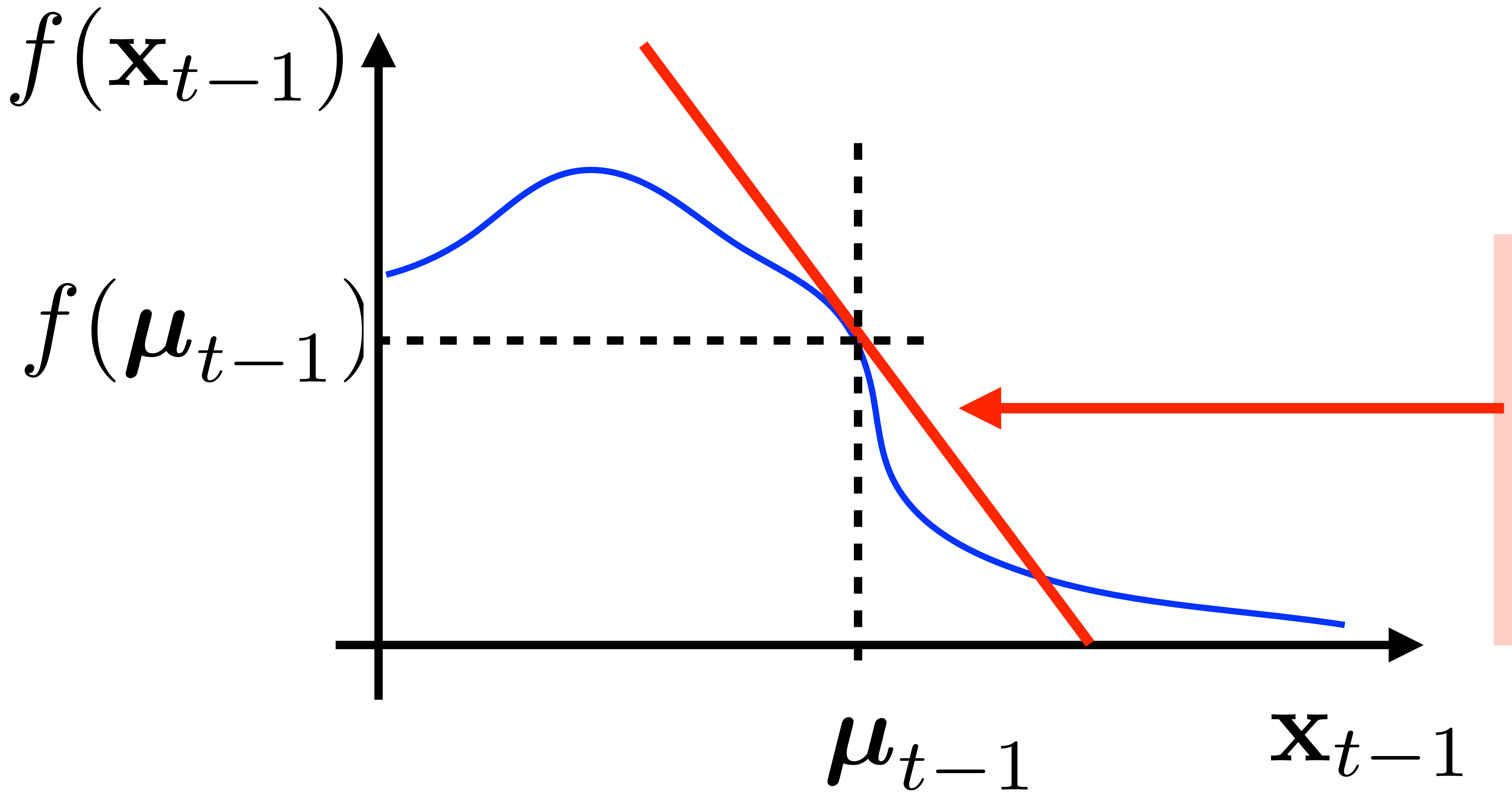
# Extended Kalman filter

**Karel Zimmermann**

# Prerequisites: Extended Kalman Filter

- First order Taylor expansion
- Jacobian




$$f(\mathbf{x}_{t-1}) \approx f(\boldsymbol{\mu}_{t-1}) + \mathbf{F}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1})$$

$$\mathbf{F}_t = \frac{\partial f(\mathbf{x} = \boldsymbol{\mu}_{t-1})}{\partial \mathbf{x}}$$

https://en.wikipedia.org/wiki/Bro

# Prerequisites: Extended Kalman Filter

## Rotation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = f(\theta) : \mathbb{R} \to \mathbb{R}^2$$

## Linear approximation

$$f(\theta) \approx f(\pi/2) + \mathbf{J} \cdot (\theta - \pi/2) = \begin{bmatrix} -y - x(\theta - \pi/2) \\ x - y(\theta - \pi/2) \end{bmatrix}$$

$$f(\pi/2) = \begin{bmatrix} x\cos\pi/2 - y\sin\pi/2 \\ x\sin\pi/2 + y\cos\pi/2 \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} -x\sin\theta - y\cos\theta \\ x\cos\theta - y\sin\theta \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

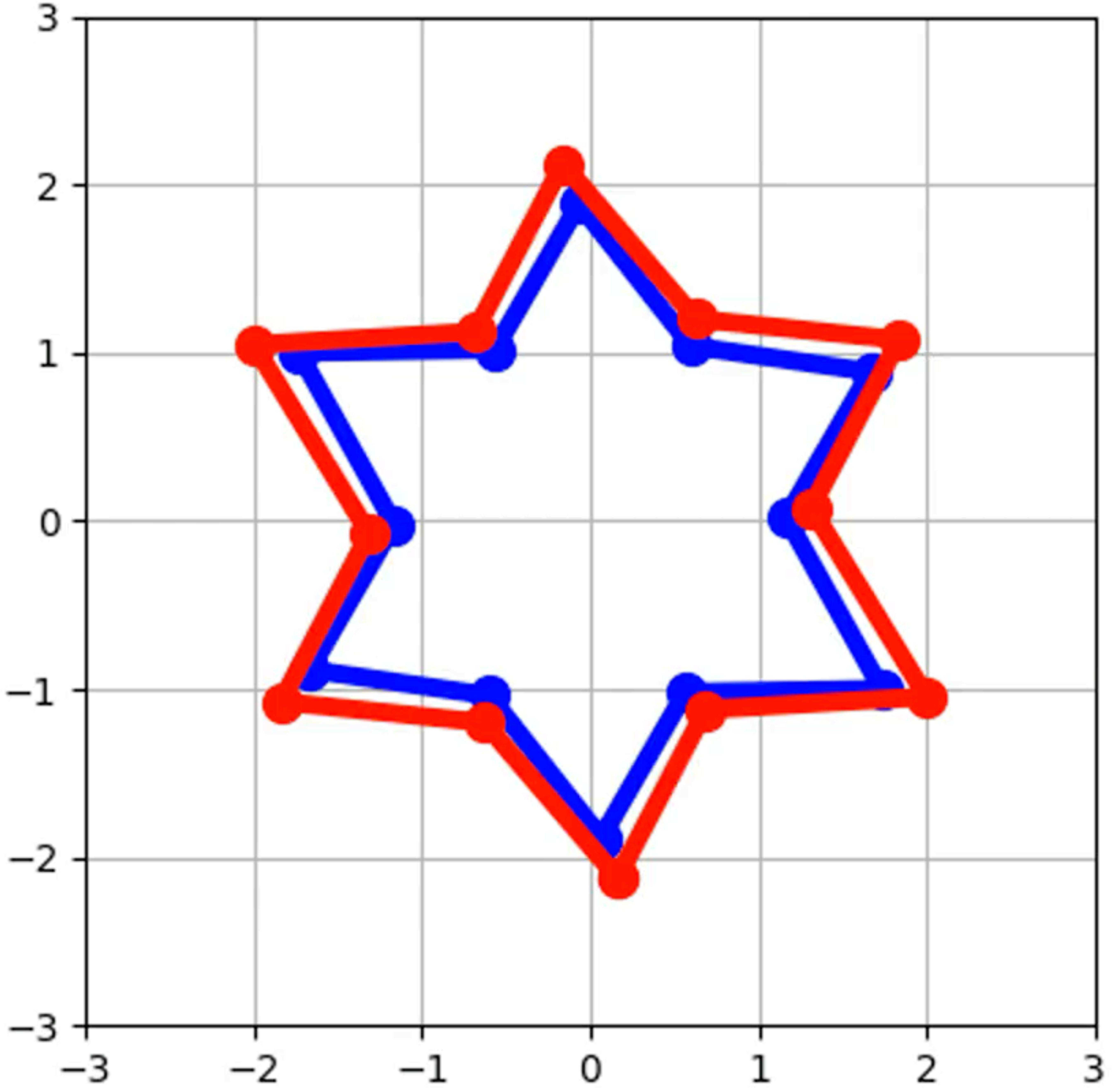
# Prerequisites: Extended Kalman Filter

Rotation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = f(\theta)$$

Linear approximation

$$f(\theta) \approx \begin{bmatrix} -y - x(\theta - \pi/2) \\ x - y(\theta - \pi/2) \end{bmatrix}$$



# Prerequisites: Extended Kalman Filter

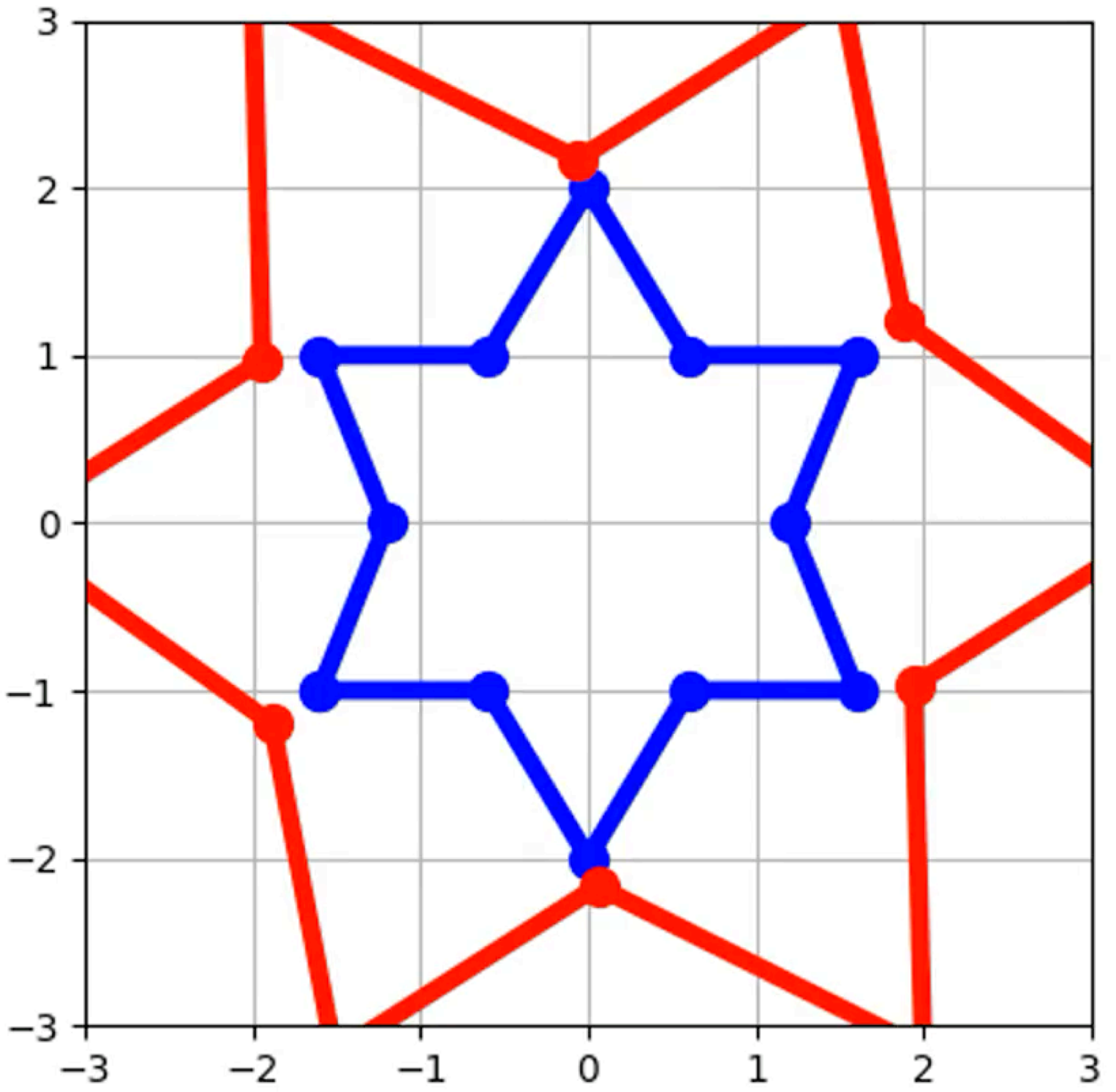
Rotation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = f(\theta)$$

Linear approximation

$$f(\theta) \approx \begin{bmatrix} -y - x(\theta - \pi/2) \\ x - y(\theta - \pi/2) \end{bmatrix}$$

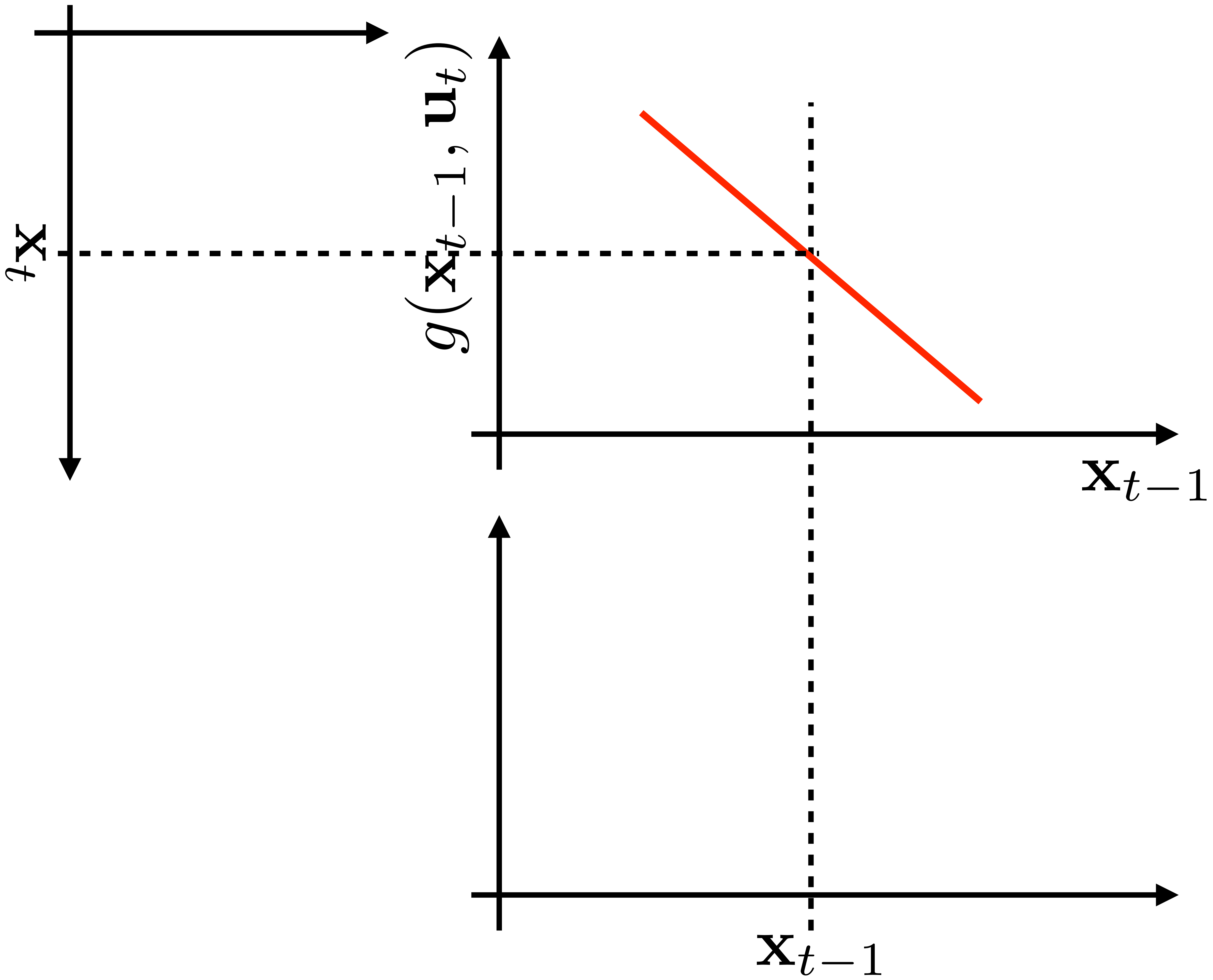
**Beware of too distant approximations !!!**



# Extended Kalman Filter

Linear system:

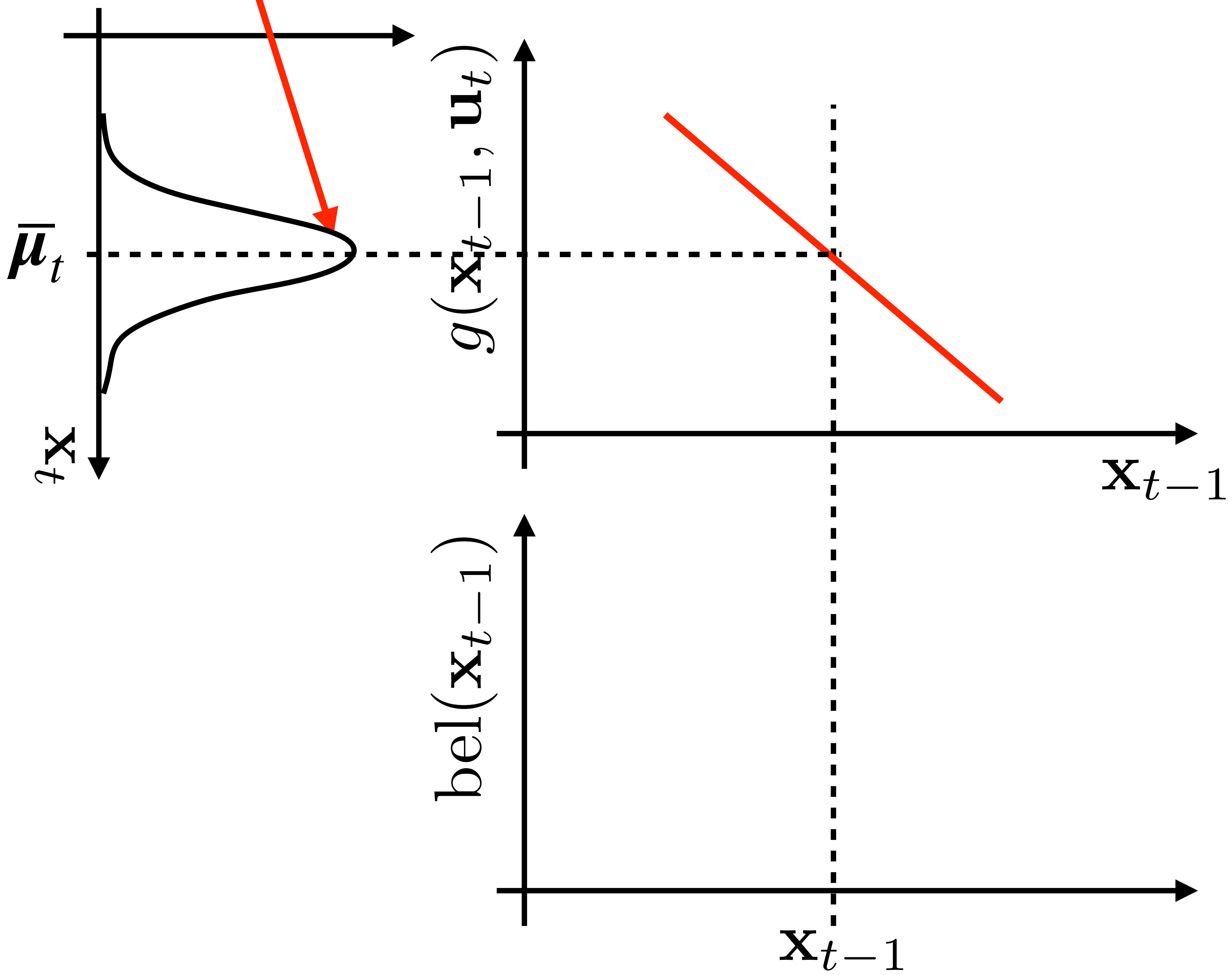
$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_t \mathbf{u}_t$$



# Extended Kalman Filter

Linear system with Gaussian noise:

$$p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t;\ \mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t, \mathbf{R}_t)$$

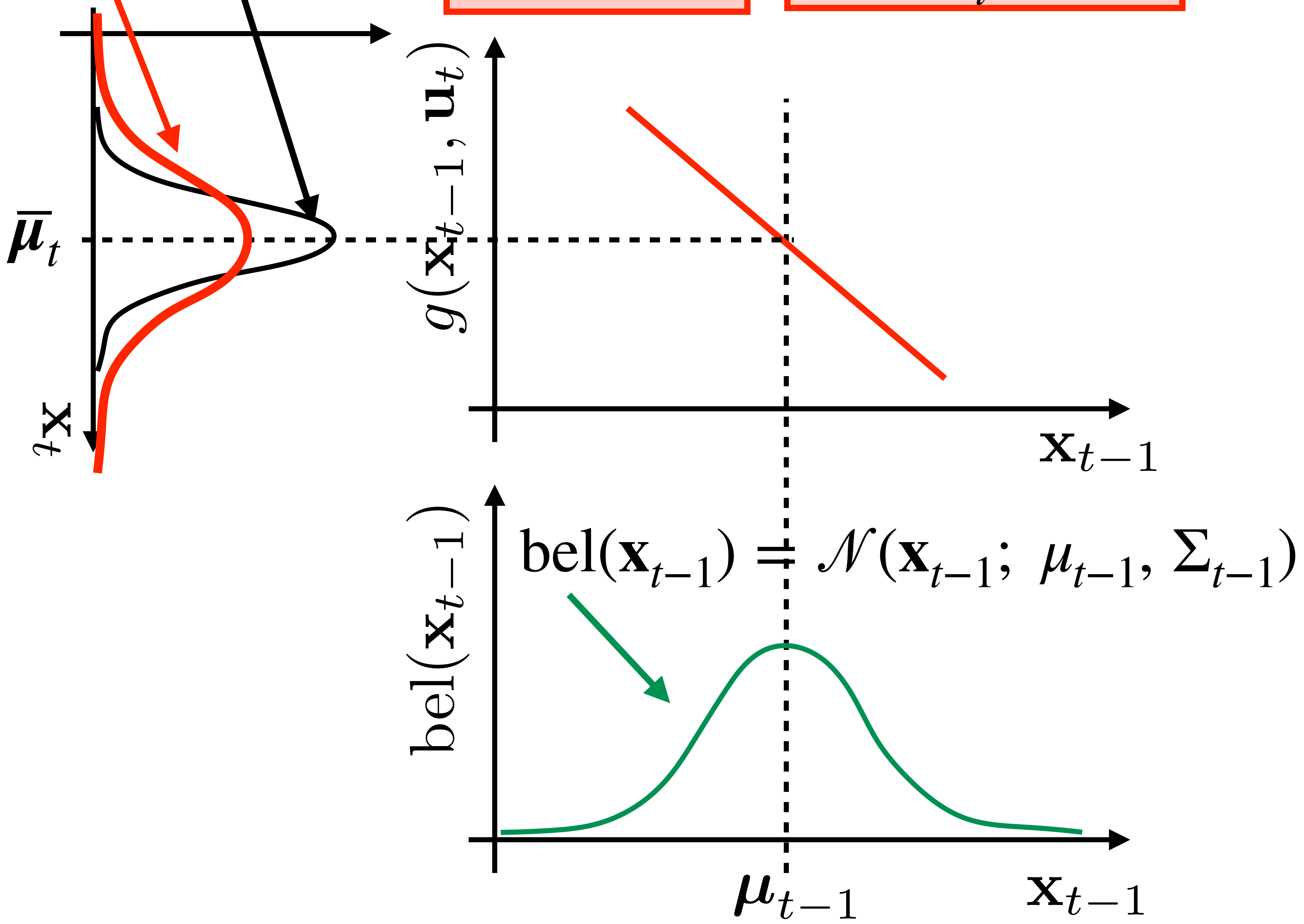


# Extended Kalman Filter

Linear system with Gaussian noise:

$$p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t;\ \mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t)$$

