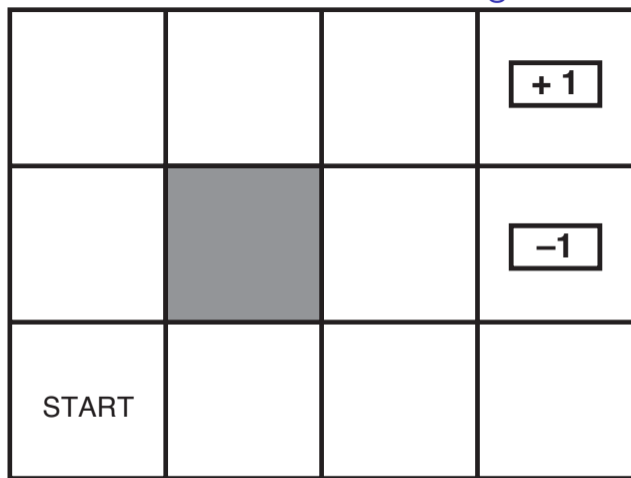
Unreliable actions in observable grid world



States $s \in \mathcal{S}$, actions $a \in \mathcal{A}$

(Transition) Model $T(s, a, s') \equiv p(s'|s, a) =$ probability that a in s leads to s'

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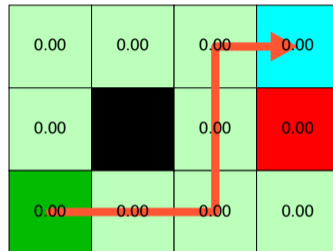
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Unreliable (results of) actions



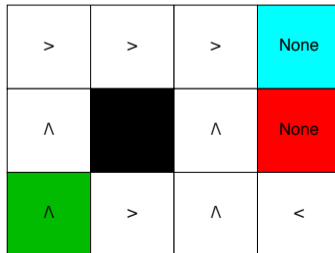
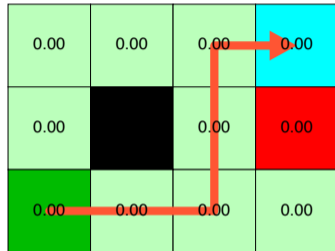
Plan? Policy

- ▶ In deterministic world: **Plan** – sequence of actions from **Start** to **Goal**.
- ▶ MDPs, we need a **policy** $\pi : \mathcal{S} \rightarrow \mathcal{A}$.
- ▶ An action for each possible state. *Why each?*
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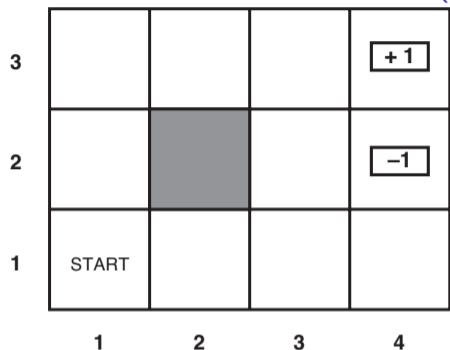
Rewards

-0.04	-0.04	-0.04	1.00
-0.04		-0.04	-1.00
-0.04	-0.04	-0.04	-0.04

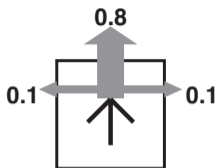
Reward : Robot/Agent takes an action a and it is **immediately** rewarded.

Reward function $r(s)$ (or $r(s, a)$, $r(s, a, s')$)
= $\begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$

Markov Decision Processes (MDPs)



(a)



(b)

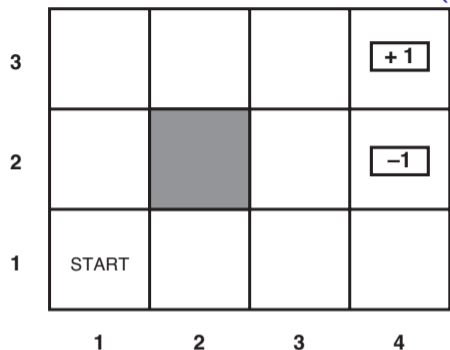
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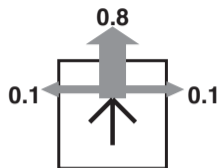
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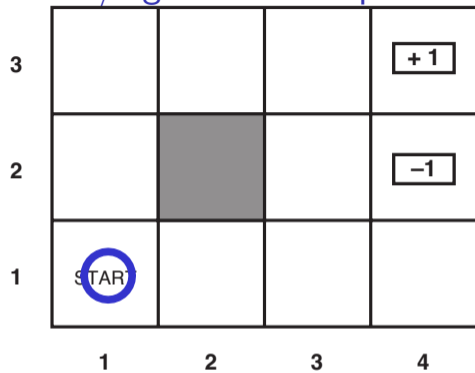
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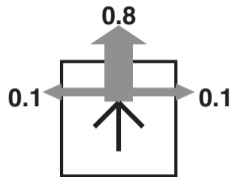
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Robot/Agent walk – Episode



