Quantum Computing

Exercises 1: Intro to Quantum Physics

1. *a) Show that the 'in' and 'out' states defined as:*

$$|i\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$$
$$|o\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$$

are orthogonal.
b) Calculate the expectation values of σ_y in the states $|u\rangle$ and $|i\rangle$, and of σ_z in the state $|o\rangle$.

a) We take their product, which in the braket notation reads as $\langle o|i\rangle$, and verify that it is 0:

$$\langle o|i\rangle = \frac{1}{\sqrt{2}}(\langle u| + i\langle d|) \cdot \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle) = \frac{1}{2}\left(\langle u|u\rangle + i\langle u|d\rangle + i\langle d|u\rangle - \langle d|d\rangle\right)$$

Now, recalling $|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|d\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, we have:

$$\langle u|u\rangle = \begin{pmatrix}1 & 0\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 1 \quad , \quad \langle u|d\rangle = \begin{pmatrix}1 & 0\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\langle d|u\rangle = \begin{pmatrix}0 & 1\end{pmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = 0 \quad , \quad \langle d|d\rangle = \begin{pmatrix}0 & 1\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = 1$$

b) We insert the Pauli matrices into the expression for the expectation value of a general operator: $\langle\psi|A|\psi\rangle$

$$\langle u|\sigma_y|u\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix} = 0$$

$$\langle l|\sigma_y|l\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = 1$$

Alternatively, rather than working with matrix multiplication you can decompose the state into the eigenstates of the desired operator. For example

$$\langle o|\sigma_z|o\rangle = \frac{1}{2}(\langle u| + i\langle d|)\sigma_z(|u\rangle - i|d\rangle)$$

Since we know that $\sigma_z|u\rangle = |u\rangle$ and $\sigma_z|d\rangle = -|d\rangle$ we have

$$\langle o|\sigma_z|o\rangle = \frac{1}{2}(\langle u| + i\langle d|)(|u\rangle + i|d\rangle) = \frac{1}{2}(1-1) = 0$$

2. *a) Normalise the state*

$$|\psi\rangle = 3i|u\rangle + (1-2i)|d\rangle.$$

b) For this (normalised) state, calculate the probability of getting both positive $(+1)$ and negative (-1) spin eigenvalues by measuring σ_z.

a) Normalization means that taking the norm of the state ,$|\psi\rangle$, is unity i.e. $\sqrt{\langle\psi|\psi\rangle} = 1$.
In this case however, we have:

$$\langle\psi|\psi\rangle = (-3i)(3i) + (1+2i)(1-2i) = 9 + 5 = 14 \neq 1.$$

We should then rescale our state by a constant N, $|\psi\rangle \Rightarrow N \cdot |\psi\rangle$ such that the result is 1:

$$\langle N\psi|N\psi\rangle = N^2 \overbrace{\langle\psi|\psi\rangle}^{=14} = 1 \Rightarrow N = \frac{1}{\sqrt{14}}$$

So our new normalised state is

$$|\psi\rangle = \frac{3i}{\sqrt{14}}|u\rangle + \frac{(1-2i)}{\sqrt{14}}|d\rangle.$$

b) $P_\psi(+) = |\langle u|\psi\rangle|^2 = |\frac{3i}{\sqrt{14}}|^2 = \frac{9}{14}$. For $P_\psi(-)$, we know that this is the only other possible outcome so $P_\psi(-) = 1 - \frac{9}{14} = \frac{5}{14}$.

3. *(Nielsen & Chuang Ex. 2.11 [Eigendecomposition of a Pauli matrix])*
Find the eigenvectors, eigenvalues and diagonal representations of σ_x.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ det(\sigma_x - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

For the eigenvalue -1 we have:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a + b = 0 \\ a + b = 0 \end{cases}$$

The eigenvector is therefore $v_- = \begin{pmatrix} a \\ -a \end{pmatrix}$. Choosing $a = 1$, we have $v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
And in an analogous way for the positive eigenvalue $+1$:
$v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
The diagonal representation is given by:

$$\sum_i \lambda_i \left| v_i \right\rangle \left\langle v_i \right|$$

In this case we have:

$$\sigma_x = \left| l \right\rangle \left\langle l \right| - \left| r \right\rangle \left\langle r \right|$$