$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}(\mathbf{x}_t;\ \underbrace{\boxed{\mathbf{A}\mu_{t-1} + \mathbf{B}\mathbf{u}_t}}_{\bar{\boldsymbol{\mu}}_t},\ \underbrace{\boxed{\mathbf{A}^\top\Sigma_{t-1}\mathbf{A}^\top + \mathbf{R}_t}}_{\bar{\boldsymbol{\Sigma}}_t})$$

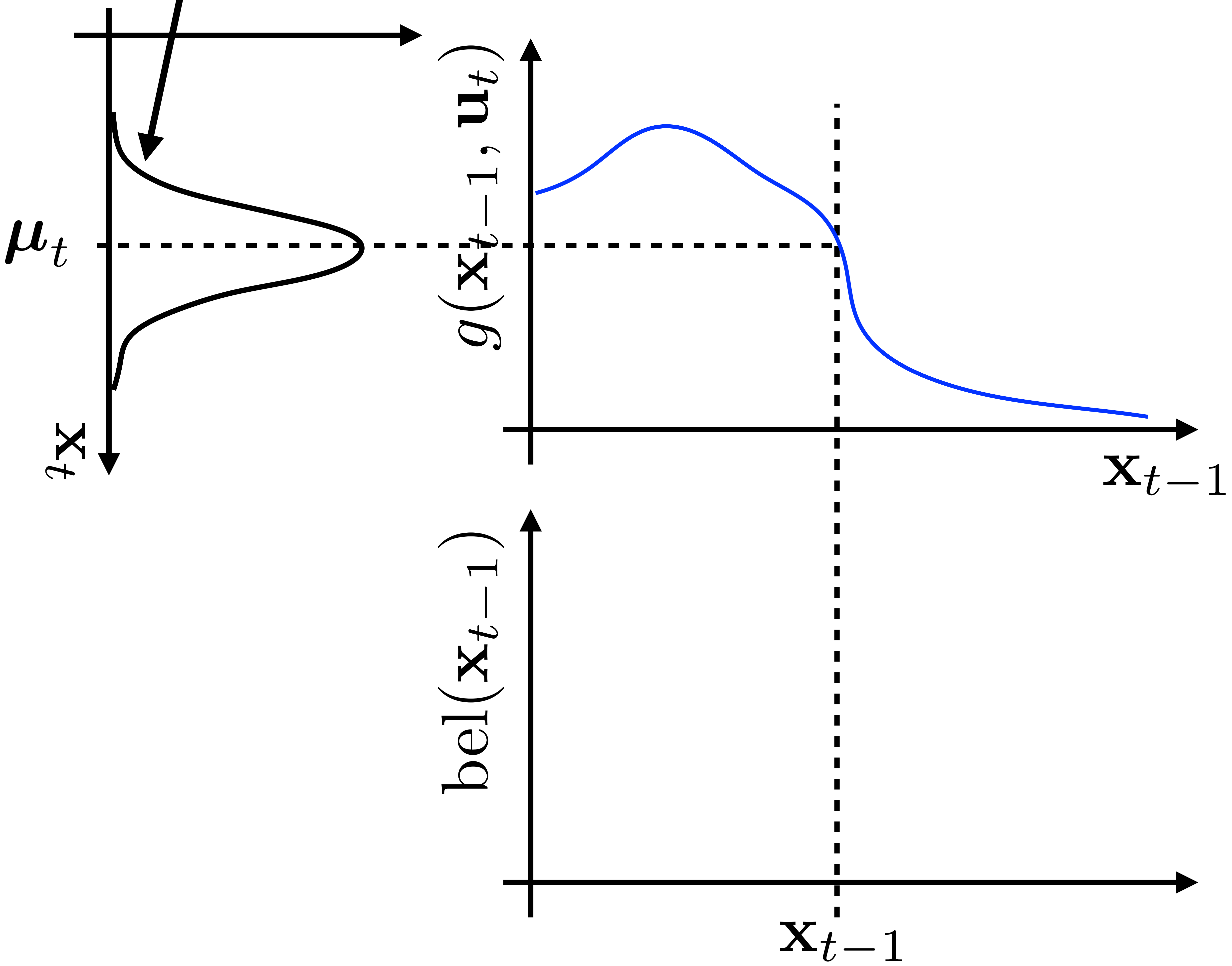



**How does it work for non-linear motion models?**

# Extended Kalman Filter

Non-linear system with Gaussian noise:

$$p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t; \quad g(\mathbf{x}_{t-1}, \mathbf{u}_t), \mathbf{R}_t)$$



# Extended Kalman Filter

Non-linear system with Gaussian noise:

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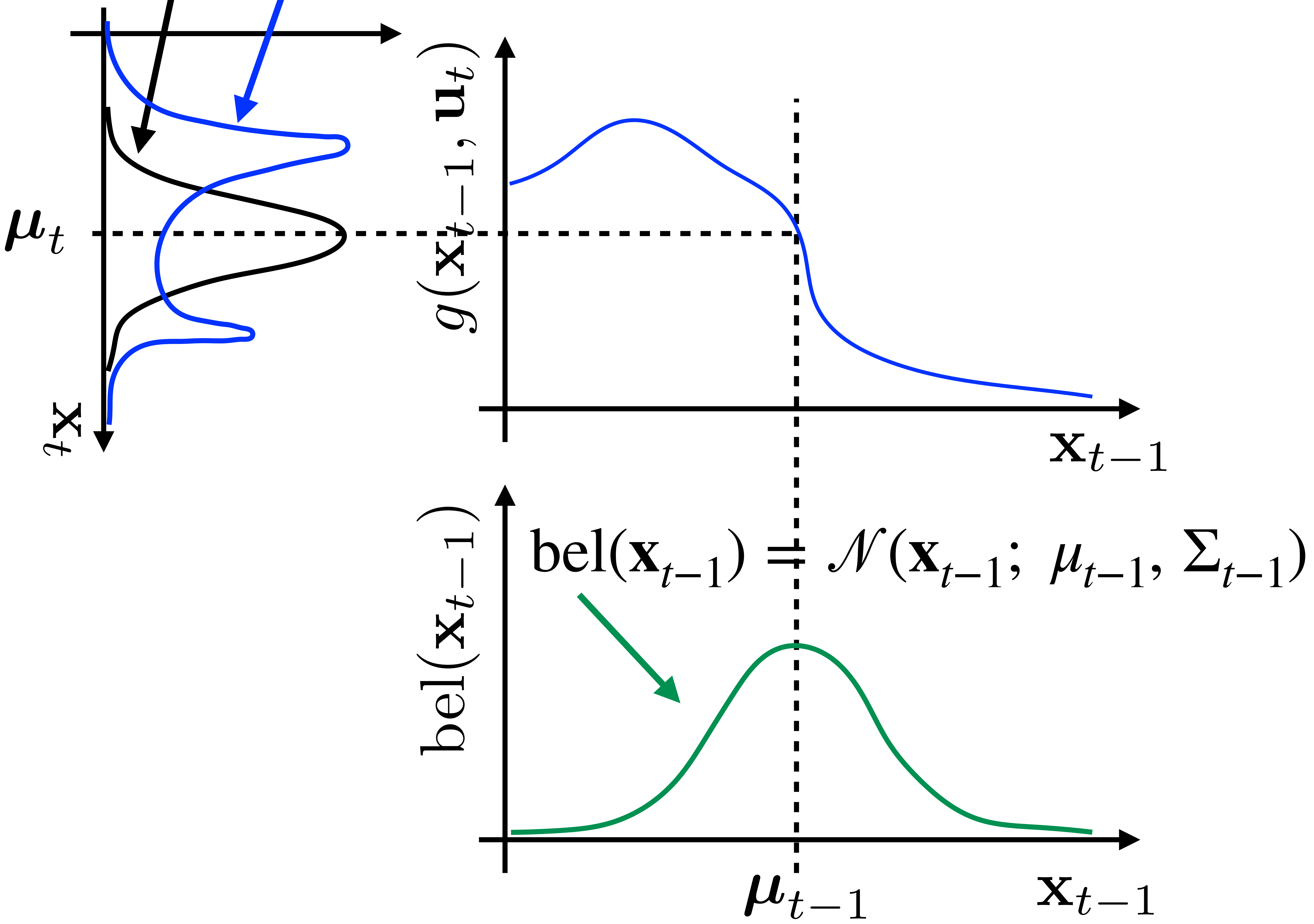
$\overline{\mathrm{bel}}(\mathbf{x}_t)$ is not gaussian !



# Extended Kalman Filter

Non-linear system with Gaussian noise:

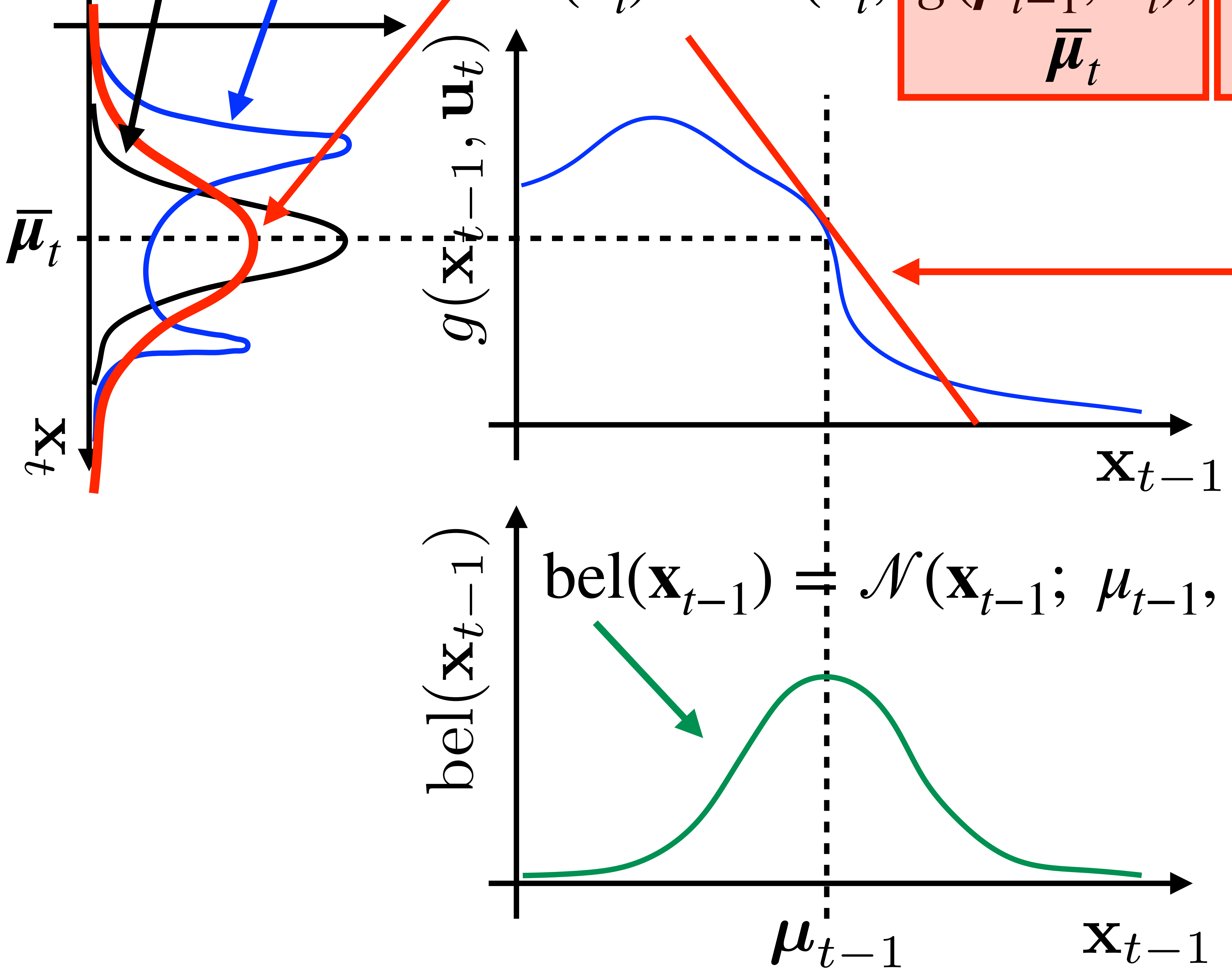
$$p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t;\ \ g(\mathbf{x}_{t-1}, \mathbf{u}_t), \mathbf{R}_t)$$

$\overline{\text{bel}}(\mathbf{x}_t)$ is not gaussian !

Linearized system with Gaussian noise:

$$\approx \mathcal{N}(\mathbf{x}_t;\ \ g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}), \mathbf{R}_t)$$

$$\overline{\text{bel}}(\mathbf{x}_t) = \mathcal{N}(\mathbf{x}_t;\ \underbrace{g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t)}_{\bar{\boldsymbol{\mu}}_t}, \underbrace{\mathbf{G}_t^\top \Sigma_{t-1} \mathbf{G}_t^\top + \mathbf{R}_t}_{\bar{\boldsymbol{\Sigma}}_t})$$




$$\text{bel}(\mathbf{x}_{t-1}) = \mathcal{N}(\mathbf{x}_{t-1};\ \mu_{t-1}, \Sigma_{t-1})$$

Linearized system:

$$g(\mathbf{u}_t, \mathbf{x}_{t-1}) \approx g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1})$$

$$\mathbf{G}_t = \frac{\partial g(\mathbf{u} = \mathbf{u}_t, \mathbf{x} = \boldsymbol{\mu}_{t-1})}{\partial \mathbf{x}}$$

# Extended Kalman Filter

**In order to use EKF we need G and H, the rest is the same !!!**

Linear system with Gaussian noise:

Linearized system with Gaussian noise:

$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t) \approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

$$p(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}_{\mathbf{z}_t}(\mathbf{C}_t\mathbf{x}_t,\ \mathbf{Q}_t) \qquad \approx \mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t\boldsymbol{\mu}_{t-1} + \mathbf{B}_t\mathbf{u}_t$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}_t\boldsymbol{\Sigma}_{t-1}\mathbf{A}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top(\mathbf{C}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - \mathbf{C}_t\overline{\boldsymbol{\mu}}_t)$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1})$$
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4. Repeat from 2

# Extended Kalman Filter

**In order to use EKF we need G and H, the rest is the same !!!**

Linear system with Gaussian noise:

Linearized system with Gaussian noise:

$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t) \approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

$$p(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}_{\mathbf{z}_t}(\mathbf{C}_t\mathbf{x}_t,\ \mathbf{Q}_t) \approx \mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

**Kalman Filter (linear):**

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:

$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t\boldsymbol{\mu}_{t-1} + \mathbf{B}_t\mathbf{u}_t$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}_t\boldsymbol{\Sigma}_{t-1}\mathbf{A}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$

3. Measurement update:

$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top(\mathbf{C}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - \mathbf{C}_t\overline{\boldsymbol{\mu}}_t)$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$

4. Repeat from 2

**Extended Kalman Filter (linearized):**

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:

$$\overline{\boldsymbol{\mu}}_t = g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1})$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{G}_t\boldsymbol{\Sigma}_{t-1}\mathbf{G}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$

3. Measurement update:

$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top(\mathbf{H}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t))$$
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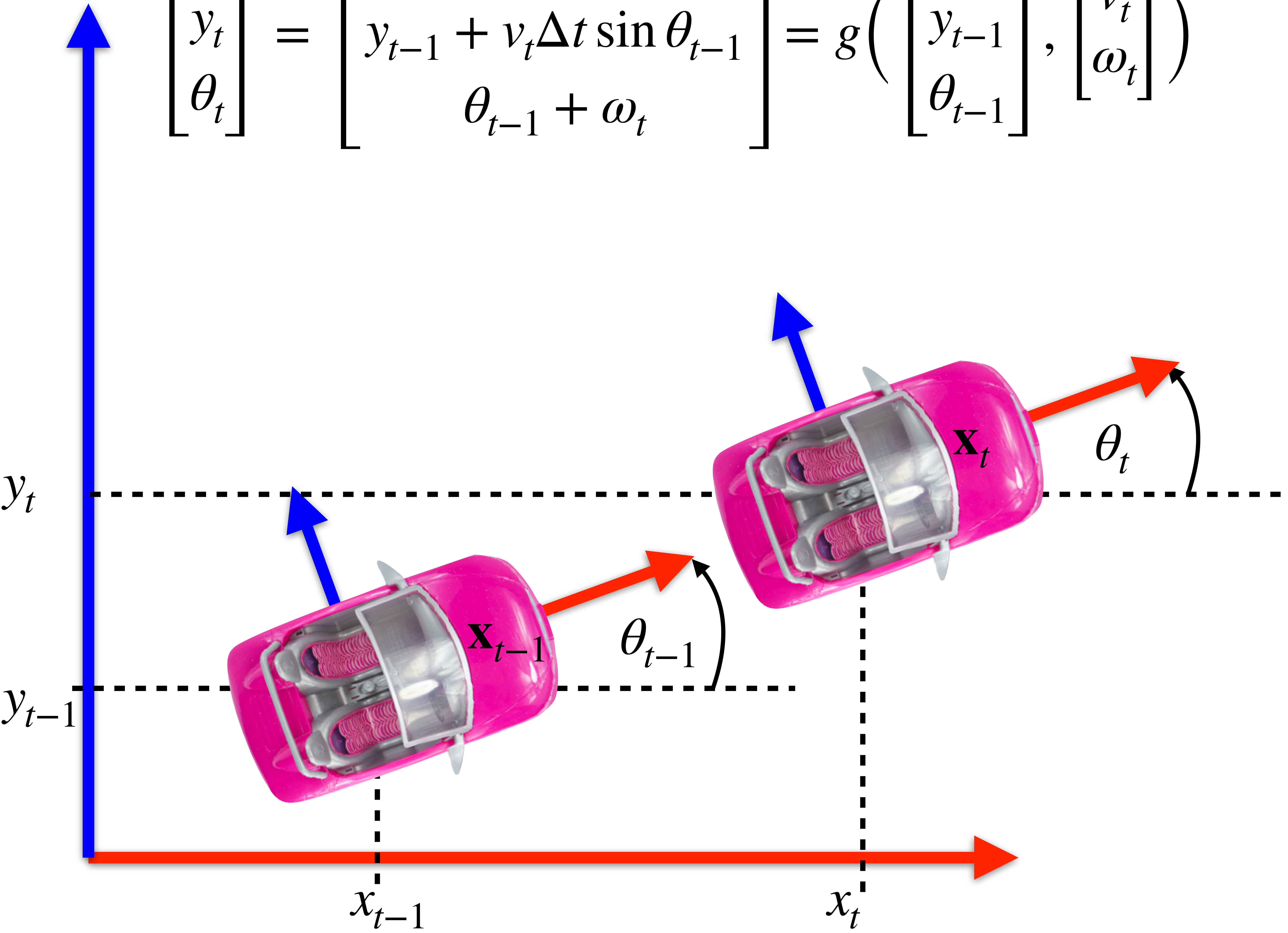
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$

4. Repeat from 2

# Differential drive model - linearization

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_t \Delta t \cos \theta_{t-1} \\ y_{t-1} + v_t \Delta t \sin \theta_{t-1} \\ \theta_{t-1} + \omega_t \end{bmatrix} = g\left( \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix}, \begin{bmatrix} v_t \\ \omega_t \end{bmatrix} \right)$$



# Differential drive model - linearization

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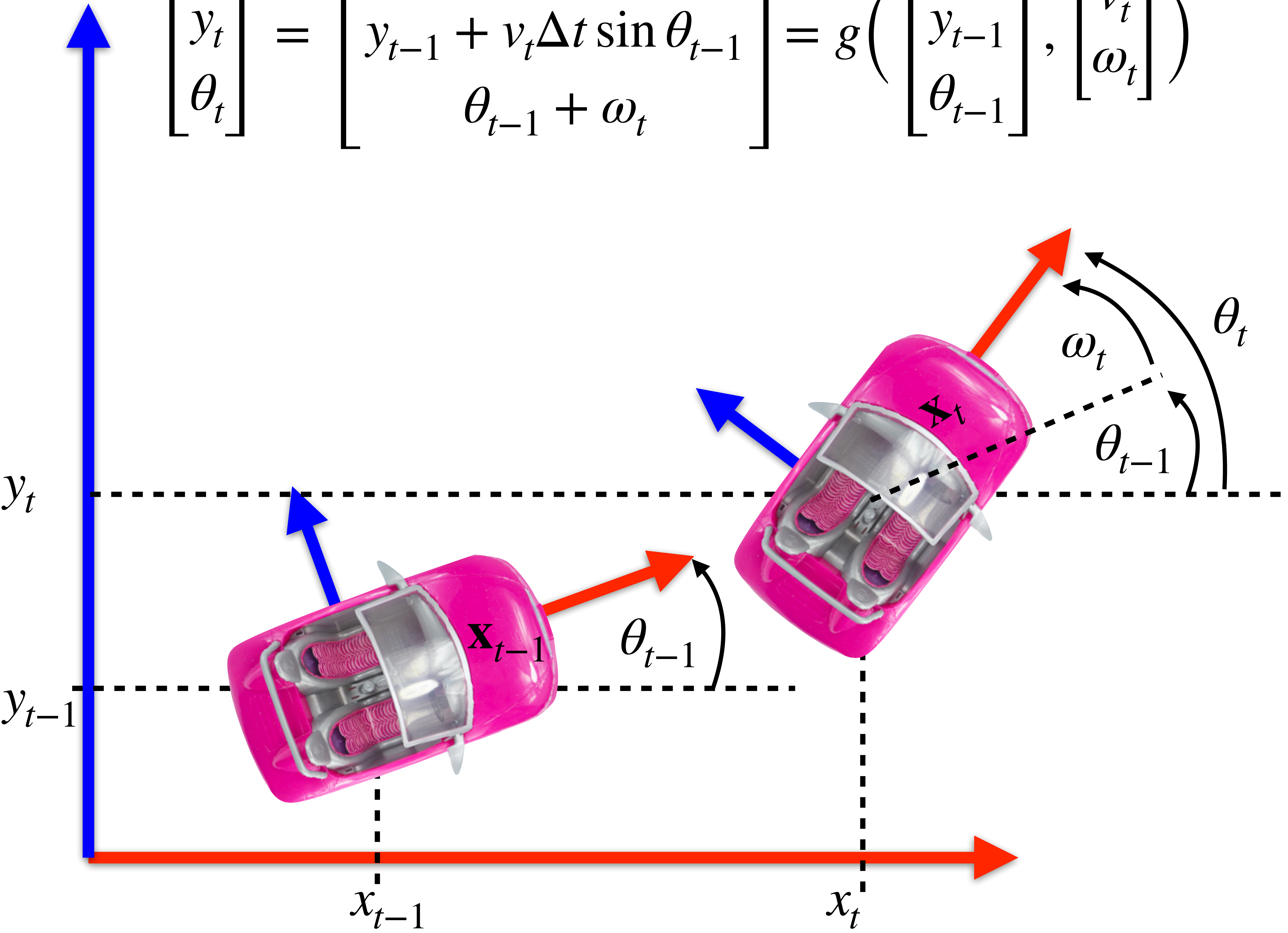