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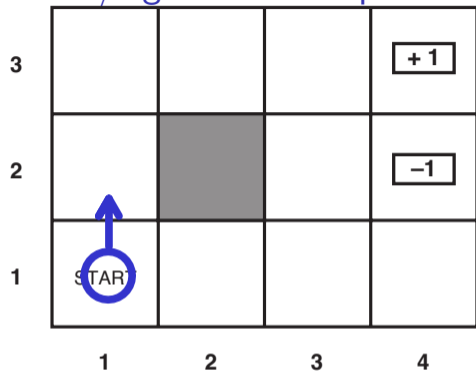


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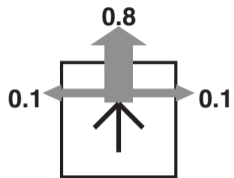
$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2 \dots$

Episode : one walk from S_0 to terminal.

Robot/Agent walk – Episode



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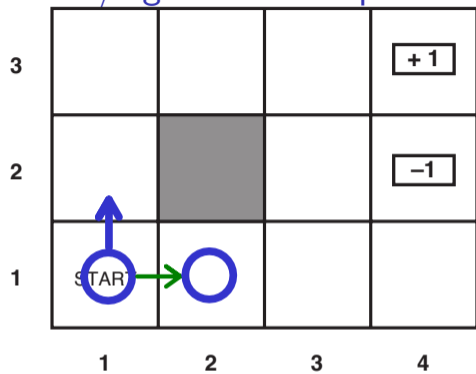


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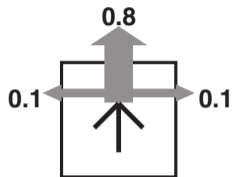
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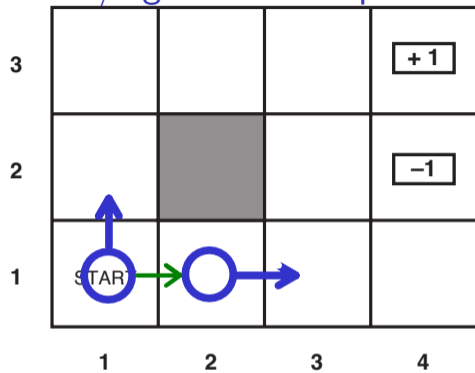


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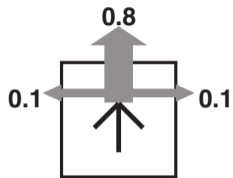
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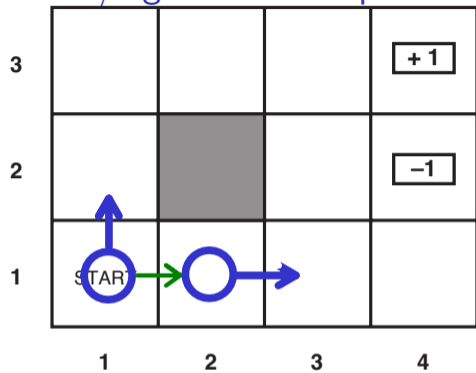


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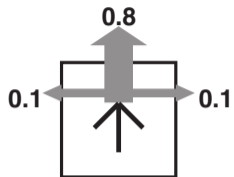
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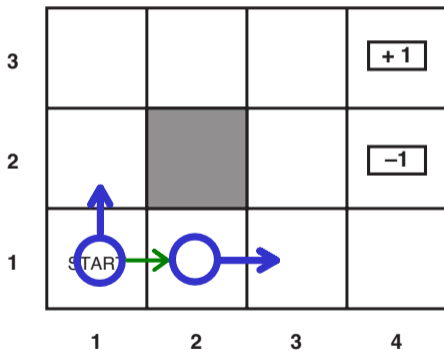
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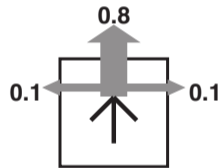
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Markovian property

- ▶ Given the present state, the future and the past are independent.
- ▶ MDP: Markov means action depends only on the current state.
- ▶ In search: successor function (transition model) depends on the current state only.