That is, the matrix representation of the operator in the basis $v_\pm$ is the diagonal matrix, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The change of basis matrix, P, that gets us from one matrix representation to the other has as columns the eigenvectors, $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. This is achived in the following way:

$$P^{-1} \sigma_x P = D.$$

4. *(Hermitian operators)*
For a hermitian matrix $\mathbf{A}$*, that is, a matrix that satisfies* $\mathbf{A} = \mathbf{A}^\dagger$*, show that:*
a) Different eigenvalues have orthogonal eigenvectors.
b) All its eigenvalues are real. Does the converse also hold, that is, if the spectrum (the set of all eigenvalues) of a matrix is in $\mathbb{R}$*, is it then a hermitian matrix?*

a) [Done in Susskind chapter 3]
Consider two eigenvectors of A

$$A|\psi\rangle = \lambda_\psi |\psi\rangle, A|\psi\rangle = \lambda_\phi |\phi\rangle$$

Now consider:

$$\langle \psi | A | \phi \rangle = \lambda_\phi \langle \psi | \phi \rangle$$

But also since $A = A^\dagger$ this can be written as:

$$\langle \psi | A^\dagger | \phi \rangle = \lambda_\psi^* \langle \psi | \phi \rangle = \lambda_\psi \langle \psi | \phi \rangle$$

Since eigenvlaues of Hermitian matrices are real as we will show below. These two expressions are equivalent so subtracting one from the other leads to:

$$(\lambda_\phi - \lambda_\psi) \langle \psi | \phi \rangle = 0$$

$\lambda_\psi \neq \lambda_\phi$ so there difference is $\neq 0$. Therefore the eigenvectors must be orthogonal for the above to be true.

b) Consider the matrix element of the adjoint of the operator: $\langle \phi | A^\dagger | \psi \rangle$
The operator can either act on the ket (that is, from the left) or on the bra, in which case it is 'daggered':

$$\langle \phi | A^\dagger | \psi \rangle = \langle A\phi | \psi \rangle$$

Now, we know that for any bra(c)ket we have: $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$
Taking this remark into account:

$$\langle \phi | A^\dagger | \psi \rangle = \langle A\phi | \psi \rangle = \langle \psi | A\phi \rangle^* = \langle \psi | A | \phi \rangle^*$$

Particularising for the case where $\phi = \psi$ and taking into account the eigenvalue equation, $\mathbf{A}|\psi\rangle = a|\psi\rangle$, we retrieve the eigenvalues of the operator:

$$\langle\psi|\mathbf{A}|\psi\rangle = \langle\psi|a|\psi\rangle = a \cdot \langle\psi|\psi\rangle = a$$

Last, by assumption, we have $A^\dagger = A$, so:

$$\langle\psi|A^\dagger|\psi\rangle = \langle\psi|A|\psi\rangle = \langle\psi|A|\psi\rangle^* \Rightarrow a = a^*$$

The converse is not true in general, e.g. $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

5. *(Unitary operators)*
Now, consider a unitary matrix, one for which

$$UU^\dagger = \mathbb{I} \iff U^\dagger U = \mathbb{I} \iff U^{-1} = U^\dagger$$

holds. Prove that its eigenvalues are of the form $e^{i\theta}$ *and that eigenvectors of different eigenvalues must be orthogonal as well.*

We start by writing the eigenvalue equation for the unitatry operator U :

$$U\,|x\rangle = \lambda_x\,|x\rangle \leftrightarrow \langle x|\,U^\dagger = \langle x|\,\lambda_x^*$$

We have normalised our eigenvector, so:

$$\langle x|x\rangle = 1 = \langle x|U^\dagger U|x\rangle = \langle x|\lambda_x^*\lambda_x|x\rangle = |\lambda_x|^2\,\langle x|x\rangle = |\lambda_x|^2 \Rightarrow |\lambda_x|^2 = 1 \Rightarrow \lambda_x = e^{i\theta}$$

Now, as in the previous example, consider two eigenvector of U :

$$U\,|x\rangle = \lambda_x\,|x\rangle \;\;,\; U\,|y\rangle = \lambda_y\,|y\rangle$$

If $\lambda_x \neq \lambda_y$, then:

$$\langle x|y\rangle = \langle x|U^\dagger U|y\rangle = \lambda_x^*\lambda_y\,\langle x|y\rangle \Rightarrow \langle x|y\rangle\,(1 - \lambda_x^*\lambda_y) = 0 \overset{\cdot\lambda_x}{\Rightarrow} \langle x|y\rangle\,(\lambda_x - \lambda_y) = 0$$