Quantum Computing

Exercises 1: Intro to Quantum Physics

1. a) Show that the left and right states defined as:

$$|l\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$$

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$$

are orthogonal:
b) Calculate the expectation values of σ_x in the states $|d\rangle$ and $|l\rangle$, and of σ_z in the state $|r\rangle$.

a) We take their product, which in the braket notation reads as $\langle l|r\rangle$, and verify that it is 0:

$$\langle l|r\rangle = \frac{1}{\sqrt{2}}(\langle u| + \langle d|) \cdot \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle) = \frac{1}{2}\big(\langle u|u\rangle + \langle u|d\rangle + \langle d|u\rangle - \langle d|d\rangle\big)$$

Now, choosing the computational basis $|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|d\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, we have:

$$\langle u|u\rangle = \begin{pmatrix}1 & 0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1 \quad , \quad \langle u|d\rangle = \begin{pmatrix}1 & 0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\langle d|u\rangle = \begin{pmatrix}0 & 1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0 \quad , \quad \langle d|d\rangle = \begin{pmatrix}0 & 1\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = 1$$

b) We insert the Pauli matrices in the expression of the expected value for a general operator: $\langle \psi|A|\psi\rangle$

$$\langle d|\sigma_x|d\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$\langle l|\sigma_x|l\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 1$$

$$\langle r|\sigma_z|r\rangle = 0$$

2. a) Normalise the state

$$|\psi\rangle = (1-i)|u\rangle + 2i|d\rangle.$$

b) For this (normalised) state, calculate the probability of getting both positive $(+1)$ and negative (-1) spin eigenvalues by measuring σ_z.

a) Normalisation means that taking the norm of the state, $|\psi\rangle$, the result is 1, that is: $\sqrt{\langle\psi|\psi\rangle} = 1$.
But taking the squared norm in this case is:

$$\langle\psi|\psi\rangle = (1-i)\cdot(1+i) + (2i)\cdot(-2i) = 2 + 4 = 6$$

We should then choose the normalisation constant N by which we will multiply the state $|\psi\rangle \to N \cdot |\psi\rangle$ such that the result is 1:

$$\langle N\cdot\psi|N\cdot\psi\rangle = N^2 \overbrace{\langle\psi|\psi\rangle}^{6} = 1 \Rightarrow N = \frac{1}{\sqrt{6}}$$

b) $P_\psi(+) = |\langle u|\psi\rangle|^2 = \langle\psi|u\rangle^*\langle u|\psi\rangle = \frac{1}{3}$, and, since the state ψ is normalised, we know that $P_\psi(-) = 1 - 1/3 = 2/3 = |\langle d|\psi\rangle|^2$.

3. [Nielsen & Chuang Ex. 2.17] (Eigendecomposition of a Pauli matrix) Find the eigenvectors, eigenvalues and diagonal representations of σ_y.

$$\sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \; ; \; det(\sigma_y - \lambda I) = \begin{vmatrix}-\lambda & -i\\i & -\lambda\end{vmatrix} = \lambda^2 + i^2 = 0 \Rightarrow \lambda = \pm 1$$

For the eigenvalue -1 we have:

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a - ib = 0 \\ ai + b = 0 \end{cases}$$

The eigenvector is therefore $v_- = \begin{pmatrix} a \\ ib \end{pmatrix}$. Choosing $a = b = 1$, we have $v_- = \begin{pmatrix} 1 \\ i \end{pmatrix}$.
And in an analogous way for the positive eigenvalue $+1$:
$v_+ = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

4. Show that the eigenvalues of hermitian matrices, $\mathbf{A} = \mathbf{A}^\dagger$, are real: $\lambda \in \mathbb{R}$.

Consider the matrix element of the adjoint of the operator: $\langle\phi|A^\dagger|\psi\rangle$
The operator can either act on the ket (that is, from the left) or on the bra, in which case it is 'daggered':

$$\langle\phi|A^\dagger|\psi\rangle = \langle A\phi|\psi\rangle$$

Now, we know that for any bra(c)ket we have: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$
A fact that comes from the very definition of the inner product which is linear in the second argument and anti-linear in the first, see for instance Nielsen and Chuang eqs. (2.13) and (2.15)
Taking this remark into account:

$$\langle\phi|A^\dagger|\psi\rangle = \langle A\phi|\psi\rangle = \langle\psi|A\phi\rangle^* = \langle\psi|A|\phi\rangle^*$$

Particularising for the case where $\phi = \psi$ and taking into account the eigenvalue equation, $\mathbf{A}|\psi\rangle = a|\psi\rangle$, we retrieve the eigenvalues of the operator:

$$\langle\psi|\mathbf{A}|\psi\rangle = \langle\psi|a|\psi\rangle = a \cdot \langle\psi|\psi\rangle = a$$

