Quantum Computing

Exercises: Quantum walks

1. At each time step, a quantum walk corresponds to a unitary map $U \in U(N)$ such that

$$\begin{aligned} U : \mathcal{H}_G &\to \mathcal{H}_G \\ |x\rangle &\mapsto a|x-1\rangle + b\,|x\rangle + c\,|x+1\rangle \end{aligned}$$

Show that U is unitary if and only if one of the following three conditions is true:

(a) $|a| = 1, b = c = 0,$

(b) $|b| = 1, a = c = 0,$

(c) $|c| = 1, a = b = 0.$

Using the unitarity of the operator we know that:

$$\langle x| \underbrace{U^\dagger U}_{\mathbf{1}} |y\rangle = \delta_{xy} \tag{1}$$

So, for instance, for the following states, we have:

$$\langle x-1| U^\dagger U |x+1\rangle = \big(a\,\langle x-2| + b\,\langle x-1| + c\,\langle x|\,\big)\big(a\,|x\rangle + b\,|x+1\rangle + c\,|x+2\rangle\,\big) = 0$$

The only term surviving being $c\,\langle x|\, a\,|x\rangle = ac = 0$

$$\langle x| U^\dagger U |x+1\rangle = \big(a\,\langle x-1| + b\,\langle x| + c\,\langle x+1|\,\big)\big(a\,|x\rangle + b\,|x+1\rangle + c\,|x+2\rangle\,\big) = 0$$

The non-vanishing terms now are

$$\left\{\begin{array}{l} b\,\langle x|\, a\,|x\rangle \Rightarrow ab \\ c\,\langle x+1|\, b\,|x+1\rangle \Rightarrow bc \end{array}\right. \Rightarrow ab + bc = 0$$

$$\langle x| U^\dagger U |x\rangle = \big(a\,\langle x-1| + b\,\langle x| + c\,\langle x+1|\,\big)\big(a\,|x-1\rangle + b\,|x\rangle + c\,|x+1\rangle\,\big) = 0$$

Lastly, the system to be solved is:

$$\begin{cases} ac = 0 \\ ab + bc = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases} \tag{2}$$

2. Demonstrate that the shift operator S, as defined in

$$S = \left(|0\rangle\langle 0| \otimes \sum_{x=-\infty}^{\infty} |x+1\rangle\langle x| \right) + \left(|1\rangle\langle 1| \otimes \sum_{x=-\infty}^{\infty} |x-1\rangle\langle x| \right)$$

is equivalent to

$$S|i,x\rangle = \begin{cases} |0,x+1\rangle & \text{if } i=0, \\ |1,x-1\rangle & \text{if } i=1. \end{cases}$$

Applying directly the first definition of the operator to the state $|i,x\rangle$, we get the second one:

$$S|i,x\rangle = \left(|0\rangle \overbrace{\langle 0||i\rangle}^{\delta_{0i}} \otimes \underbrace{\sum_{k=-\infty}^{\infty} |k+1\rangle \overbrace{\langle k||x\rangle}^{\delta_{kx}}}_{|x+1\rangle} \right) + \left(|1\rangle \overbrace{\langle 1||i\rangle}^{\delta_{1i}} \otimes \underbrace{\sum_{k=-\infty}^{\infty} |k-1\rangle \overbrace{\langle k||x\rangle}^{\delta_{kx}}}_{|x-1\rangle} \right)$$

$$= \begin{cases} |0\rangle \otimes |x+1\rangle & \text{if } i=0, \\ |1\rangle \otimes |x-1\rangle & \text{if } i=1. \end{cases} \quad = \begin{cases} |0,x+1\rangle & \text{if } i=0, \\ |1,x-1\rangle & \text{if } i=1. \end{cases}$$

3. In the lecture notes, starting at the state $|\psi_0\rangle = |0\rangle|0\rangle$, we have seen how to obtain the succesive states up to $|\psi_3\rangle$ by using the unitary operator $U = S(H \otimes I)$. Derive $|\psi_4\rangle$ for the walker on the finite subset of $\mathbb{Z}$.