(a)



(b)

Desired robot/agent behavior specified through rewards

- ▶ Before: shortest/cheapest path
- ▶ Solution found by search.
- ▶ Environment/problem is defined through the reward function.
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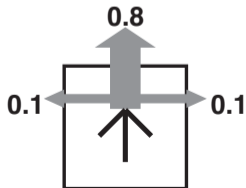
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We come back to this in more detail when discussing RL.

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- A: A-a, B-b, C-c
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Utilities of sequences; what is a better walk?

- ▶ State reward at time/step t , R_t .
- ▶ State at time t , S_t . State sequence $[S_0, S_1, S_2, \dots,]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \dots] \succ [R, R'_1, R'_2, R'_3, \dots] \Leftrightarrow [R_1, R_2, R_3, \dots] \succ [R'_1, R'_2, R'_3, \dots]$$

If stationary preferences :

Utility (h -history)

$$U_h([S_0, S_1, S_2, \dots,]) = R_1 + R_2 + R_3 + \dots$$

If the horizon is finite - limited number of steps - preferences are nonstationary (depends on how many steps left).

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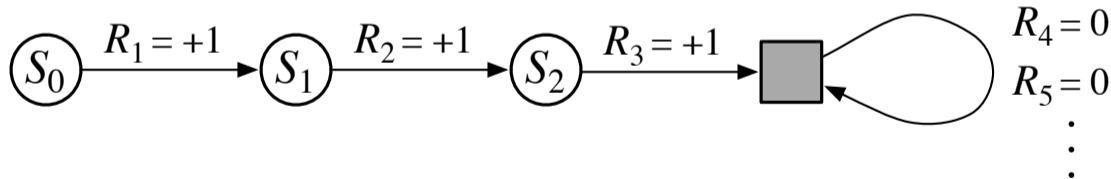
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Finite walk – Episode – and its Return (by introducing Terminal state)

- ▶ Executing policy - sequence of states and **rewards**.
- ▶ **Episode** starts at t , ends at T (ending in a terminal state).
- ▶ **Return** (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \dots + R_T$$



Horizon too far, infinite – Discount rewards

Problem: Infinite lifetime \Rightarrow additive utilities are infinite.

- ▶ Finite horizon: termination at a fixed time \Rightarrow nonstationary policy, $\pi(s)$ depends on the time left.
- ▶ Absorbing (terminal) state. (sooner or later walk ends here)
- ▶ Discounted return , $\gamma < 1, R_t \leq R_{\max}$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

Returns are successive steps related to each other

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MDPs recap

Markov decision processes (MDPs):

- ▶ Set of states \mathcal{S}
- ▶ Set of actions \mathcal{A}
- ▶ Transitions $p(s'|s, a)$ or $T(s, a, s')$
- ▶ Reward function $r(s, a, s')$; and discount γ
- ▶ Alternative to last two: $p(s', r|s, a)$.

MDP quantities:

- ▶ (deterministic) Policy $\pi(s)$ – choice of action for each state
- ▶ Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

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Expected Return of a policy π

- ▶ Executing policy $\pi \rightarrow$ sequence of states (and rewards).
- ▶ Utility of a state sequence.
 - ▶ But actions are unreliable - environment is stochastic.
 - ▶ Expected return of a policy π .

Starting at time t , i.e. S_t ,

$$U^\pi(S_t) \doteq E^\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \right]$$

Expected Return of a policy π

- ▶ Executing policy $\pi \rightarrow$ sequence of states (and rewards).
- ▶ Utility of a state sequence.
- ▶ But actions are unreliable - environment is stochastic.
- ▶ Expected return of a policy π .