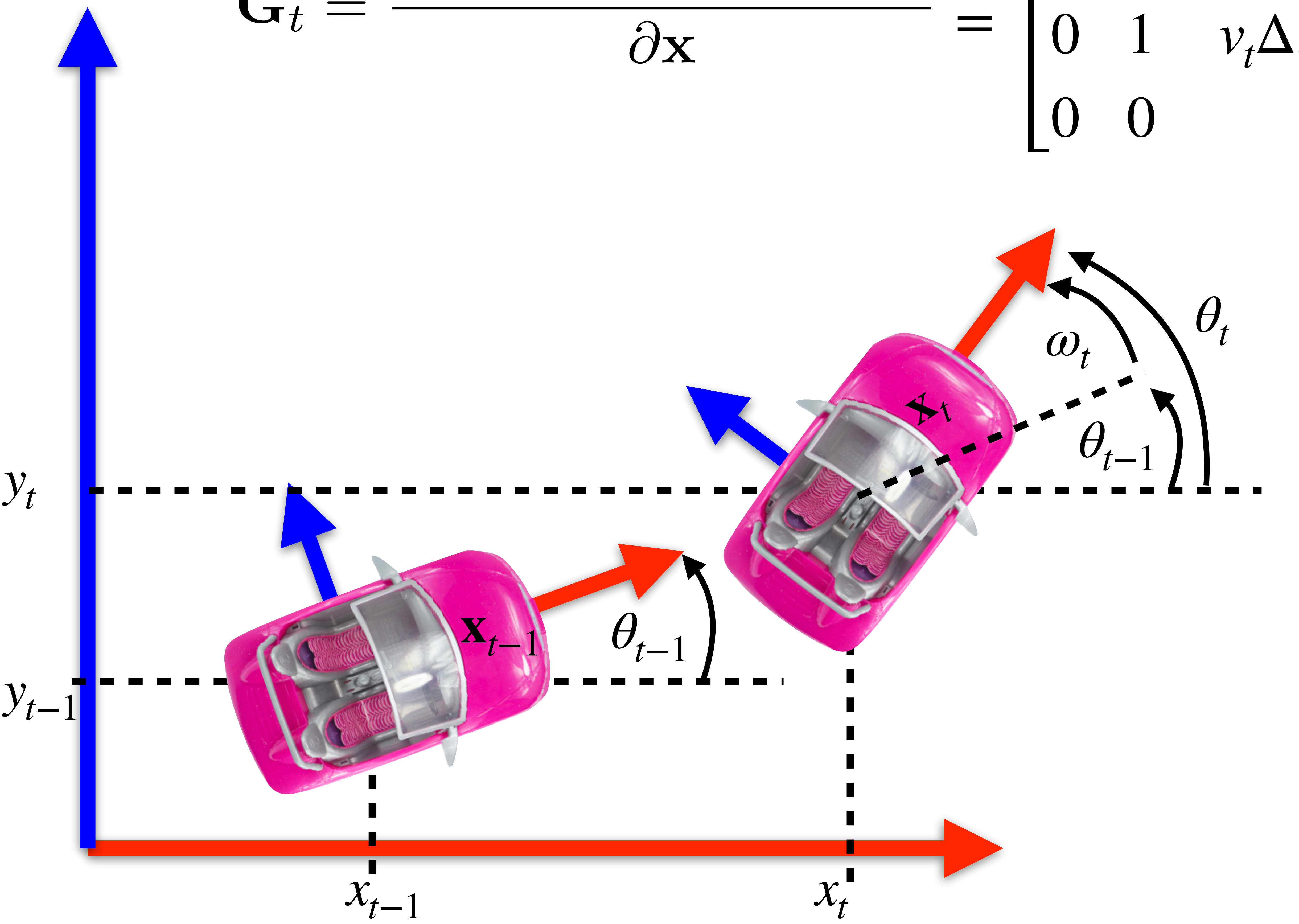
# Differential drive model - linearization

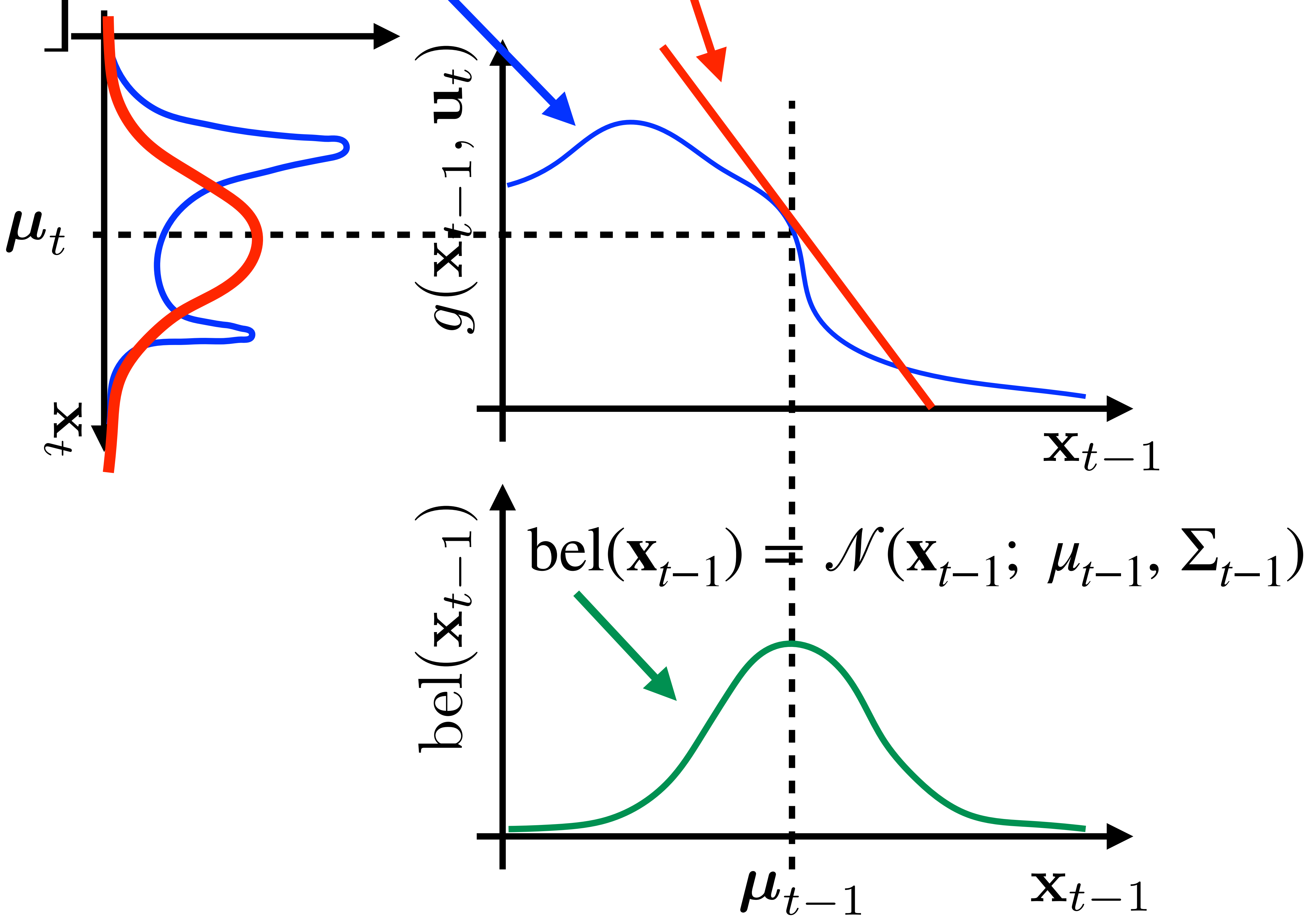
**What is dimensionality of g if u is assumed to be known?**

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_t \Delta t \cos\theta_{t-1} \\ y_{t-1} + v_t \Delta t \sin\theta_{t-1} \\ \theta_{t-1} + \omega_t \end{bmatrix} = g\left( \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix}, \begin{bmatrix} v_t \\ \omega_t \end{bmatrix} \right) \approx g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1})$$

$$g(\mathbf{x}, \mathbf{u}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\mathbf{G}_t = \frac{\partial g(\mathbf{u} = \mathbf{u}_t, \mathbf{x} = \boldsymbol{\mu}_{t-1})}{\partial \mathbf{x}} = \begin{bmatrix} 1 & 0 & -v_t \Delta t \sin \mu^\theta_{t-1} \\ 0 & 1 & v_t \Delta t \cos \mu^\theta_{t-1} \\ 0 & 0 & 1 \end{bmatrix}$$



# Extended Kalman Filter

**In order to use it we need G and H, the rest is the same !!!**

Linear system with Gaussian noise: Linearized system with Gaussian noise:

$$p(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t) \approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

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1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
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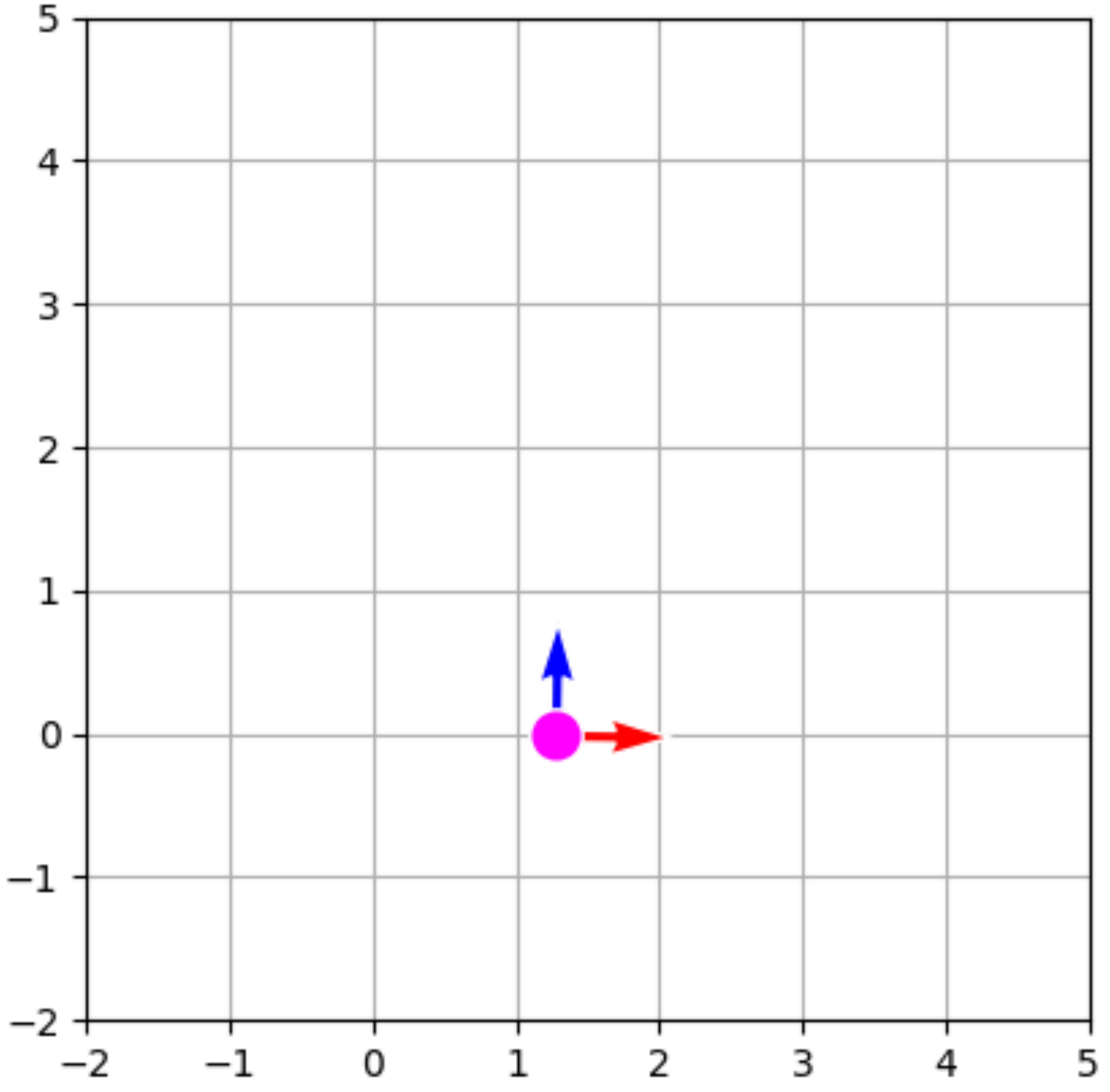
# EKF SLAM: absolute marker, differential-drive motion model




**State**

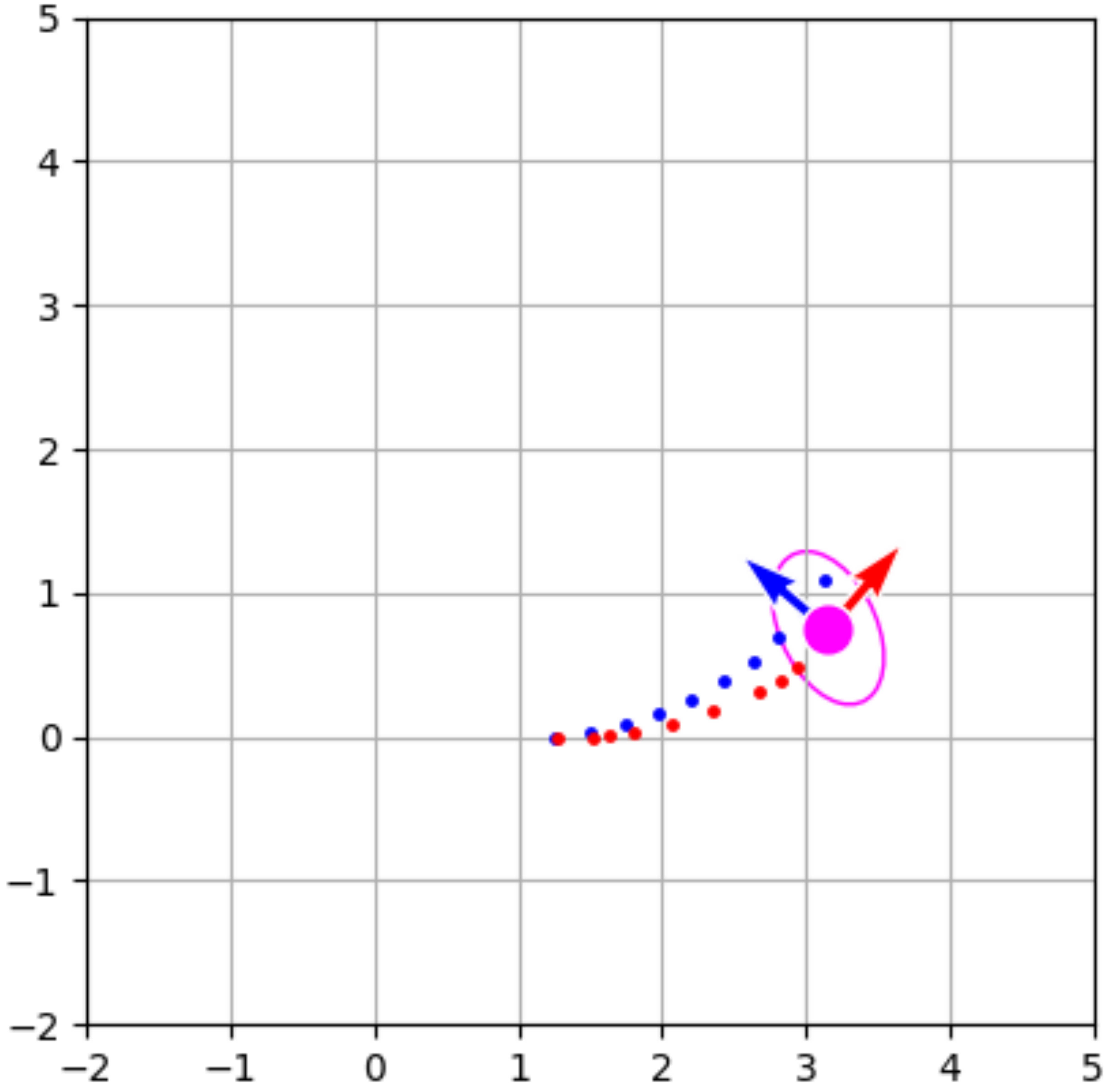
$\mathbf{x}_t$ … estimated robot pose

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

…… ground truth trajectory

…… estimated trajectory

# EKF SLAM: absolute marker, differential-drive motion model

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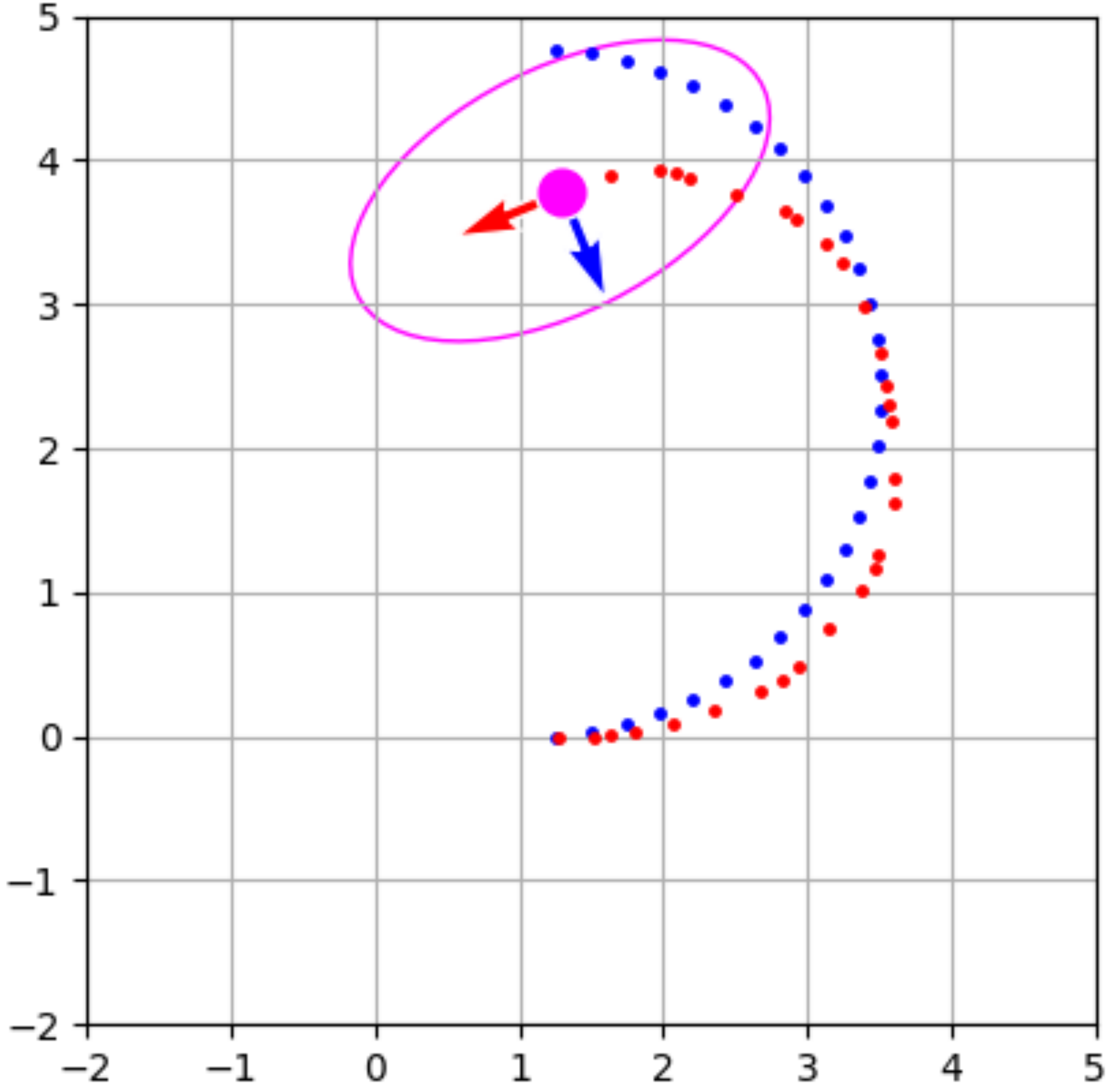
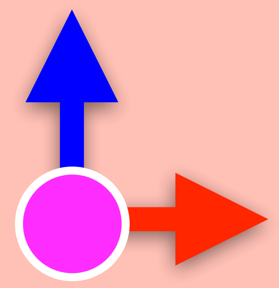
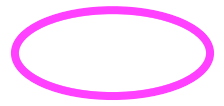



**State**



$\mathbf{x}_t$ … estimated robot pose



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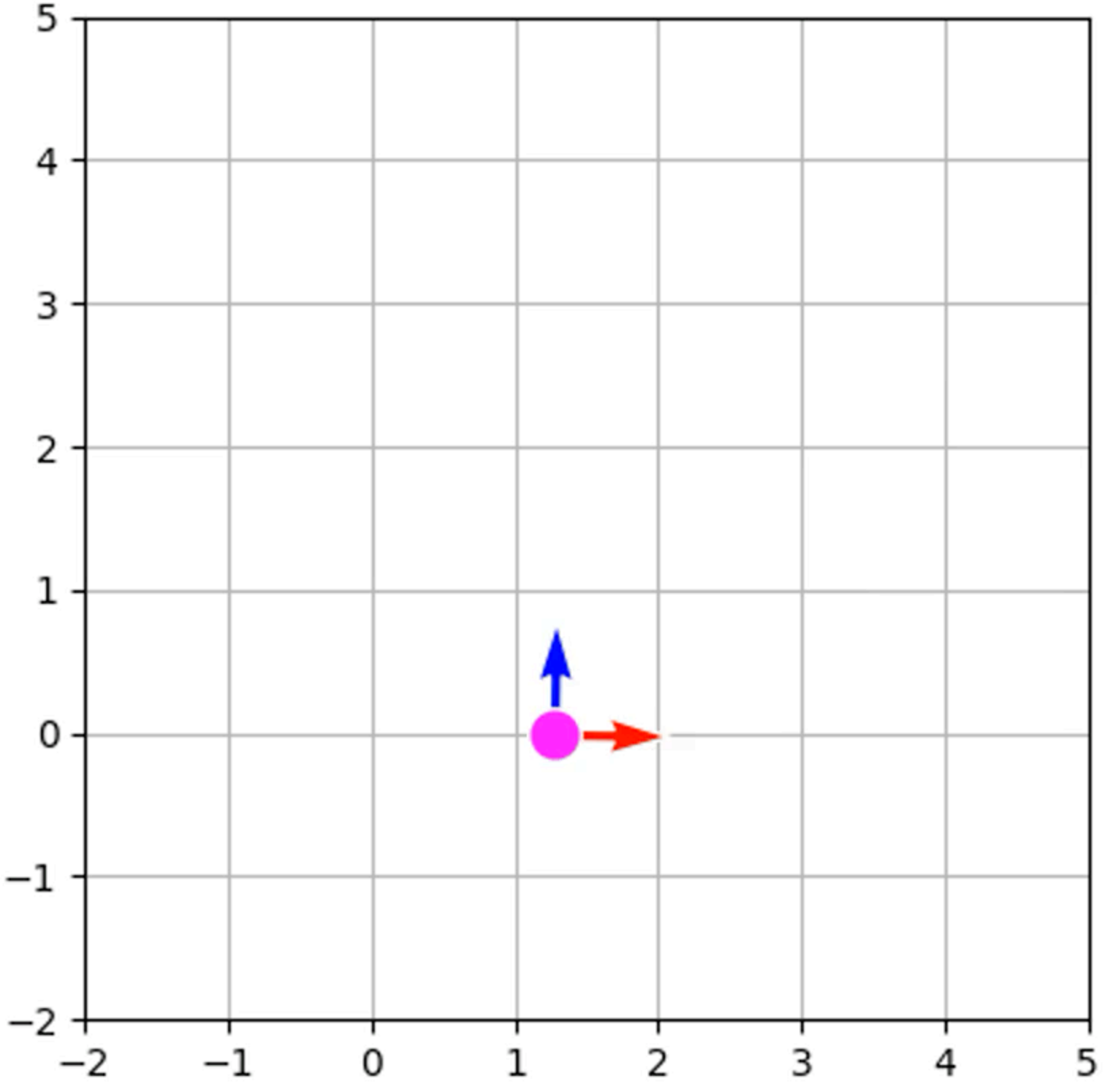


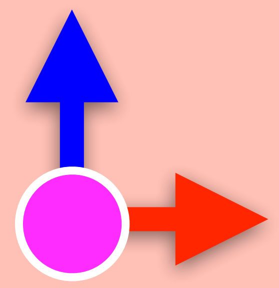


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…… ground truth trajectory

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**Linear system:**

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2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t\boldsymbol{\mu}_{t-1} + \mathbf{B}_t\mathbf{u}_t$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}_t\boldsymbol{\Sigma}_{t-1}\mathbf{A}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
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$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

**Linearized system:**

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = \boxed{g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1})}$$
$$\overline{\boldsymbol{\Sigma}}_t = \boxed{\mathbf{G}_t}\boldsymbol{\Sigma}_{t-1}\boxed{\mathbf{G}_t^\top} + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top(\mathbf{H}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t))$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{H}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

# Extended Kalman Filter

**In order to use it we need G and H, the rest is the same !!!**

Linear system with Gaussian noise:

Linearized system with Gaussian noise:

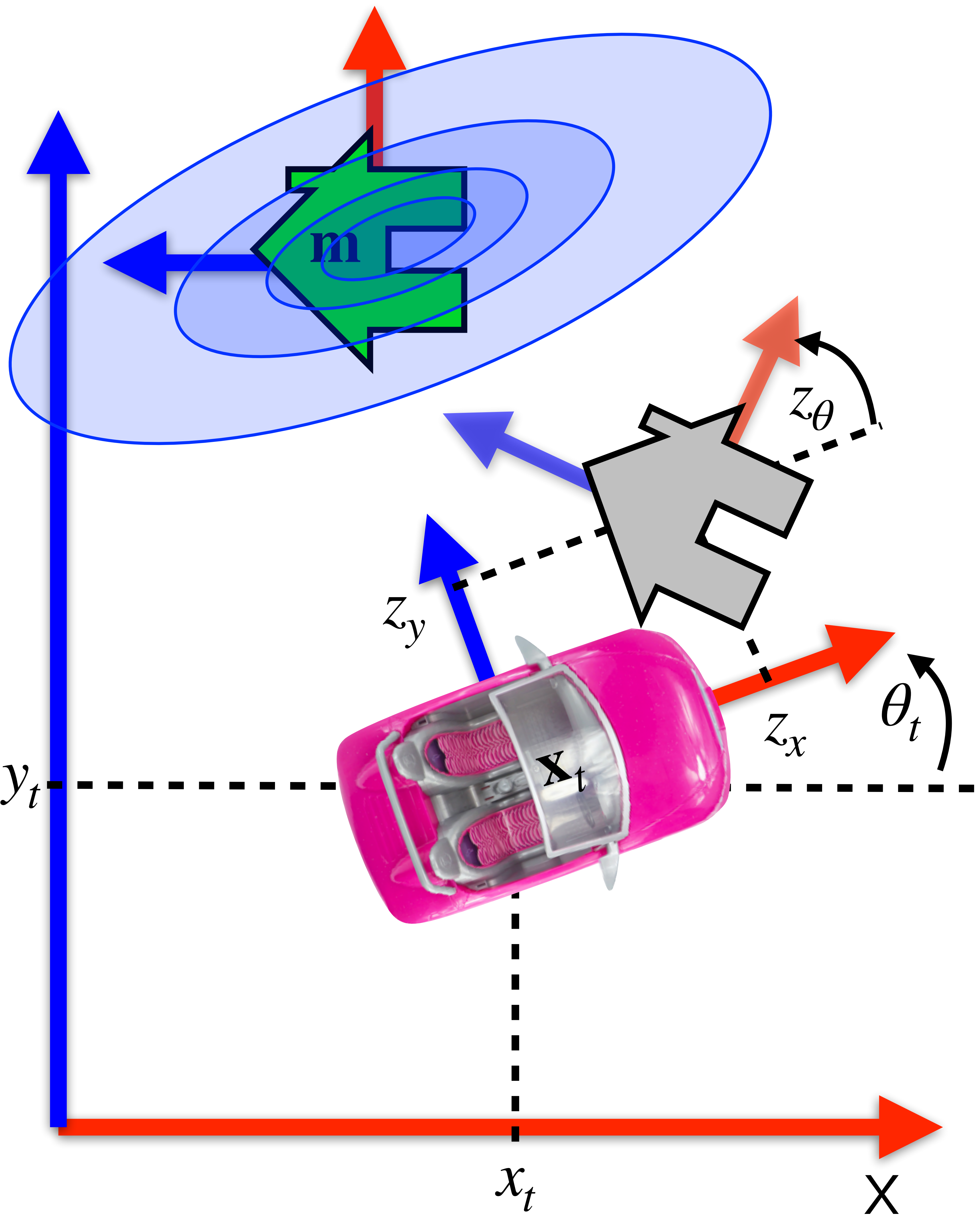
$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t) \approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

$$p(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}_{\mathbf{z}_t}(\mathbf{C}_t\mathbf{x}_t,\ \mathbf{Q}_t) \qquad \approx \mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t\boldsymbol{\mu}_{t-1} + \mathbf{B}_t\mathbf{u}_t$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}_t\boldsymbol{\Sigma}_{t-1}\mathbf{A}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top(\mathbf{C}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - \mathbf{C}_t\overline{\boldsymbol{\mu}}_t)$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1})$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{G}_t\boldsymbol{\Sigma}_{t-1}\mathbf{G}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top(\mathbf{H}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t))$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{H}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

# Absolute marker detector in EKF




$$p\left(\underbrace{\begin{bmatrix} z_t^x \\ z_t^y \\ z_t^\theta \end{bmatrix}}_{\mathbf{z}_t^{\mathbf{m}}} \middle| \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{\mathbf{x}_t}\right) = \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\mathrm{w2r}(\mathbf{m}, \mathbf{x_t})}_{h^{\mathbf{m}}(\mathbf{x}_t)},\ Q_t^{\mathbf{m}}\right)$$

**What is dimensionality of h if m is assumed to be known?**

$$h^{\mathbf{m}}(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^3$$

$$h^{\mathbf{m}}(\mathbf{x}_t) = \begin{bmatrix} +\cos\theta_t \cdot (m^x - x_t) + \sin\theta_t \cdot (m^y - y_t) \\ -\sin\theta_t \cdot (m^x - x_t) + \cos\theta_t \cdot (m^y - y_t) \\ m^\theta - \theta_t \end{bmatrix}$$

$$\approx h(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t) \quad \text{around point } \bar{\boldsymbol{\mu}}_t = \begin{bmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \end{bmatrix}$$

$$\mathbf{H}_t = \begin{bmatrix} -\cos\bar{\theta}_t & -\sin\bar{\theta}_t & -\sin\bar{\theta}_t \cdot (m^x - \bar{x}_t) + \cos\bar{\theta}_t \cdot (m^y - \bar{y}_t) \\ +\sin\bar{\theta}_t & -\cos\bar{\theta}_t & -\cos\bar{\theta}_t \cdot (m^x - \bar{x}_t) - \sin\bar{\theta}_t \cdot (m^y - \bar{y}_t) \\ 0 & 0 & -1 \end{bmatrix}$$

# Extended Kalman Filter

Linear system with Gaussian noise:

$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t,\ \mathbf{R}_t)$$

$$p(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}_{\mathbf{z}_t}(\mathbf{C}_t\mathbf{x}_t,\ \mathbf{Q}_t)$$

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}_t\boldsymbol{\mu}_{t-1} + \mathbf{B}_t\mathbf{u}_t$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}_t\boldsymbol{\Sigma}_{t-1}\mathbf{A}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top(\mathbf{C}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{C}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - \mathbf{C}_t\overline{\boldsymbol{\mu}}_t)$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{C}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

Linearized system with Gaussian noise:

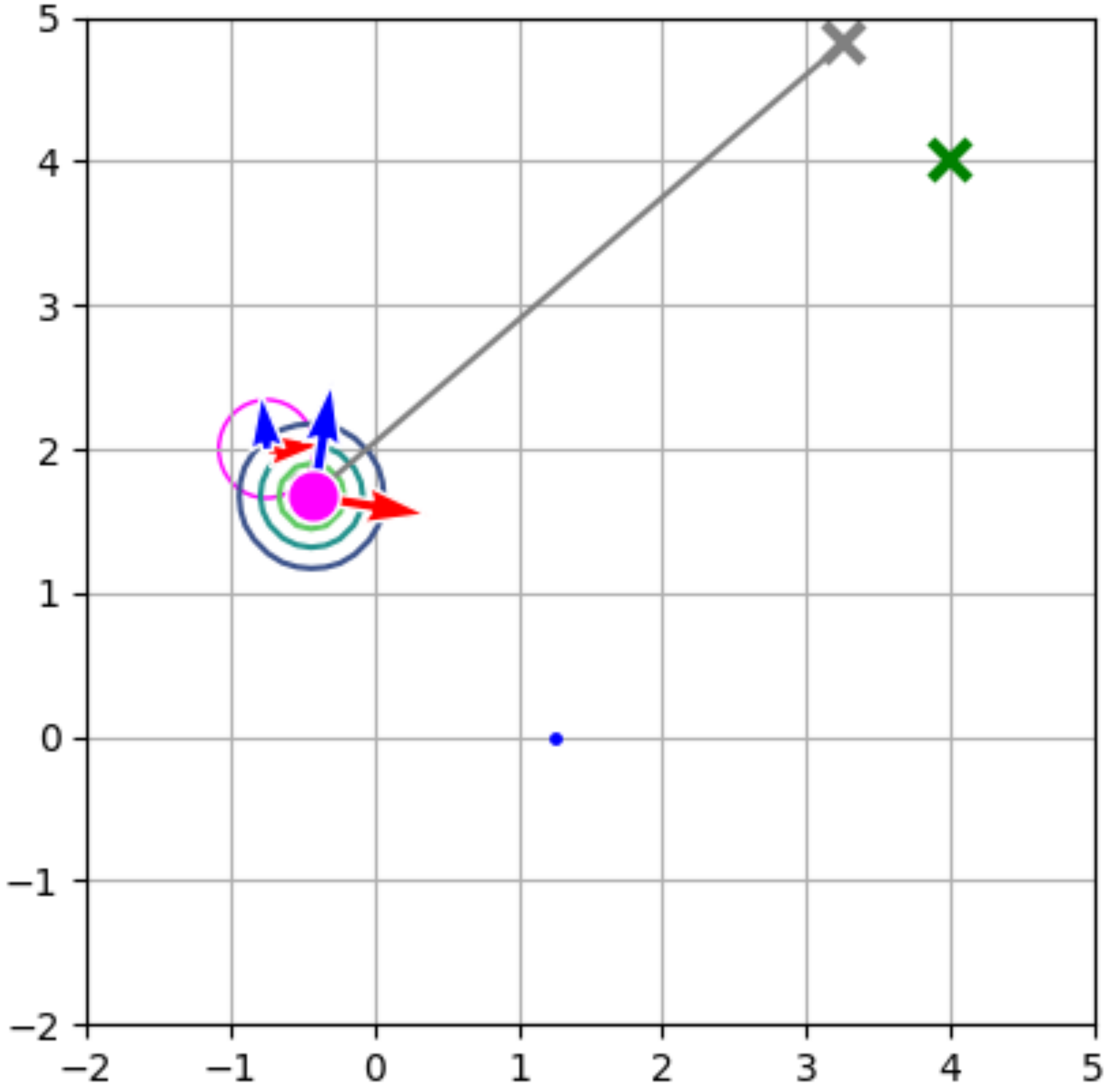
$$\approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

$$\approx \mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

1. Initialization: $\mathrm{bel}(\mathbf{x}_0)$
2. Prediction step:
$$\overline{\boldsymbol{\mu}}_t = g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1})$$
$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{G}_t\boldsymbol{\Sigma}_{t-1}\mathbf{G}_t^\top + \mathbf{R}_t$$
$$\overline{\mathrm{bel}}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\overline{\boldsymbol{\mu}}_t, \overline{\boldsymbol{\Sigma}}_t)$$
3. Measurement update:
$$\mathbf{K}_t = \overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top(\mathbf{H}_t\overline{\boldsymbol{\Sigma}}_t\mathbf{H}_t^\top + \mathbf{Q}_t)^{-1}$$
$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}_t(\mathbf{z}_t - h(\overline{\boldsymbol{\mu}}_t))$$
$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t\mathbf{H}_t)\overline{\boldsymbol{\Sigma}}_t$$
$$\mathrm{bel}(\mathbf{x}_t) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$$
4. Repeat from 2

**Lets run it !!!**

# EKF SLAM: absolute marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

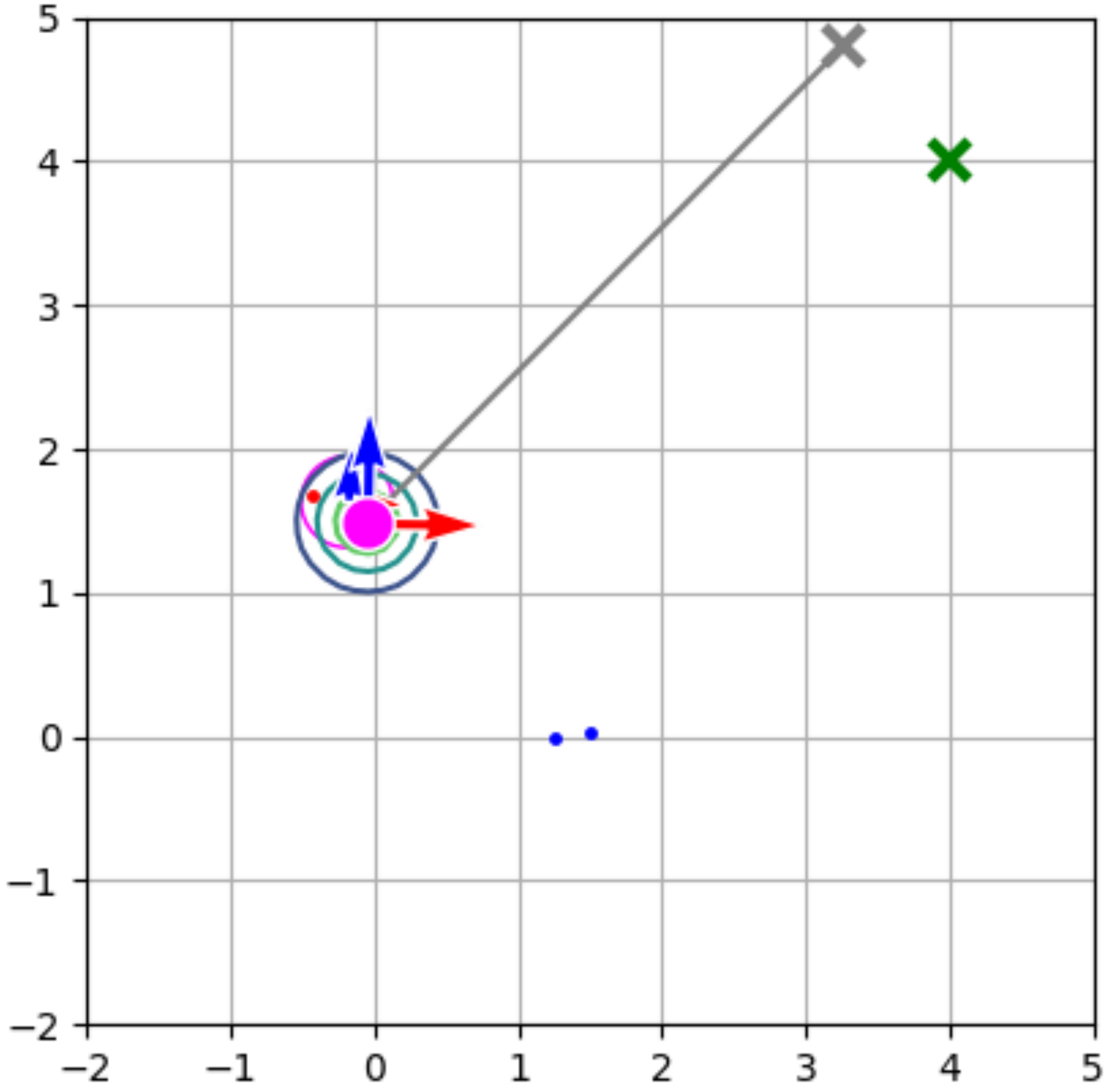
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

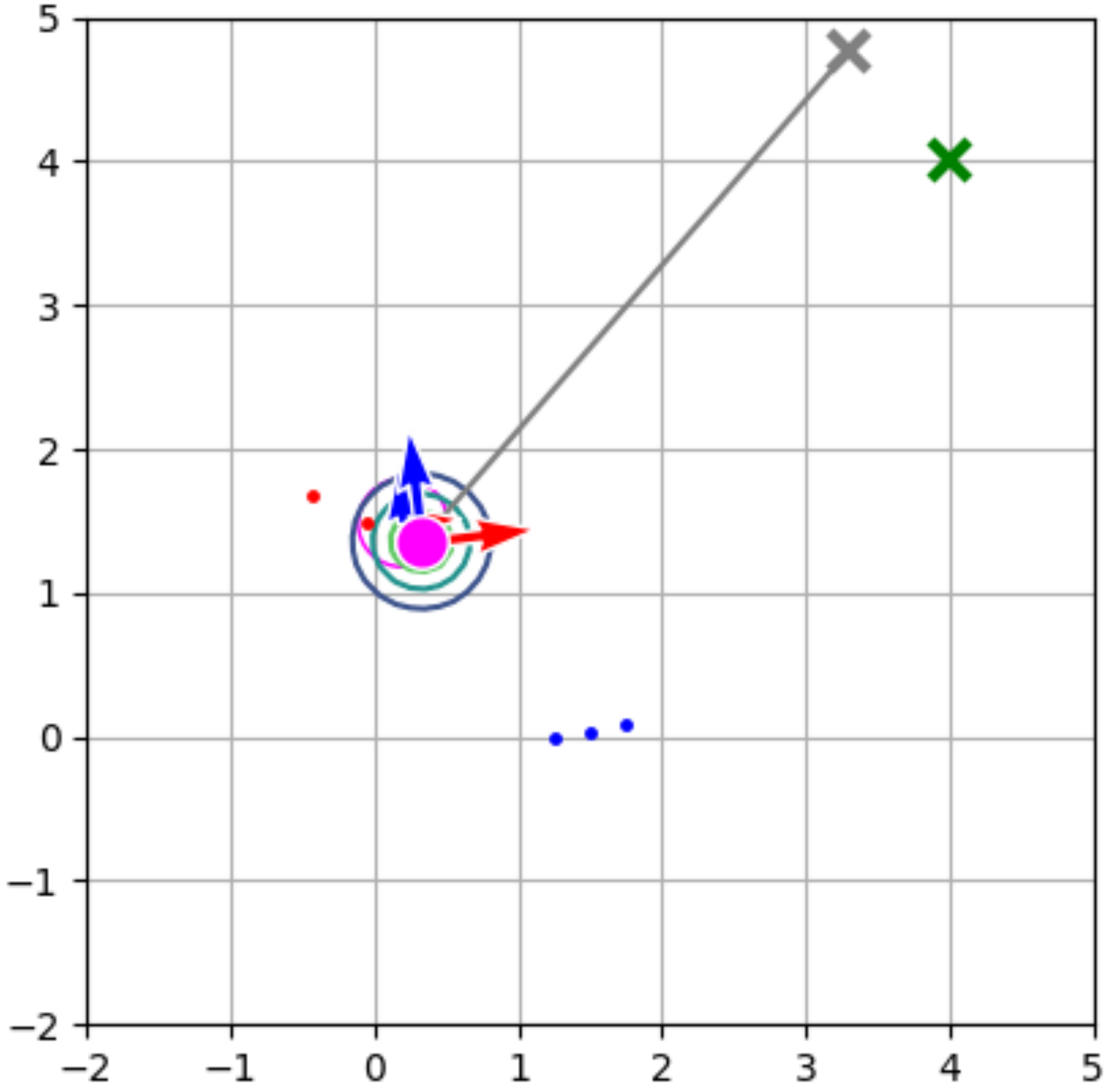
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

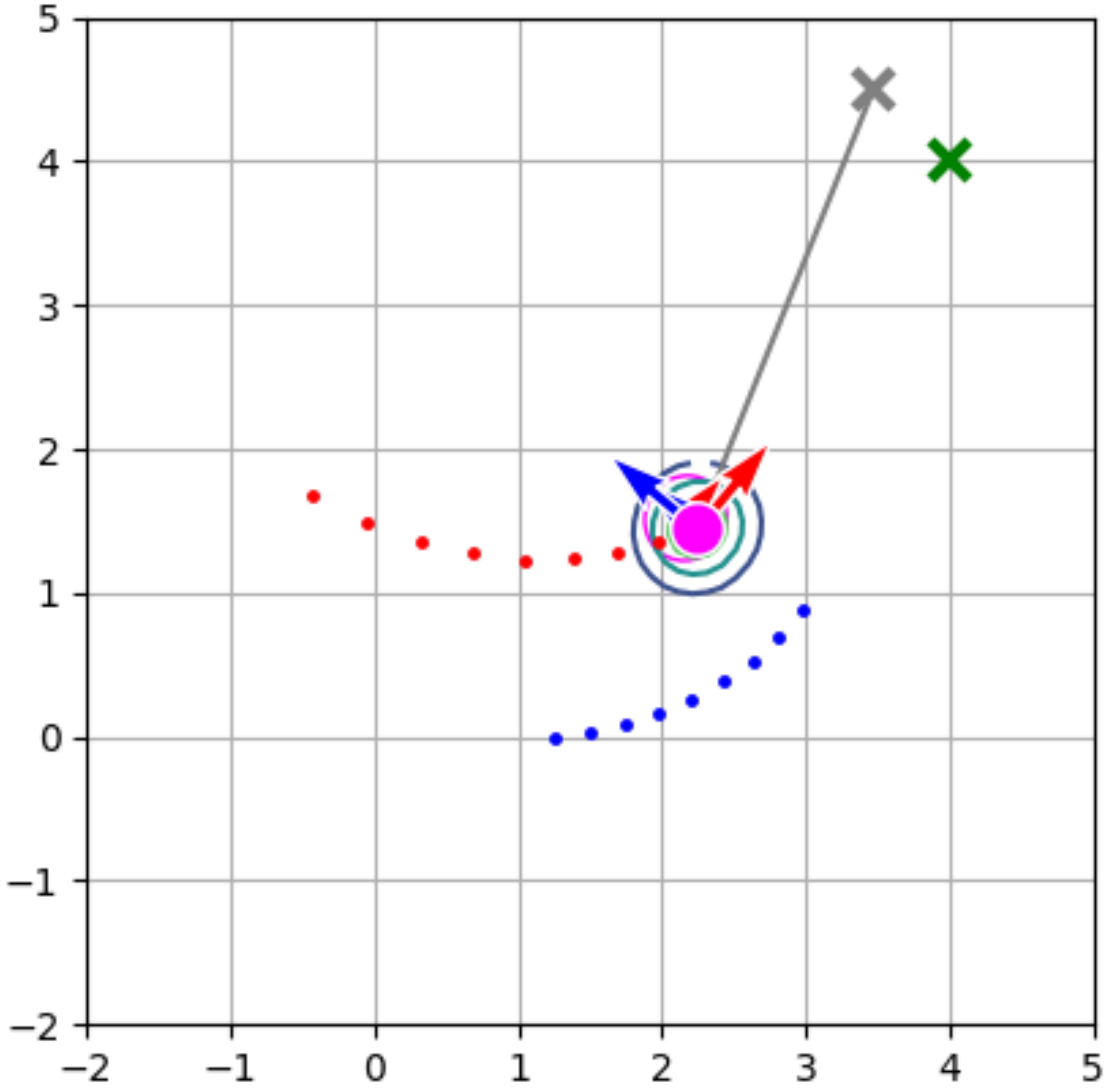
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\text{bel}}(\mathbf{x}_t)$ … prediction step

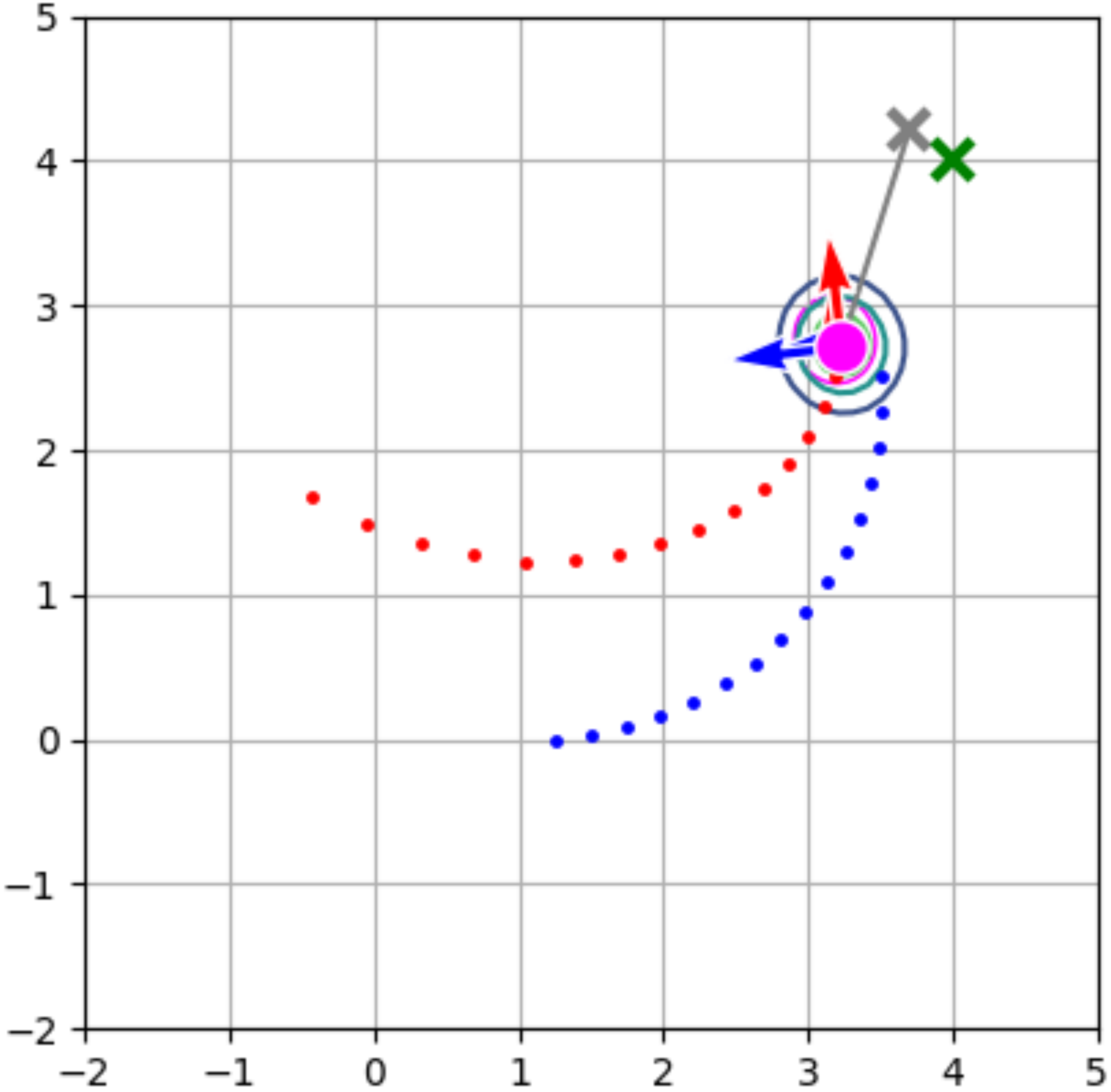
$\text{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_\mathrm{i}}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