The previous states $|\psi_{1..3}\rangle$ can be found also in R. Portugal, Quantum walks and search algorithms (3.19).

$$|\psi_4\rangle = U|\psi_3\rangle = \frac{1}{2\sqrt{2}}[2U|01\rangle + U|11\rangle + \dots]$$

$U|01\rangle = S(H \otimes I)|01\rangle = S|\frac{|0\rangle+|1\rangle}{\sqrt{2}}1\rangle = \frac{1}{\sqrt{2}}[S|01\rangle + S|11\rangle] = \frac{1}{\sqrt{2}}[|02\rangle + |10\rangle]$

$U|11\rangle = S(H \otimes I)|11\rangle = S|\frac{|0\rangle-|1\rangle}{\sqrt{2}}1\rangle = \frac{1}{\sqrt{2}}[S|01\rangle - S|11\rangle] = \frac{1}{\sqrt{2}}[|02\rangle - |10\rangle]$

$U|03\rangle = \frac{1}{\sqrt{2}}[|02\rangle - |10\rangle]$

$U|1-3\rangle = \frac{1}{\sqrt{2}}[|0-2\rangle - |1-4\rangle]$

$U|0-1\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |0-2\rangle]$

$$|\psi_4\rangle = \frac{1}{4}[|10\rangle + 3|02\rangle + |12\rangle - |00\rangle - |1-4\rangle + |04\rangle]$$

4. Show that the formula from the lecture notes, $H\,|k\rangle = 2\cos(k)\,|k\rangle$ holds, by performing the discrete Fourier transform in the computational basis states.

For the walker on the line, every state $|x\rangle$ is only connected to its adjacent states $|x\pm1\rangle$, that is, its adjacency matrix, A, is defined by:

$$\begin{cases} \langle x|\,A\,|x\pm1\rangle = 1 \\ \langle x|\,A\,|y\rangle = 0\ ,\ \ y\neq x\pm1 \end{cases} \tag{3}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Changing from the computational basis to the Fourier basis, we have:

$$|k\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1} e^{ikx}\,|x\rangle\ \ ,\ \ \langle k| = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1} e^{-ikx}\,\langle x|\ \ ,\ \text{where } k=\frac{2\pi\kappa}{N}$$

Now, taking the matrix element of the adjacency matrix and identifying it with the hamiltonian:

$$\langle k|\underbrace{A}_{=H}|k'\rangle = \frac{1}{N}\sum_{x=0}^{N-1} e^{-ikx}\,\langle x|\,H\sum_{x=0}^{N-1} e^{ik'x}\,|x\rangle = \frac{1}{N}\sum_{x=0}^{N-1} e^{-ikx}e^{ik'(x+1)} + e^{-ikx}e^{ik'(x-1)} =$$

$$= \frac{1}{N}\sum_{x=0}^{N-1} e^{-ix(k-k')}e^{ik'} + e^{-ix(k-k')}e^{-ik'} = \underbrace{\left(e^{ik'}+e^{-ik'}\right)}_{2\cos k}\underbrace{\frac{1}{N}\sum_{x=0}^{N-1} e^{-ix(k-k')}}_{\delta_{\kappa\kappa'}}$$

Where in the last equation, we have made use of the partial sum, s_n, of a geometric series:

$$s_n = ar^0 + ar^1 + \dots + ar^{n-1} \tag{4}$$

$$= \sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1} \tag{5}$$

$$= \begin{cases} a\left(\frac{1-r^n}{1-r}\right),\ \text{for } r\neq 1 \\ an,\ \text{for } r=1 \end{cases} \tag{6}$$

$$\frac{1}{N}\sum_{x=0}^{N-1} e^{-ix(k-k')} = \frac{1}{N}\sum_{x=0}^{N-1}\left(e^{-i(k-k')}\right)^x = \begin{cases} \frac{1-e^{\frac{2\pi i}{\cancel{N}}(\kappa-\kappa')\cancel{N}}}{1-e^{\frac{2\pi i}{N}(\kappa-\kappa')}} = 0 \\ \frac{N}{N} = 1 \end{cases} = \delta_{\kappa\kappa'}$$