Starting at time t , i.e. S_t ,

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Starting at time t , i.e. S_t ,

$$U^\pi(S_t) \doteq \mathbb{E}^\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \right]$$

(State) Value functions given policy π

Expected return from that state (state, action)

Value function

$$v^\pi(s) \doteq \mathbb{E}^\pi [G_t \mid S_t = s] = \mathbb{E}^\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s \right]$$

Action-value function (q-function)

$$q^\pi(s, a) \doteq \mathbb{E}^\pi [G_t \mid S_t = s, A_t = a] = \mathbb{E}^\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a \right]$$

Optimal policy π^* , and optimal value $v^*(s)$

$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

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Example 1, Robot *deterministic*: $r(s) = \{-0.04, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3
0	0.88	0.92	0.96	1.00
1	0.84		0.92	-1.00
2	0.80	0.84	0.88	0.84
	0	1	2	3

	0	1	2	3
0	>	>	>	None
1	\wedge		\wedge	None
2	\wedge	>	\wedge	<
	0	1	2	3

Optimal policy π^* , and optimal value $v^*(s)$

$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

Example 2, Robot *non-deterministic*: $r(s) = \{-0.04, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3
0	0.81	0.87	0.92	1.00
1	0.76		0.66	-1.00
2	0.71	0.66	0.61	0.39
	0	1	2	3

	0	1	2	3
0	>	>	>	None
1	\wedge		\wedge	None
2	\wedge	<	<	<
	0	1	2	3

Optimal policy π^* , and optimal value $v^*(s)$

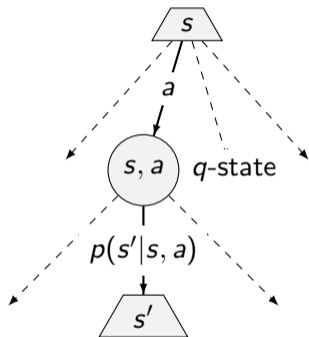
$v^*(s)$ = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

Example 3, Robot *non-deterministic*: $r(s) = \{-0.01, 1, -1\}$, $\gamma = 0.999999$, $\epsilon = 0.03$

	0	1	2	3
0	0.95	0.96	0.98	1.00
1	0.94		0.89	-1.00
2	0.92	0.91	0.90	0.80
	0	1	2	3

	0	1	2	3
0	>	>	>	1.00
1	∧		<	-1.00
2	∧	<	<	V
	0	1	2	3

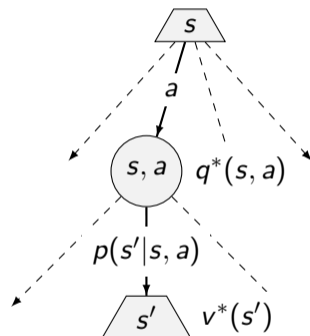
MDP search tree



MDP search tree

The value of a q -state (s, a) :

$$q^*(s, a) = \sum_{s'} p(s'|a, s) [r(s, a, s') + \gamma v^*(s')]$$



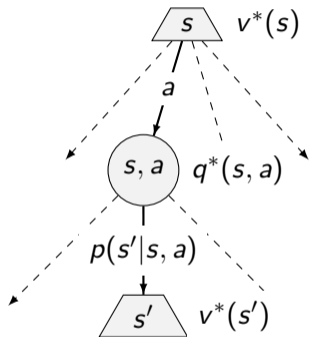
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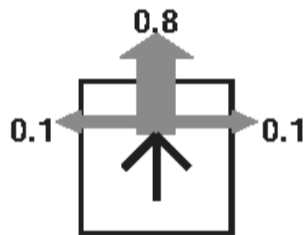
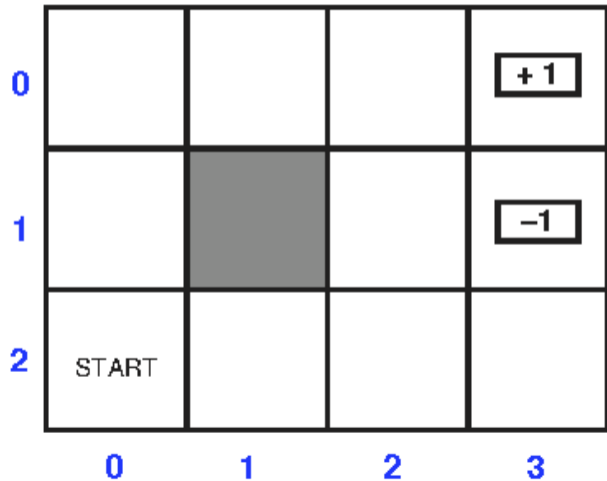
The value of a state s :

$$v^*(s) = \max_a q^*(s, a)$$



Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a, s) [r(s, a, s') + \gamma v^*(s')]$$



Value iteration – turn Bellman equation into Bellman update

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a, s) [r(s, a, s') + \gamma v^*(s')]$$

▶ Start with arbitrary $V_0(s)$ (except for terminals)

▶ Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$$

▶ Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent \Rightarrow globally optimal.