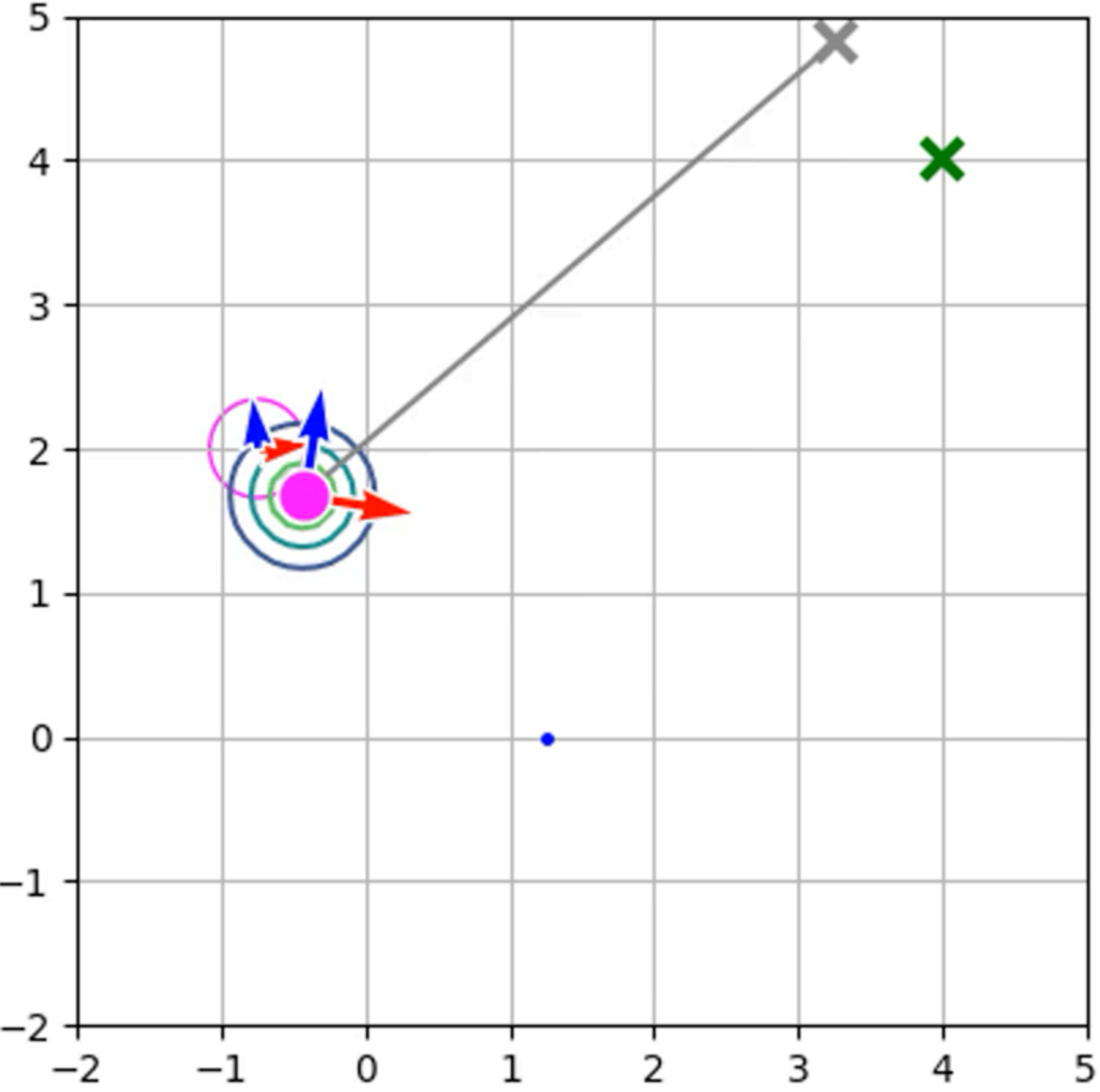
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

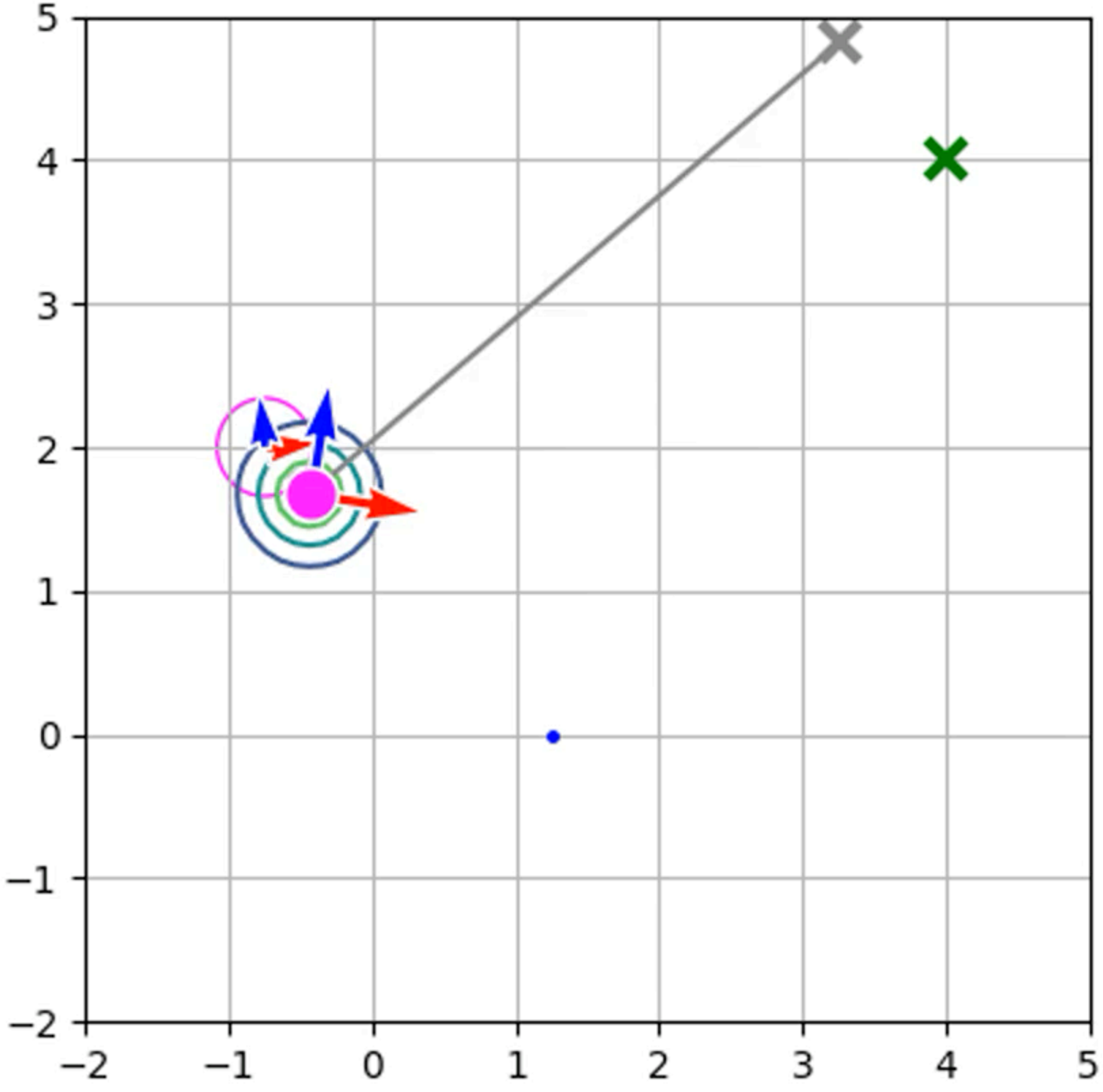
…… ground truth trajectory

…… estimated trajectory

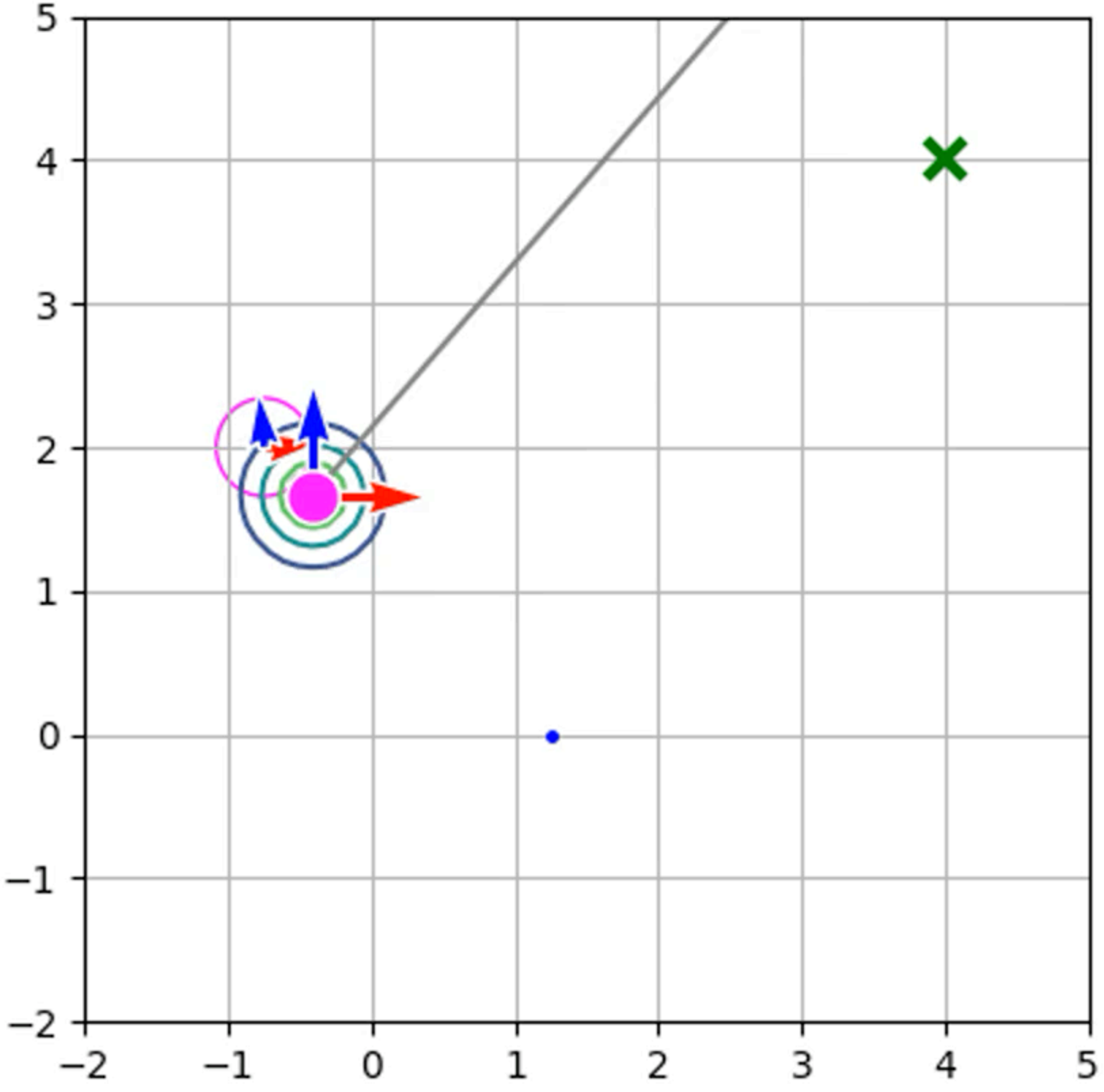
Wrong ini state x0

# EKF SLAM: absolute marker, differential-drive motion model

noiseless measurements




noisy measurements



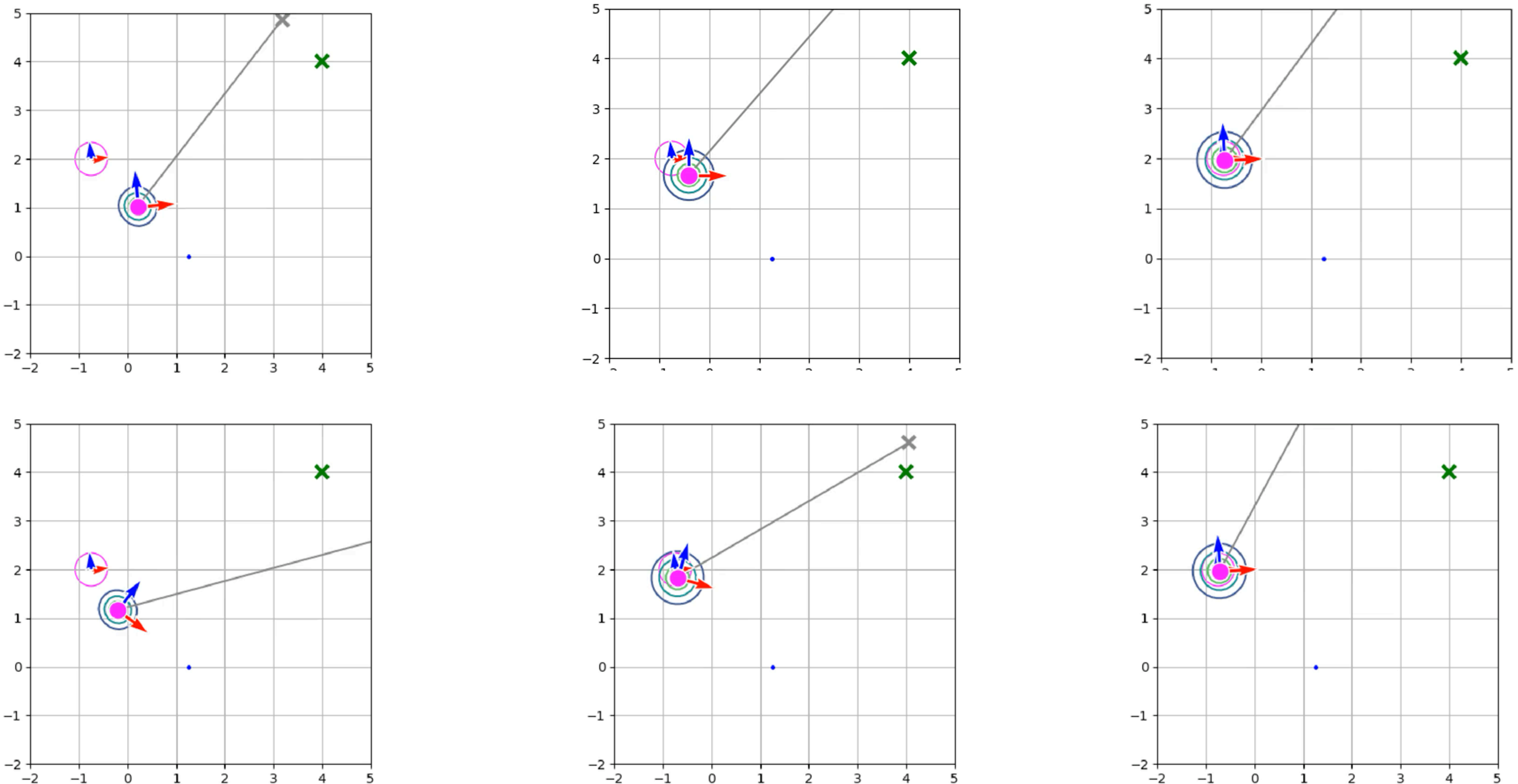
# EKF SLAM: absolute marker, differential-drive motion model

small covariance Q

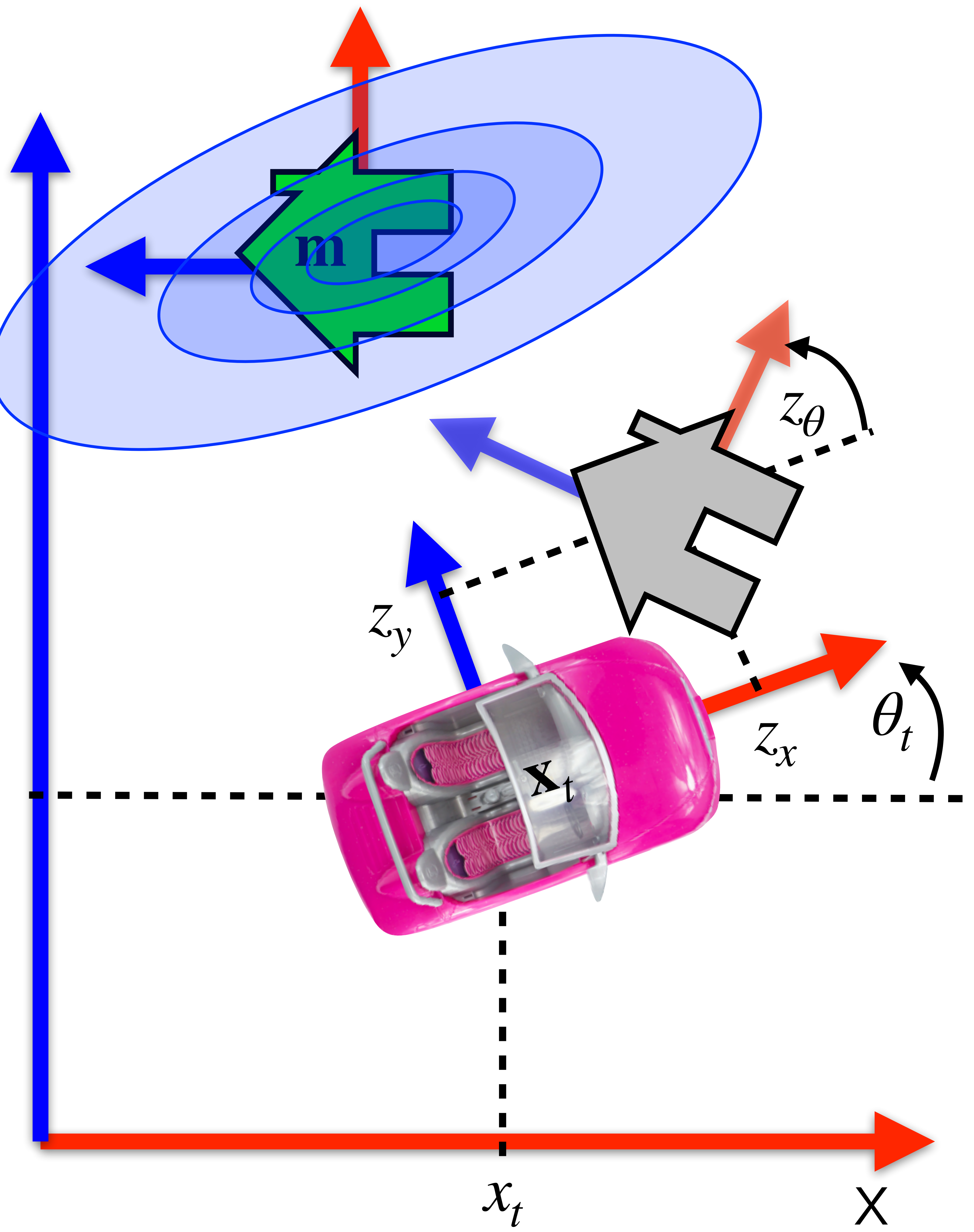
huge covariance Q

small noise

large noise



# Relative marker detector in EKF SLAM




$$p\Bigg( \underbrace{\begin{bmatrix} z_t^x \\ z_t^y \\ z_t^\theta \end{bmatrix}}_{\mathbf{z}_t^{\mathbf{m}}} \;\Bigg|\; \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ m^x \\ m^y \\ m^\theta \end{bmatrix}}_{\mathbf{x}_t} \Bigg) = \mathcal{N}\Big(\mathbf{z}_t^{\mathbf{m}};\; \underbrace{\mathrm{w2r}(\mathbf{m}, \mathbf{x_t})}_{h^{\mathbf{m}}(\mathbf{x}_t)}\;,\; Q_t^{\mathbf{m}}\Big)$$

**What is dimensionality of h if m is also unknown?**

$$h^{\mathbf{m}}(\mathbf{x}) \;:\; \mathbb{R}^6 \rightarrow \mathbb{R}^3$$

$$h^{\mathbf{m}}(\mathbf{x}_t) = \begin{bmatrix} +\cos\theta_t \cdot (m^x - x_t) + \sin\theta_t \cdot (m^y - y_t) \\ -\sin\theta_t \cdot (m^x - x_t) + \cos\theta_t \cdot (m^y - y_t) \\ m^\theta - \theta_t \end{bmatrix}$$

$$\approx h(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t) \quad \text{around point } \bar{\boldsymbol{\mu}}_t = \begin{bmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \\ \bar{m}_t^x \\ \bar{m}_t^y \\ \bar{m}_t^\theta \end{bmatrix}$$

# Marker measurement model in EKF SLAM

$$p\left(\underbrace{\begin{bmatrix} z_t^x \\ z_t^y \\ z_t^\theta \end{bmatrix}}_{\mathbf{z}_t^{\mathbf{m}}} \;\middle|\; \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ m^x \\ m^y \\ m^\theta \end{bmatrix}}_{\mathbf{x}_t}\right) = \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\mathrm{w2r}(\mathbf{m}, \mathbf{x_t})}_{h^{\mathbf{m}}(\mathbf{x}_t)},\ Q_t^{\mathbf{m}}\right)$$