Last, by assumption, we have $A^\dagger = A$, so:

$$\langle\psi|A^\dagger|\psi\rangle = \langle\psi|A|\psi\rangle = \langle\psi|A|\psi\rangle^* \Rightarrow a = a^*$$

5. [Susskind & Friedman Ex. 5.2] For any observables $\mathbf{A}$ and $\mathbf{B}$, and state $|\psi\rangle$, derive Heisenberg's uncertainty relation: $\Delta\mathbf{A} \cdot \Delta\mathbf{B} \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$, where $(\Delta\mathbf{A})^2 = \sum_a (a - \langle\mathbf{A}\rangle)^2 P(a)$, is the standard deviation of the operator $\mathbf{A}$.

Following the reasoning in Susskind 5.4 → 5.7, we first prove that $(\Delta\mathbf{A}) = \langle\bar{\mathbf{A}}^2\rangle$:

$$(\Delta\mathbf{A})^2 = \sum_a (a - \langle\mathbf{A}\rangle)^2 P(a) = \sum_a (a - \langle\mathbf{A}\rangle)^2 |\langle a|\psi\rangle|^2 = \sum_a (a - \langle\mathbf{A}\rangle)^2 \langle a|\psi\rangle^* \langle a|\psi\rangle = \sum_a (a - \langle\mathbf{A}\rangle)^2 \langle\psi|a\rangle\langle a|\psi\rangle =$$

$$= \langle\psi| \underbrace{\sum_a (a - \langle\mathbf{A}\rangle)^2 |a\rangle\langle a|}_{(\mathbf{A}-\langle\mathbf{A}\rangle)^2} \psi\rangle = \langle\bar{\mathbf{A}}^2\rangle$$

Where the last claim in the brace can be shown using the completeness relation: $\mathbf{A} = \sum_a a|a\rangle\langle a|$:

$$(\mathbf{A} - \langle\mathbf{A}\rangle)^2 = (\sum_a a - \langle\mathbf{A}\rangle|a\rangle\langle a|)(\sum_\alpha \alpha - \langle\mathbf{A}\rangle|\alpha\rangle\langle\alpha|) = \sum_{a,b} (a - \langle\mathbf{A}\rangle)(\alpha - \langle\mathbf{A}\rangle)|a\rangle \underbrace{\langle a|\alpha\rangle}_{\delta_{a\alpha}} \langle\alpha| = \sum_a (a - \langle\mathbf{A}\rangle)^2 |a\rangle\langle a|$$

Secondly, we prove $[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = [\mathbf{A}, \mathbf{B}]$, by computing explicitly the commutator:

$$[\bar{\mathbf{A}}, \bar{\mathbf{B}}] = (\mathbf{A} - \langle\mathbf{A}\rangle)(\mathbf{B} - \langle\mathbf{B}\rangle) - (\mathbf{B} - \langle\mathbf{B}\rangle)(\mathbf{A} - \langle\mathbf{A}\rangle) = \mathbf{AB} - \mathbf{A}\langle\mathbf{B}\rangle - \langle\mathbf{A}\rangle\mathbf{B} + \langle\mathbf{A}\rangle\langle\mathbf{B}\rangle - \mathbf{BA} + \mathbf{B}\langle\mathbf{A}\rangle + \langle\mathbf{B}\rangle - \langle\mathbf{A}\rangle\langle\mathbf{B}\rangle$$

Since expected values are just scalars, they commute with operators, and many cancelations take place, giving the result.
Last, by using Cauchy-Schwarz inequality $2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$ and defining the states: $|X\rangle = \bar{\mathbf{A}}|\Psi\rangle$, $|Y\rangle = i\bar{\mathbf{B}}|\Psi\rangle$, one obtains the result wanted by following equations (5.11)→(5.13) in Susskind.

6. Derive the evolution operator: $U(t) = e^{-\frac{i}{\hbar}Ht}$, by solving the Schrödinger equation: $i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$.

The Schrödinger equation , $i\hbar\psi' = H\psi$, is a first order linear homogenous ODE. Its solution is then given by:

$$\int \frac{\psi'}{\psi} dt = -\int \frac{i}{\hbar} H dt \rightarrow \log \psi = -\frac{i}{\hbar} Ht + k \rightarrow \psi(t) = e^{-\frac{i}{\hbar} Ht + k}$$

The constant k in the last term gives us the initial state of the system, $|\psi(0)\rangle$. The evolution between this state and any other in time $|\psi(t)\rangle$, is given by:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} Ht} |\psi(0) \iff |\psi(t)\rangle = U(t) |\psi(0)\rangle$$