Value iteration algorithm is an example of Dynamic Programming method.

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Value iteration algorithm is an example of **Dynamic Programming** method.

Value iteration - Complexity of one estimation sweep

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$$

A: $O(AS)$

B: $O(S^2)$

C: $O(AS^2)$

D: $O(A^2S^2)$

Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V_k(s')$$

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
2	0.71	0.66	0.61	0.39	2
	0	1	2	3	

Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \leq R(s) \leq R_{\max}$$

Max norm:

$$\|V\|_{\infty} = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \dots, s_{\infty}]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma}$$

Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V_k(s')$$

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Convergence cont'd

$V_{k+1} \leftarrow BV_k \dots B$ as the Bellman update $V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$

$$\|BV_k - BV'_k\|_\infty \leq \gamma \|V_k - V'_k\|_\infty$$

$$\|BV_k - V_{\text{true}}\|_\infty \leq \gamma \|V_k - V_{\text{true}}\|_\infty$$

Rewards are bounded, at the beginning then Value error is

$$\|V_0 - V_{\text{true}}\|_\infty \leq \frac{2R_{\text{max}}}{1-\gamma}$$

We run N iterations and reduce the error by factor γ in each and want to stop the error is below ϵ :

$$\gamma^N 2R_{\text{max}} / (1 - \gamma) \leq \epsilon \text{ Taking logs, we find: } N \geq \frac{\log(2R_{\text{max}}/\epsilon(1-\gamma))}{\log(1/\gamma)}$$

To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.

If we stop when

$$\|V_{k+1} - V_k\|_\infty \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also: $\|V_{k+1} - V_{\text{true}}\|_\infty \leq \epsilon$ Proof on the next slide

Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\|_{\infty} \leq \epsilon$ is the same as $\|V_{k+1} - V_{\infty}\|_{\infty} \leq \epsilon$

Assume $\|V_{k+1} - V_k\|_{\infty} = \text{err}$

In each of the following iteration steps we reduce the error by the factor γ (because $\|BV_k - V_{\text{true}}\|_{\infty} \leq \gamma\|V_k - V_{\text{true}}\|_{\infty}$). Till ∞ , the total sum of reduced errors is:

$$\text{total} = \gamma \text{err} + \gamma^2 \text{err} + \gamma^3 \text{err} + \gamma^4 \text{err} + \dots = \frac{\gamma \text{err}}{(1 - \gamma)}$$

We want to have $\text{total} < \epsilon$.

$$\frac{\gamma \text{err}}{(1 - \gamma)} < \epsilon$$

From it follows that

$$\text{err} < \frac{\epsilon(1 - \gamma)}{\gamma}$$

Hence we can stop if $\|V_{k+1} - V_k\|_{\infty} < \epsilon(1 - \gamma)/\gamma$

Value iteration algorithm

function VALUE-ITERATION(env, ϵ) **returns:** state values V

input: env - MDP problem, ϵ

$V' \leftarrow 0$ in all states

repeat

$V \leftarrow V'$

$\delta \leftarrow 0$

for each state s in S do

$V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V(s')$

if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$

until $\delta < \epsilon(1 - \gamma)/\gamma$

▷ iterate values until convergence

▷ keep the last known values (deepcopy)

▷ reset the max difference

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Sync vs. async Value iteration

function VALUE-ITERATION(env, ϵ) **returns:** state values V

input: env - MDP problem, ϵ

$V' \leftarrow 0$ in all states

repeat

$V = V'$

$\delta \leftarrow 0$

for each state s in S **do**

$V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V(s')$

if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$

until $\delta < \epsilon(1 - \gamma)/\gamma$

▷ iterate values until convergence

▷ **don't** keep the last known values

▷ reset the max difference

What we have learned

- ▶ Uncertain outcome of an action
- ▶ Optimal policy (strategy, sequence of decisions) maximizes *expected* return (utility, sum of rewards)
- ▶ (State) Value function given policy
- ▶ Value iteration method - through local (optimal) updated to global optimality

References

Some figures from [1] (chapter 17) but notation slightly changed in order to adapt notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course. Note that the book [2] is available on-line.

[1] Stuart Russell and Peter Norvig.

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<http://aima.cs.berkeley.edu/>.

[2] Richard S. Sutton and Andrew G. Barto.

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<http://www.incompleteideas.net/book/the-book-2nd.html>.