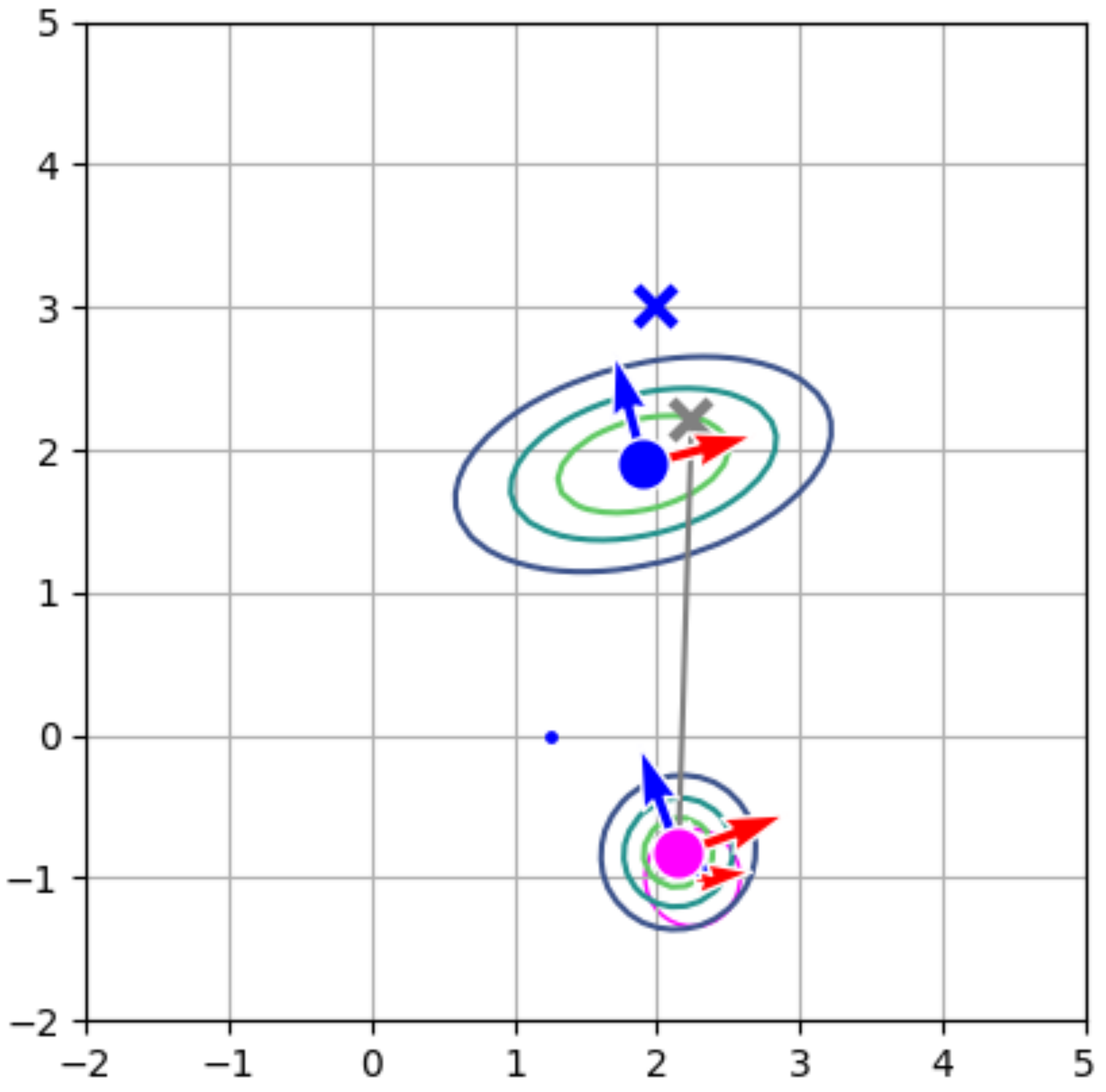
$$= \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\begin{bmatrix} +\cos\theta_t \cdot (m^x - x_t) + \sin\theta_t \cdot (m^y - y_t) \\ -\sin\theta_t \cdot (m^x - x_t) + \cos\theta_t \cdot (m^y - y_t) \\ m^\theta - \theta_t \end{bmatrix}}_{h^{\mathbf{m}}(\mathbf{x}_t)},\ Q_t^{\mathbf{m}}\right)$$

$$\approx \mathcal{N}(\mathbf{z}_t;\ h(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

around point $\bar{\boldsymbol{\mu}}_t = \begin{bmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \\ \bar{m}_t^x \\ \bar{m}_t^y \\ \bar{m}_t^\theta \end{bmatrix}$

$$\begin{array}{cc} & \begin{array}{cccccc} \dfrac{\partial}{\partial x_t} & \dfrac{\partial}{\partial y_t} & \dfrac{\partial}{\partial \theta_t} & \dfrac{\partial}{\partial m^x} & \dfrac{\partial}{\partial m^y} & \dfrac{\partial}{\partial m^\theta} \end{array} \\ \mathbf{H}_t = & \begin{bmatrix} -\cos\bar{\theta}_t & -\sin\bar{\theta}_t & -\sin\bar{\theta}_t \cdot (\bar{m}^x - \bar{x}_t) + \cos\bar{\theta}_t \cdot (\bar{m}^y - \bar{y}_t) & \cos\bar{\theta}_t & \sin\bar{\theta}_t & 0 \\ +\sin\bar{\theta}_t & -\cos\bar{\theta}_t & -\cos\bar{\theta}_t \cdot (\bar{m}^x - \bar{x}_t) - \sin\bar{\theta}_t \cdot (\bar{m}^y - \bar{y}_t) & -\sin\bar{\theta}_t & +\cos\bar{\theta}_t & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

# EKF SLAM: relative marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

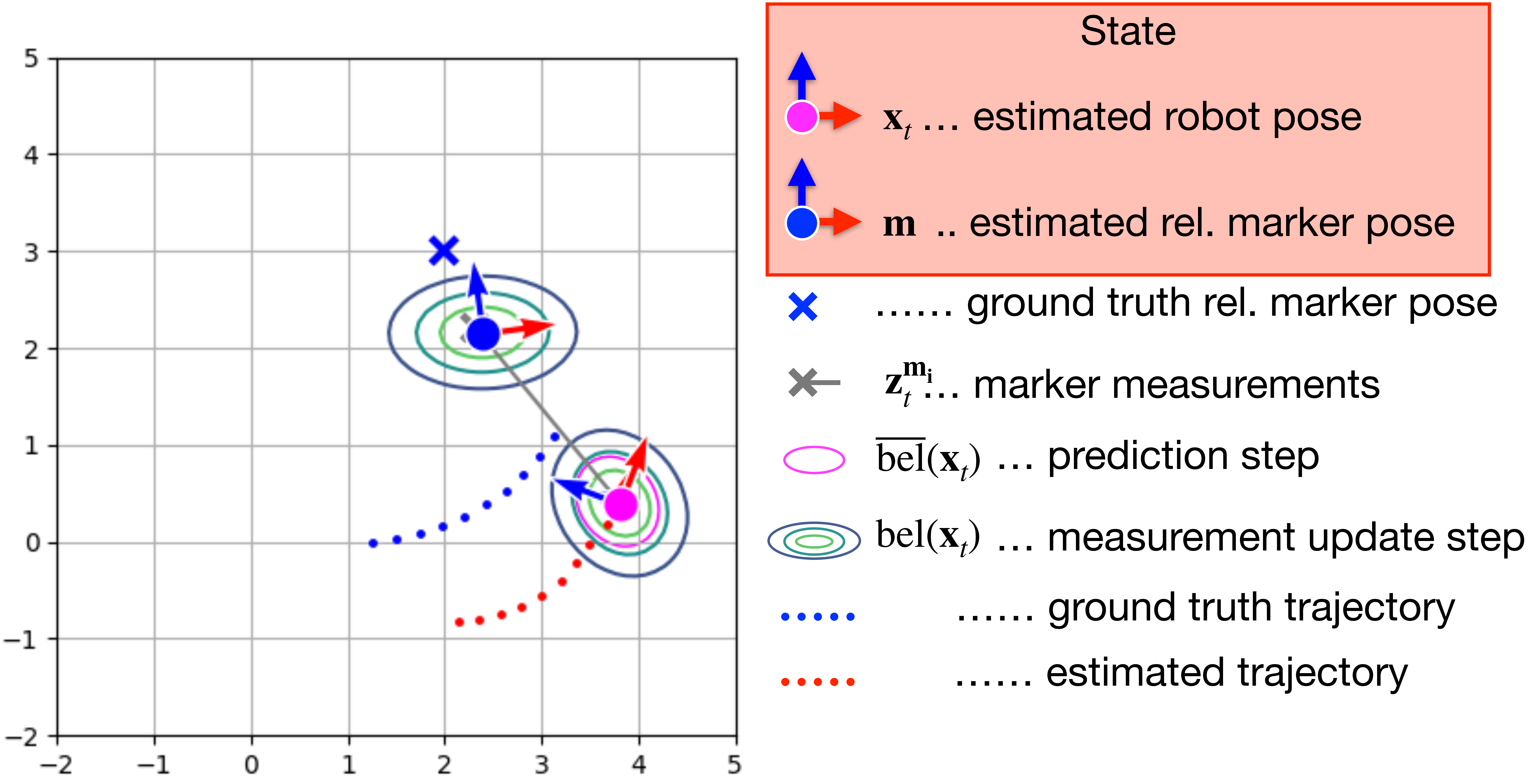
$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

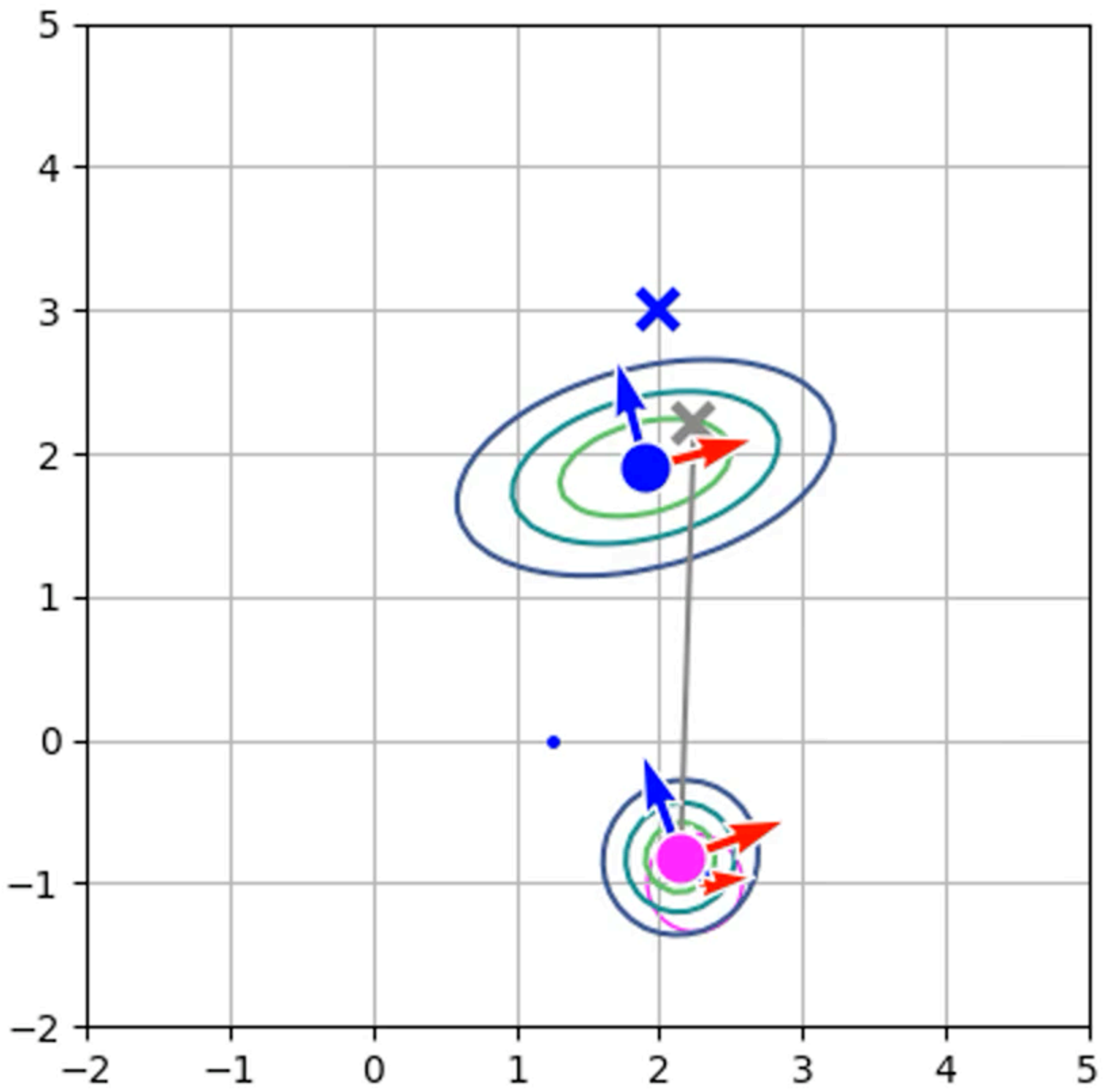
…… ground truth trajectory

…… estimated trajectory

# EKF SLAM: relative marker, differential-drive motion model



# EKF SLAM: relative marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

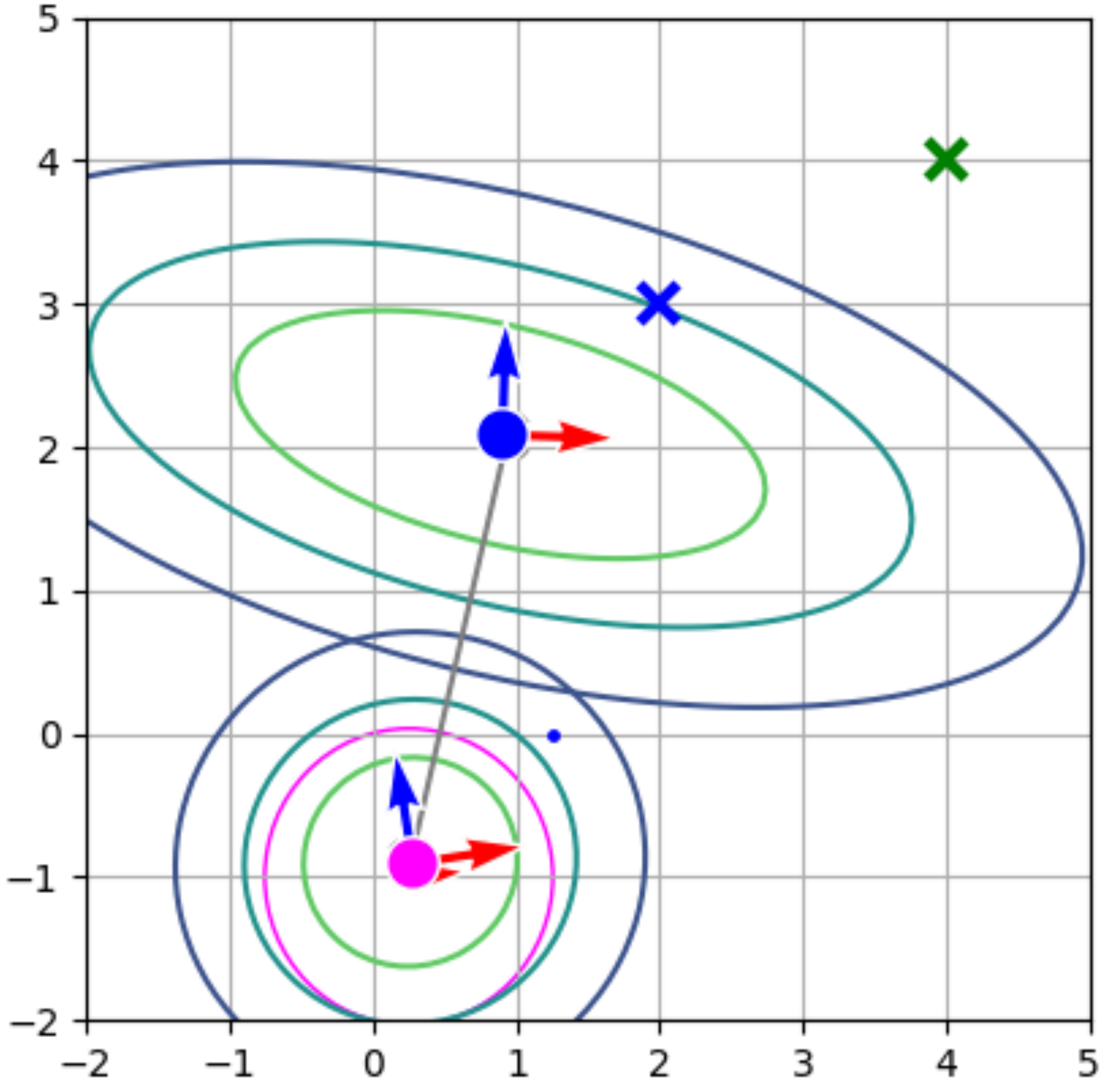
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

Relative and absolute markers together

# EKF SLAM: abs marker, relative marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

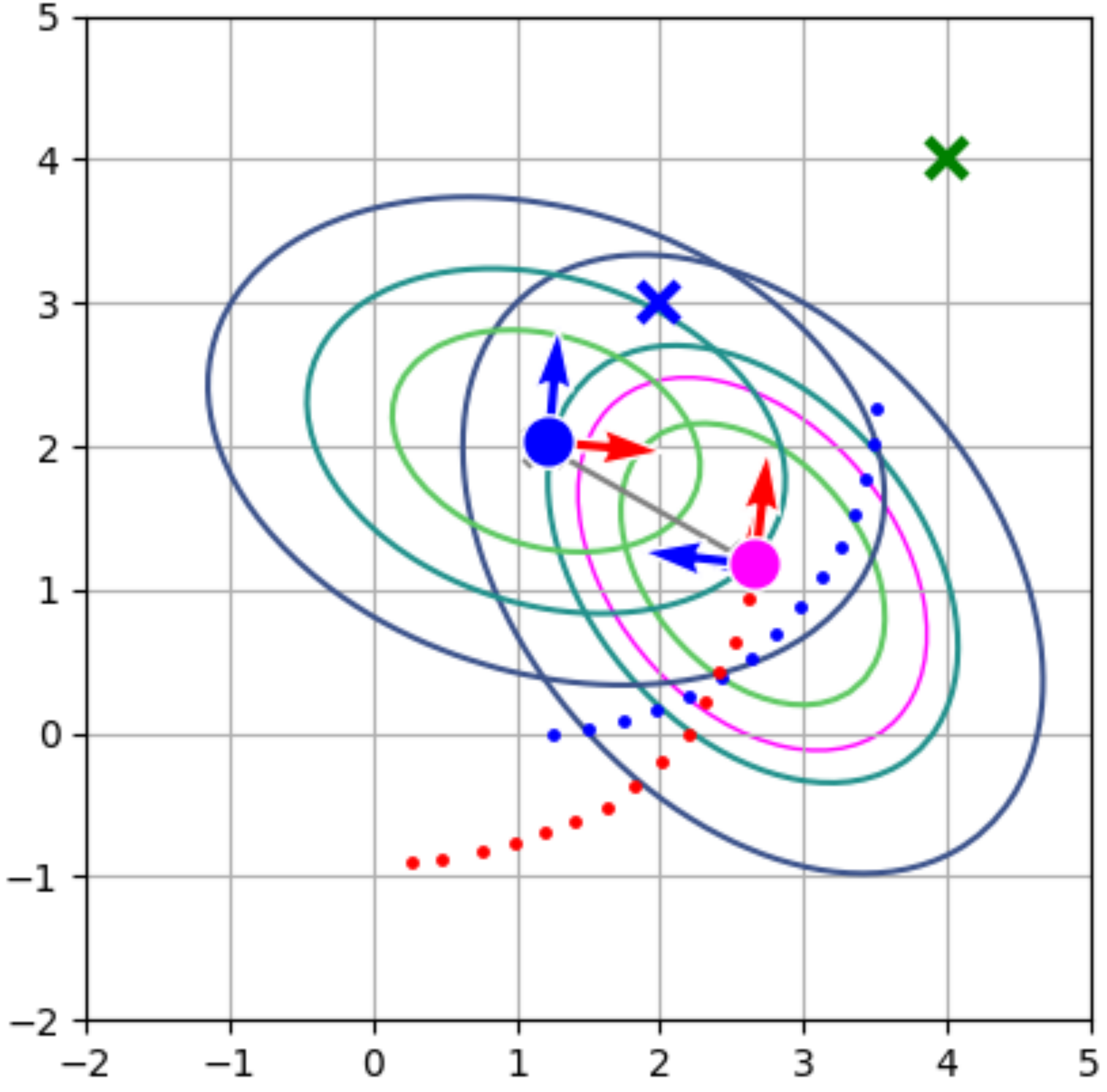
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

# EKF SLAM: abs marker, relative marker, differential-drive motion model

the last relative marker detection




**State**

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

...... ground truth rel. marker pose

...... ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

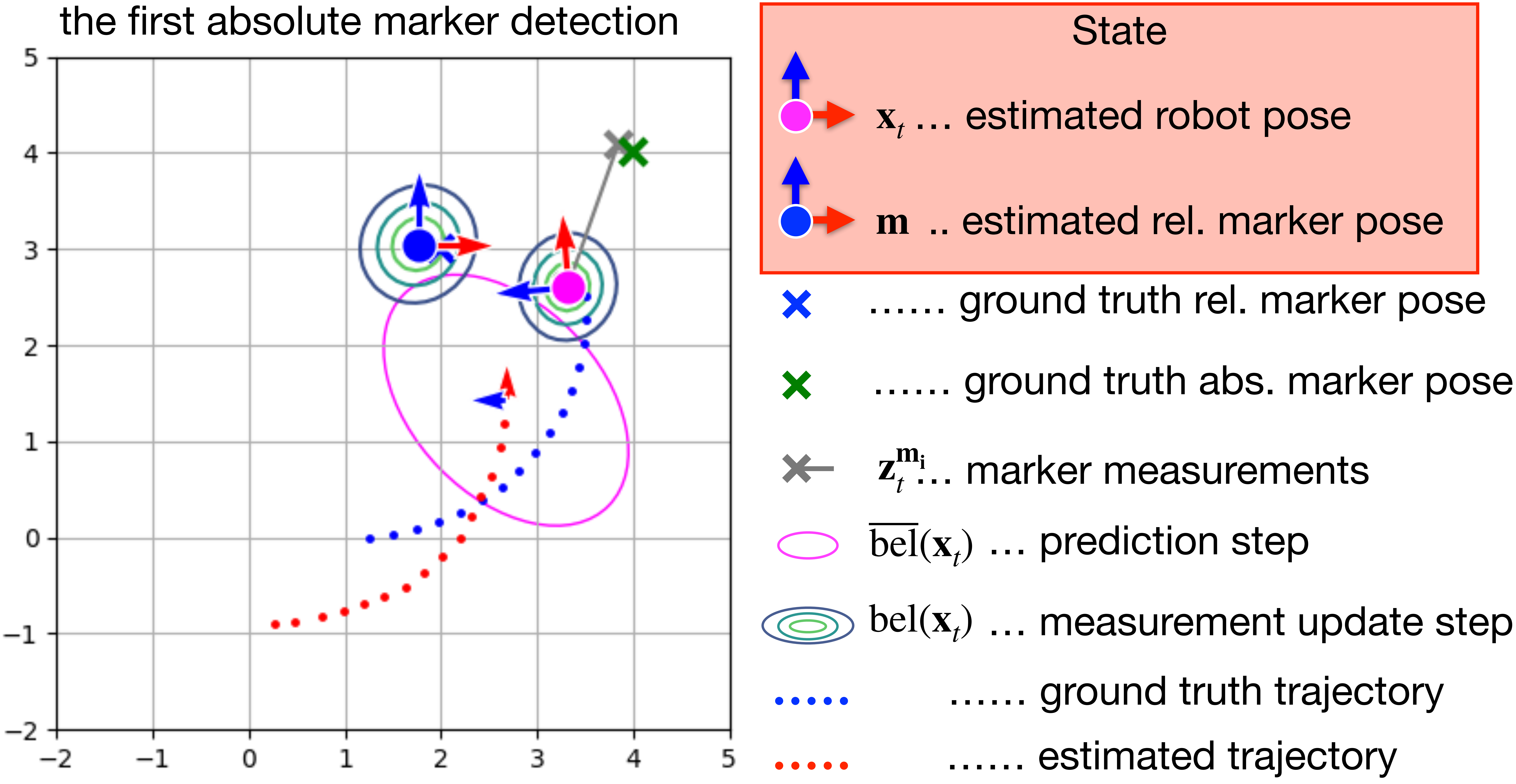
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

...... ground truth trajectory

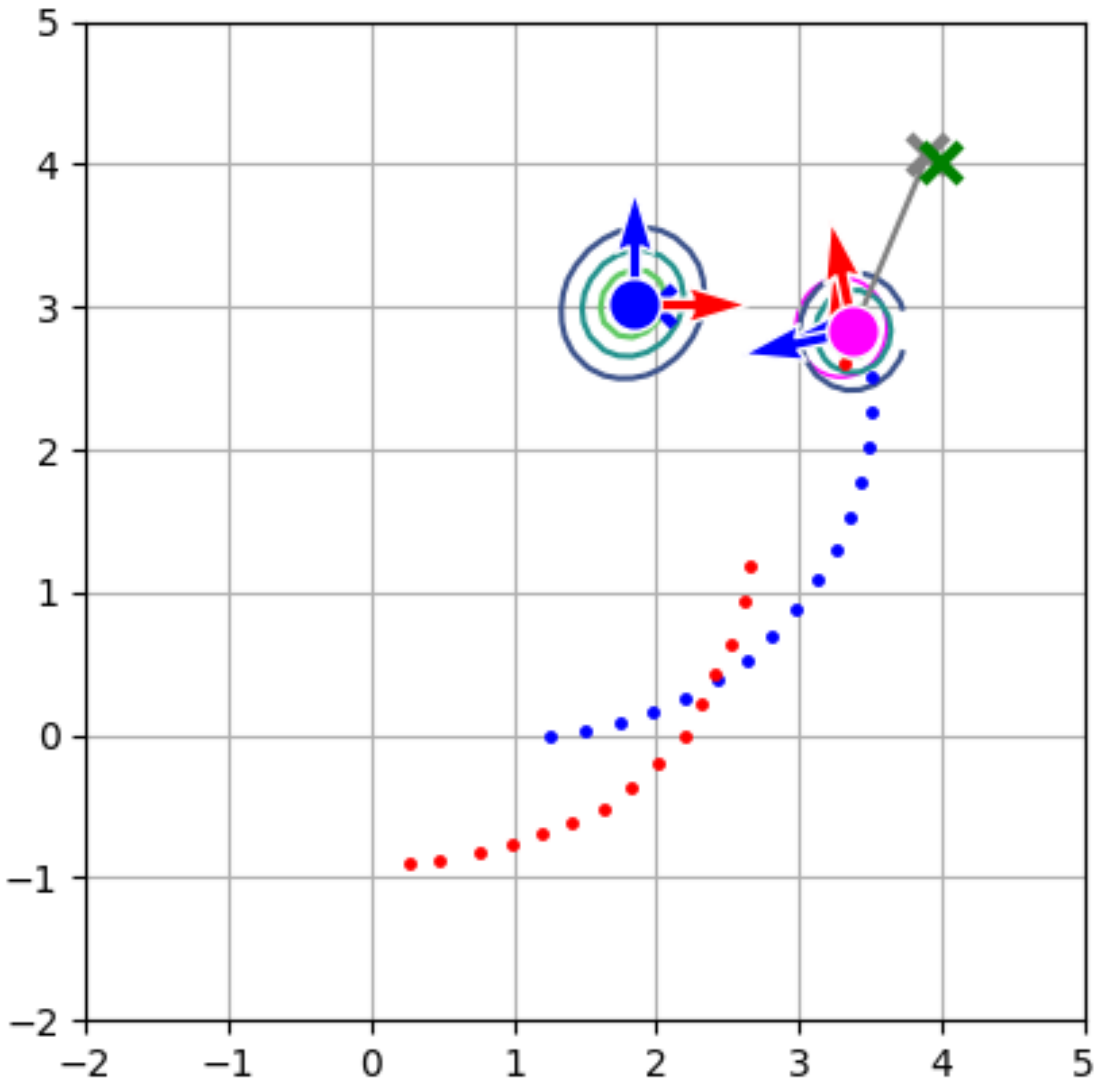
...... estimated trajectory

# EKF SLAM: abs marker, relative marker, differential-drive motion model

the first absolute marker detection



EKF SLAM: abs marker, relative marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

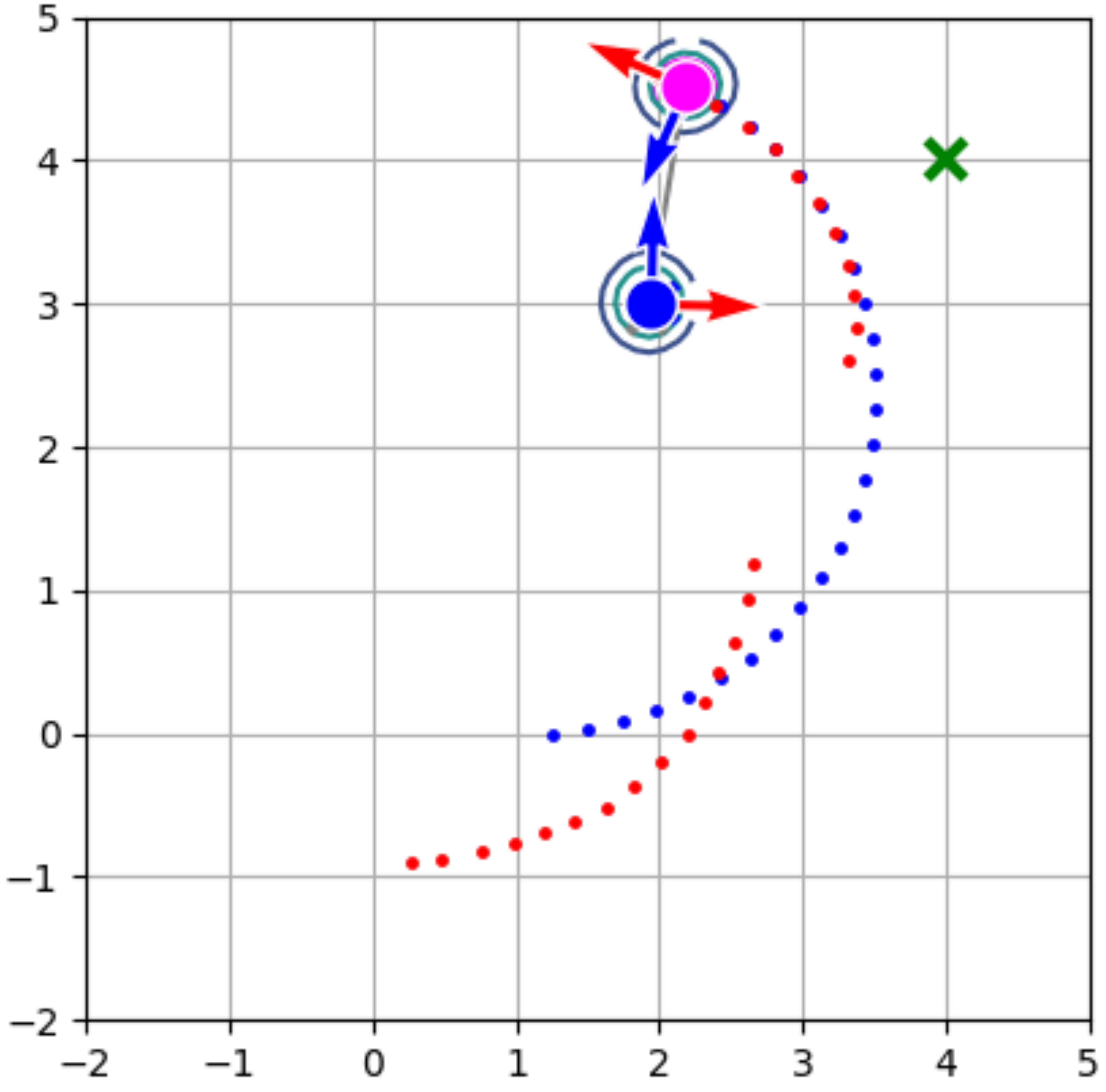
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

# EKF SLAM: abs marker, relative marker, differential-drive motion model

# EKF SLAM: abs marker, relative marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

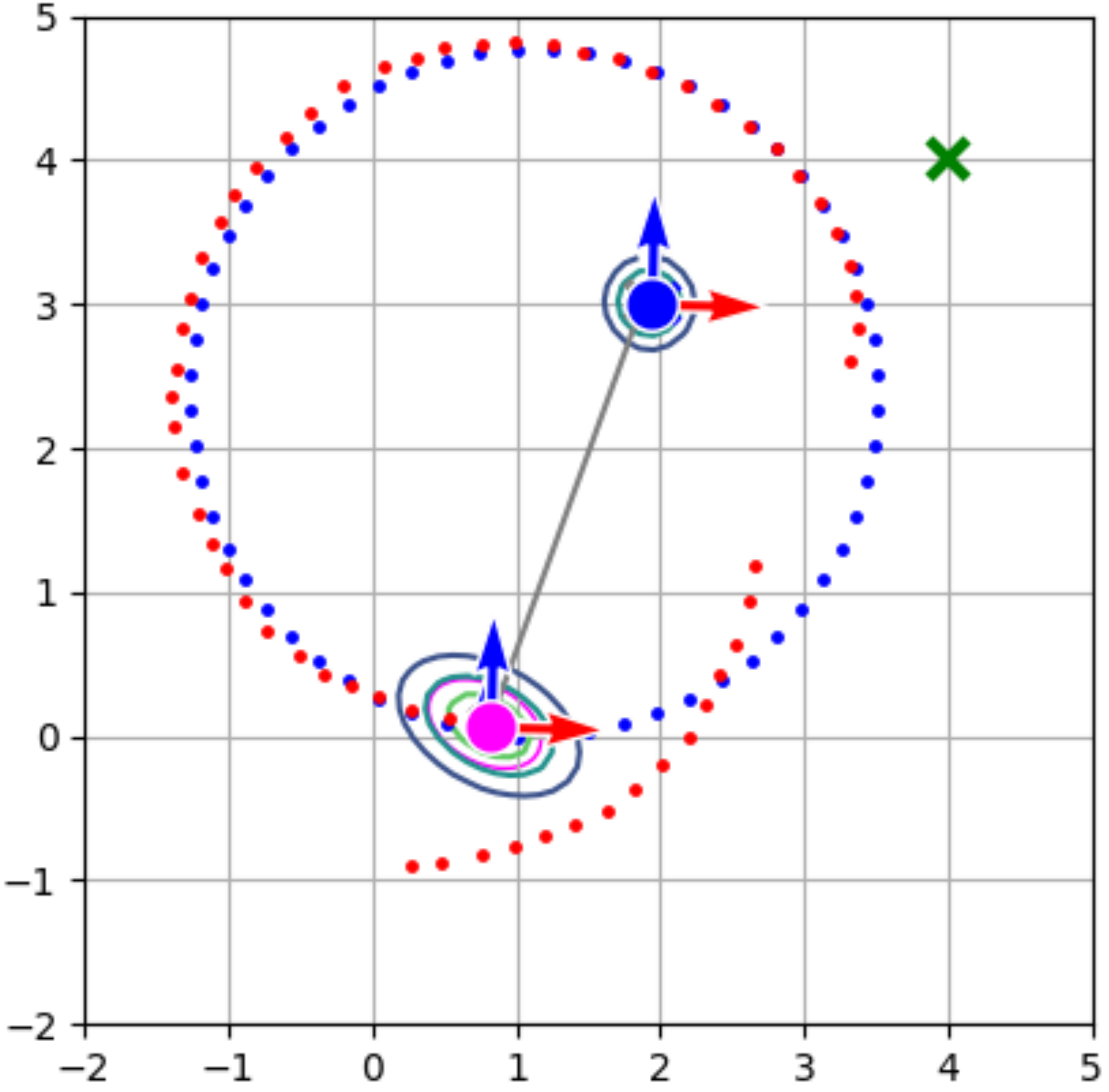
$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

EKF SLAM: abs marker, relative marker, differential-drive motion model




State

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

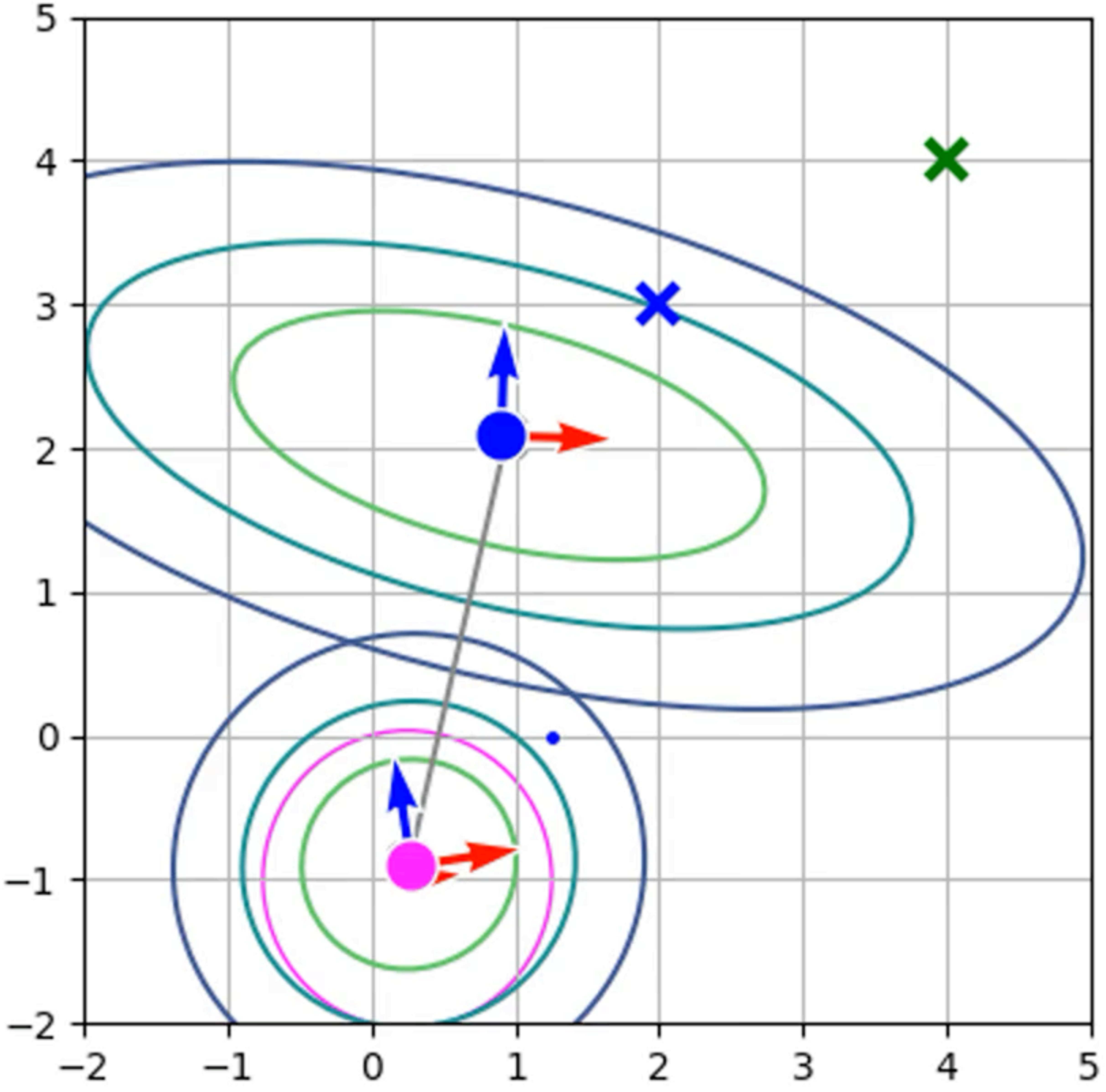
$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

…… estimated trajectory

# EKF SLAM: abs marker, relative marker, differential-drive motion model




**State**

$\mathbf{x}_t$ … estimated robot pose

$\mathbf{m}$ .. estimated rel. marker pose

…… ground truth rel. marker pose

…… ground truth abs. marker pose

$\mathbf{z}_t^{\mathbf{m}_i}$… marker measurements

$\overline{\mathrm{bel}}(\mathbf{x}_t)$ … prediction step

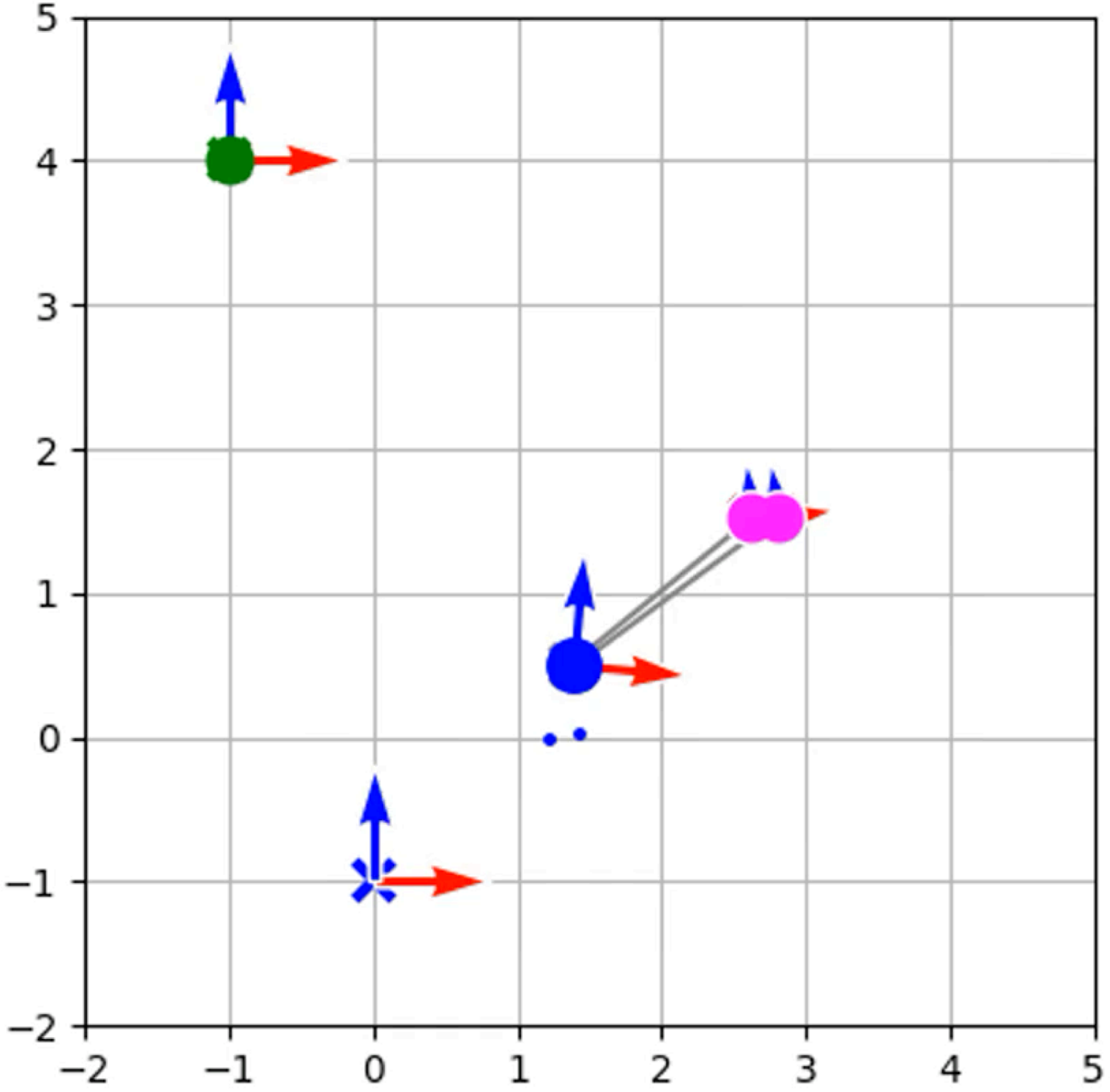
$\mathrm{bel}(\mathbf{x}_t)$ … measurement update step

…… ground truth trajectory

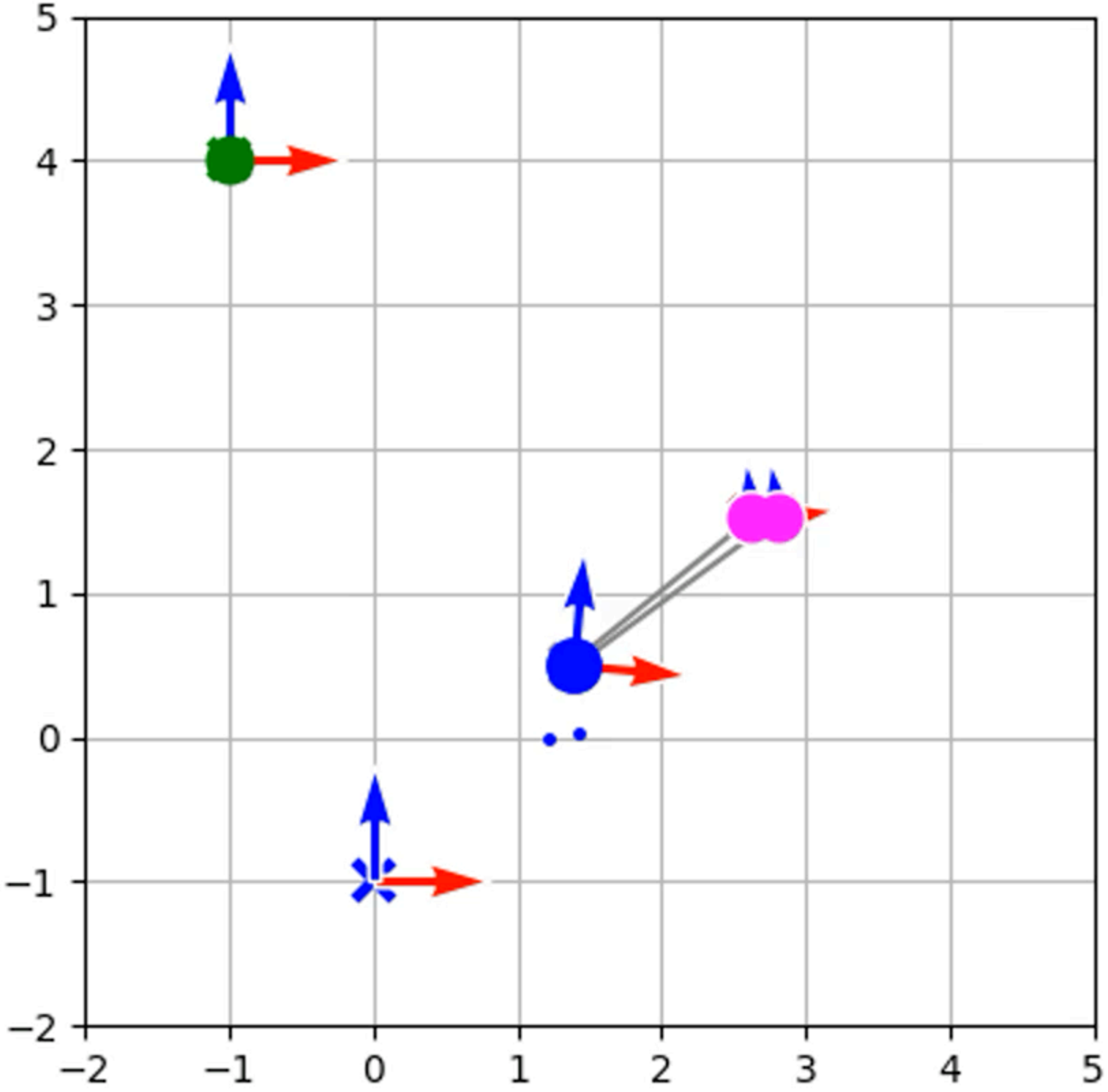
…… estimated trajectory

# SLAM: abs marker, relative marker, differential-drive motion model

## EKF SLAM




## Graph SLAM



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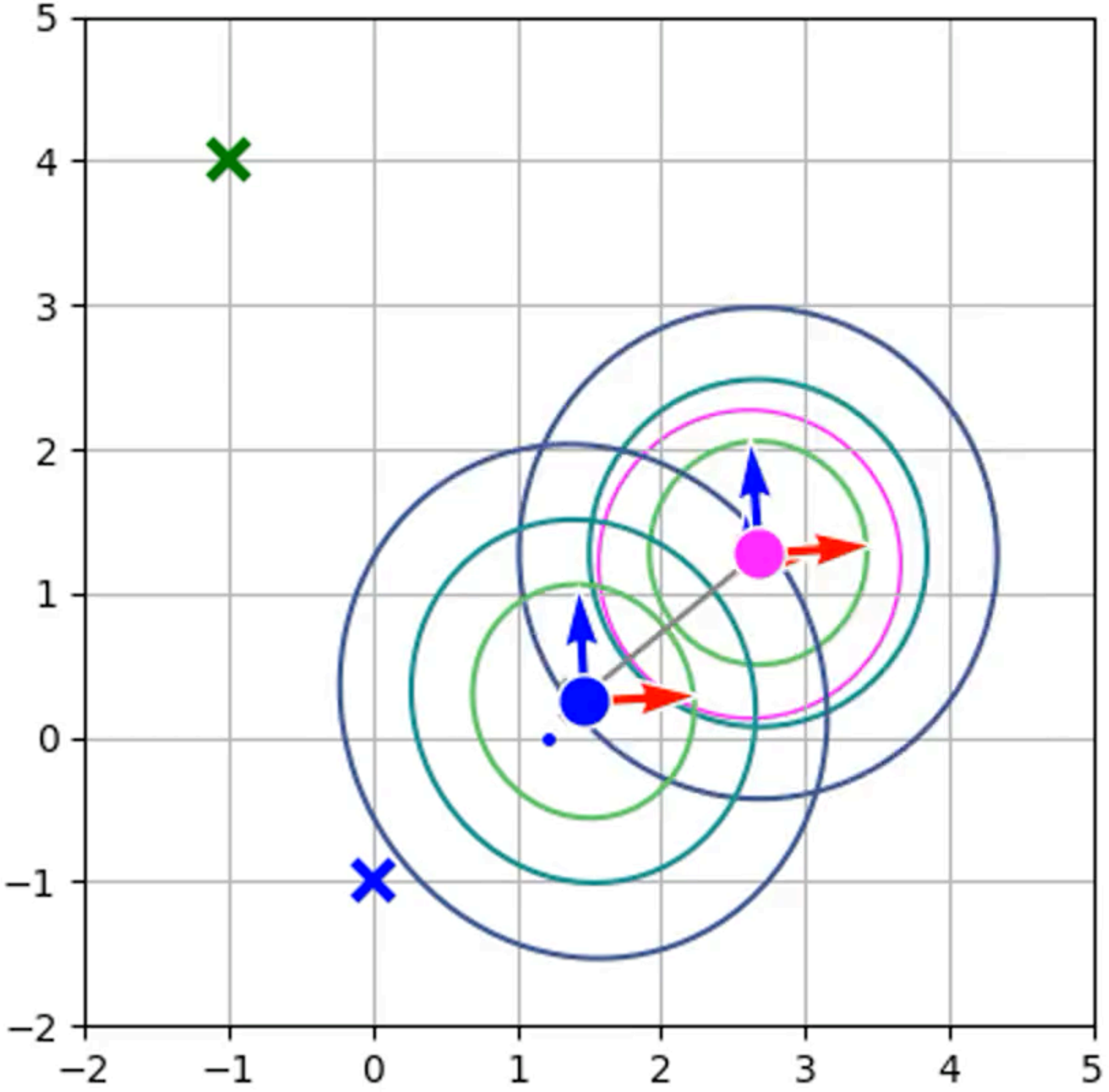
## EKF SLAM

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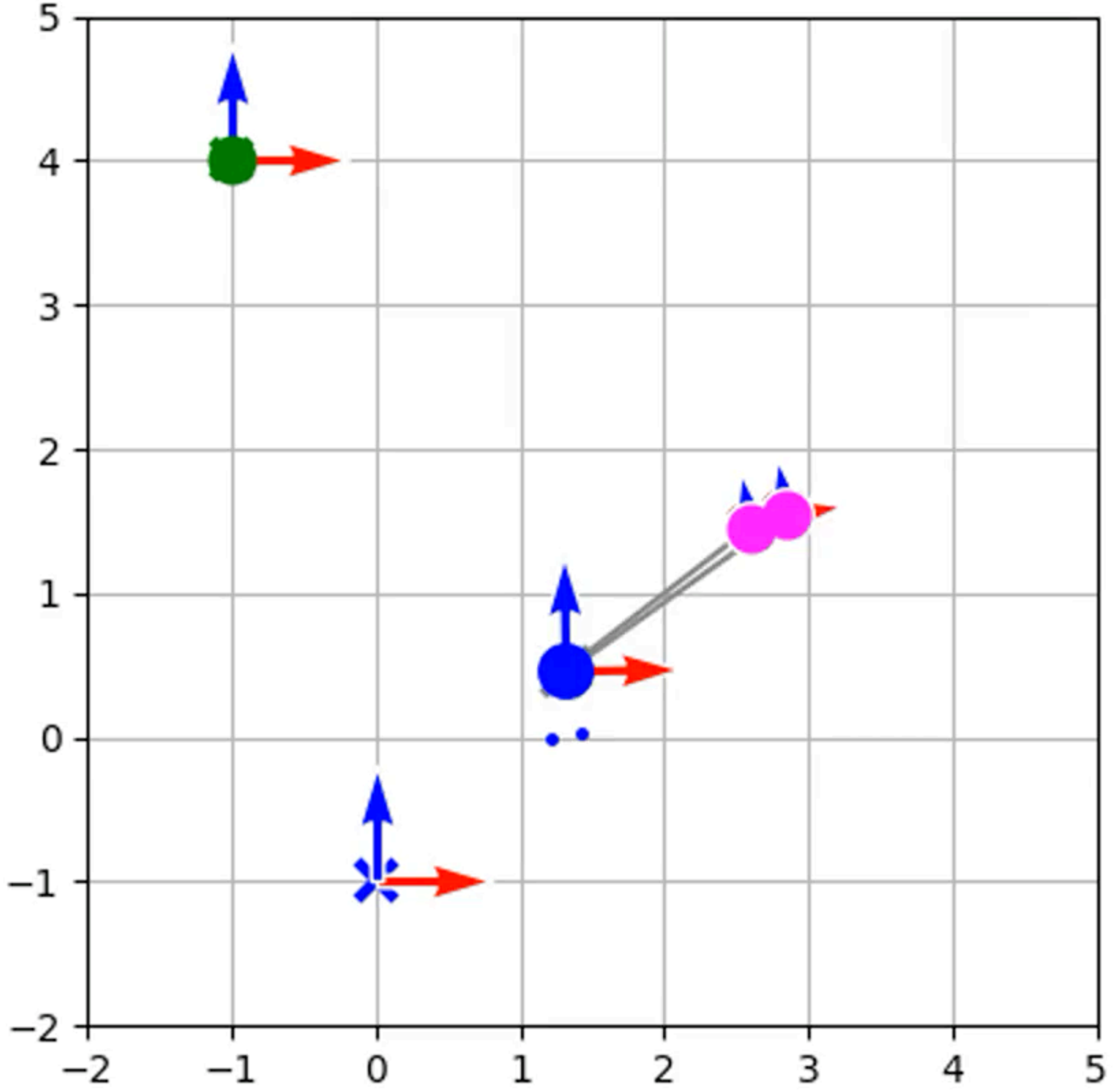

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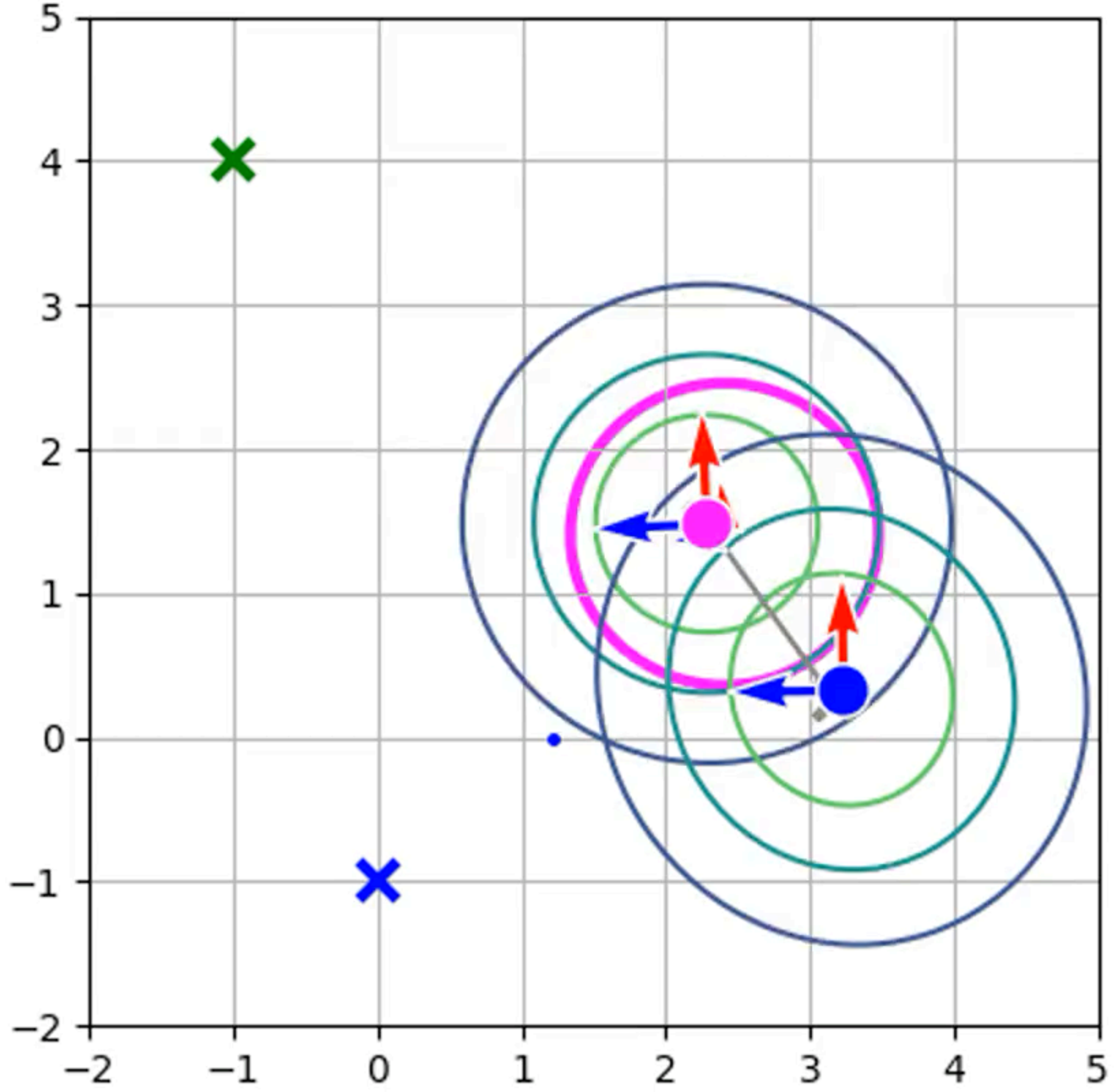
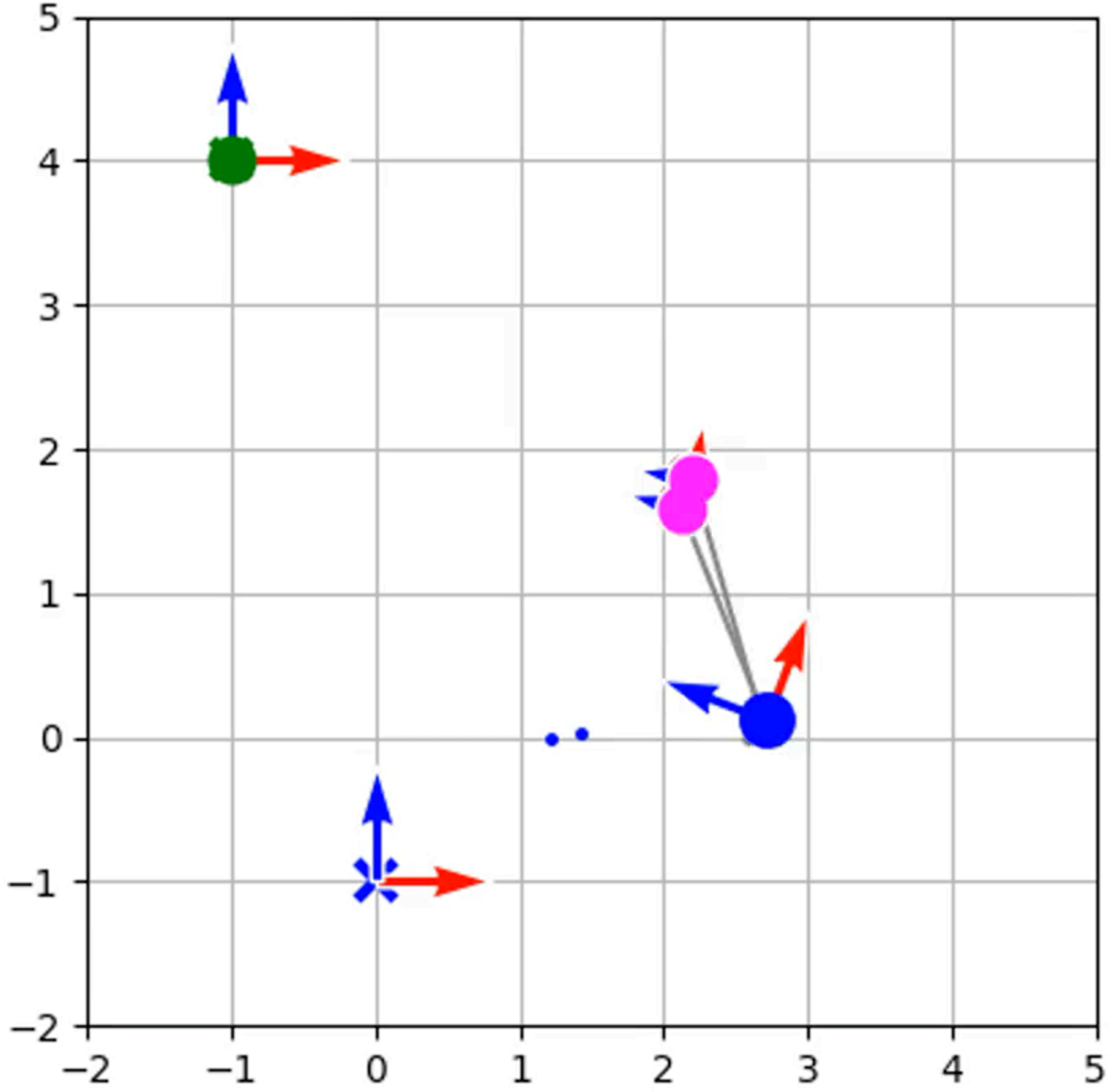
EKF SLAM




Graph SLAM




Wrongly initialized angle

# GraphSLAM vs Extended Kalman Filter (EKF)

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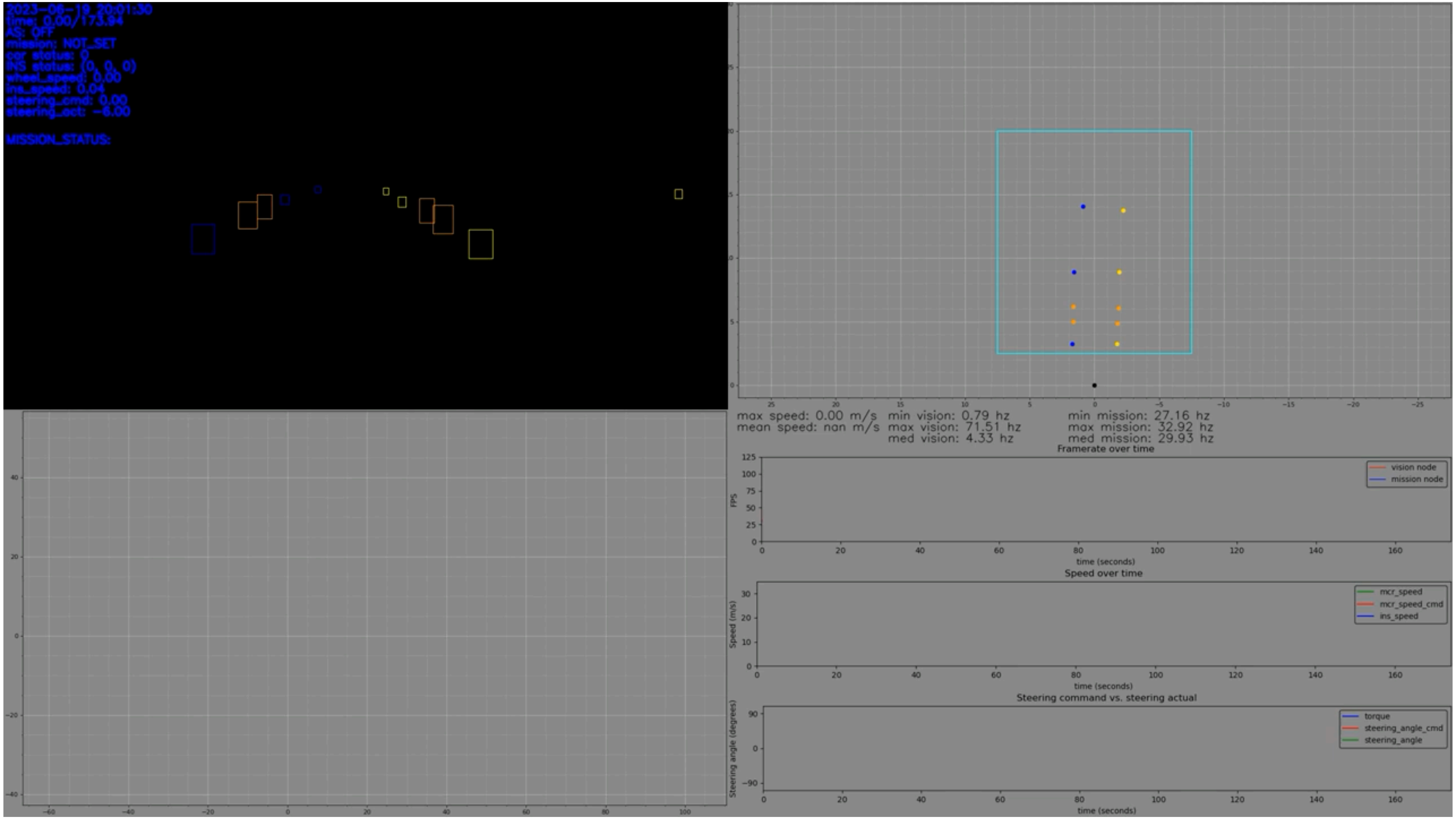
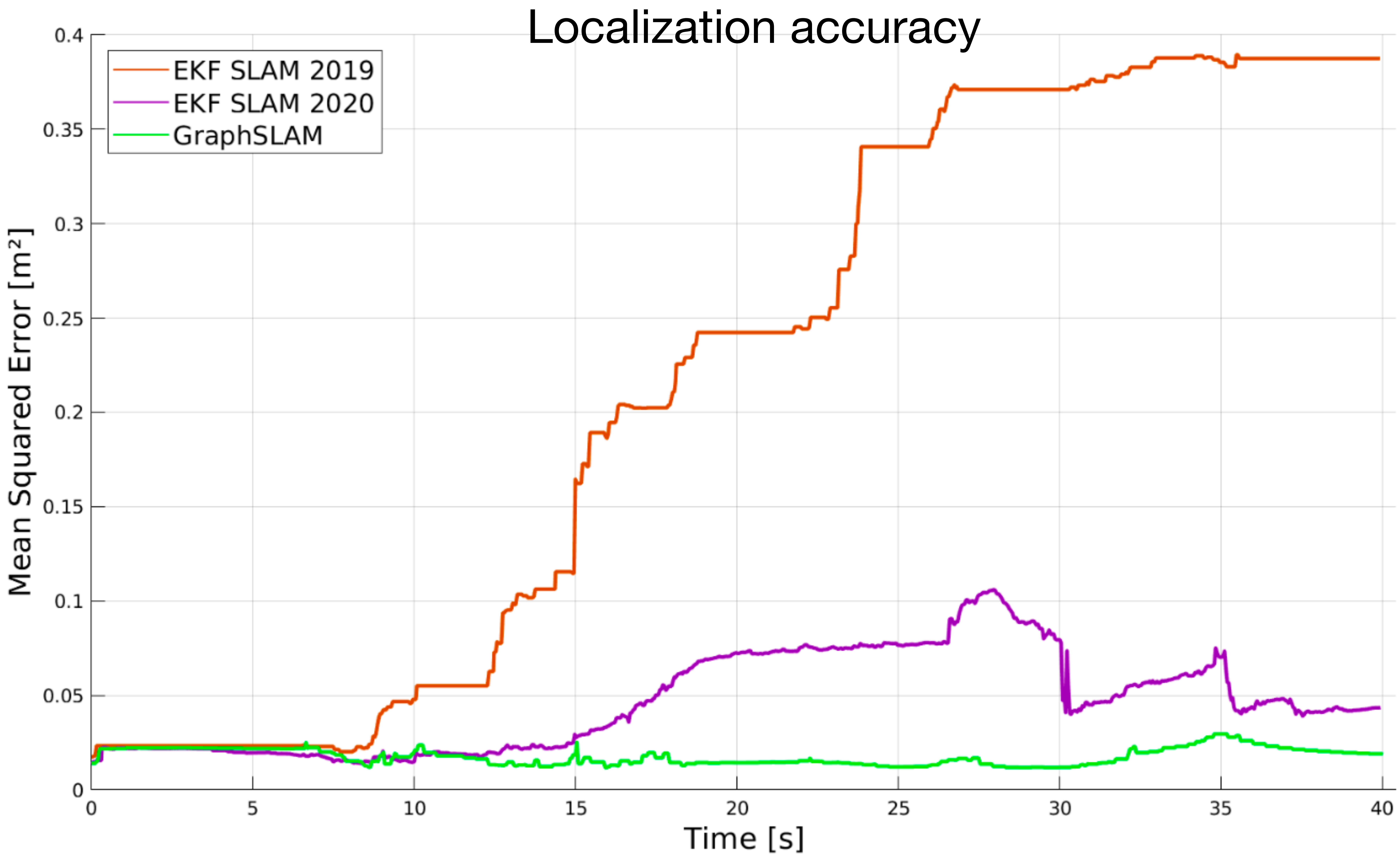


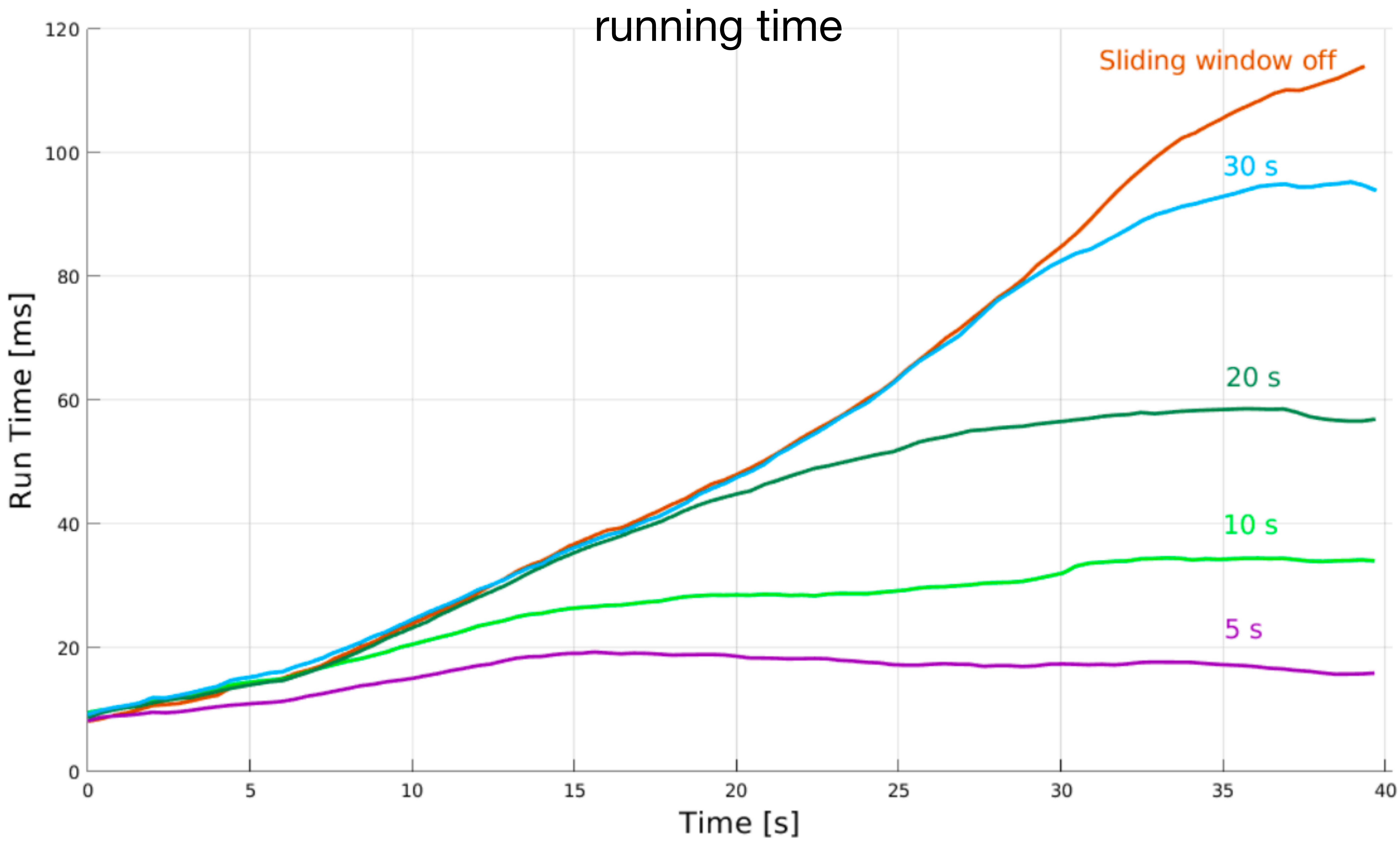

If willing to contribute, ask Roman: sipроman@fel.cvut.cz

# GraphSLAM vs Extended Kalman Filter (EKF)



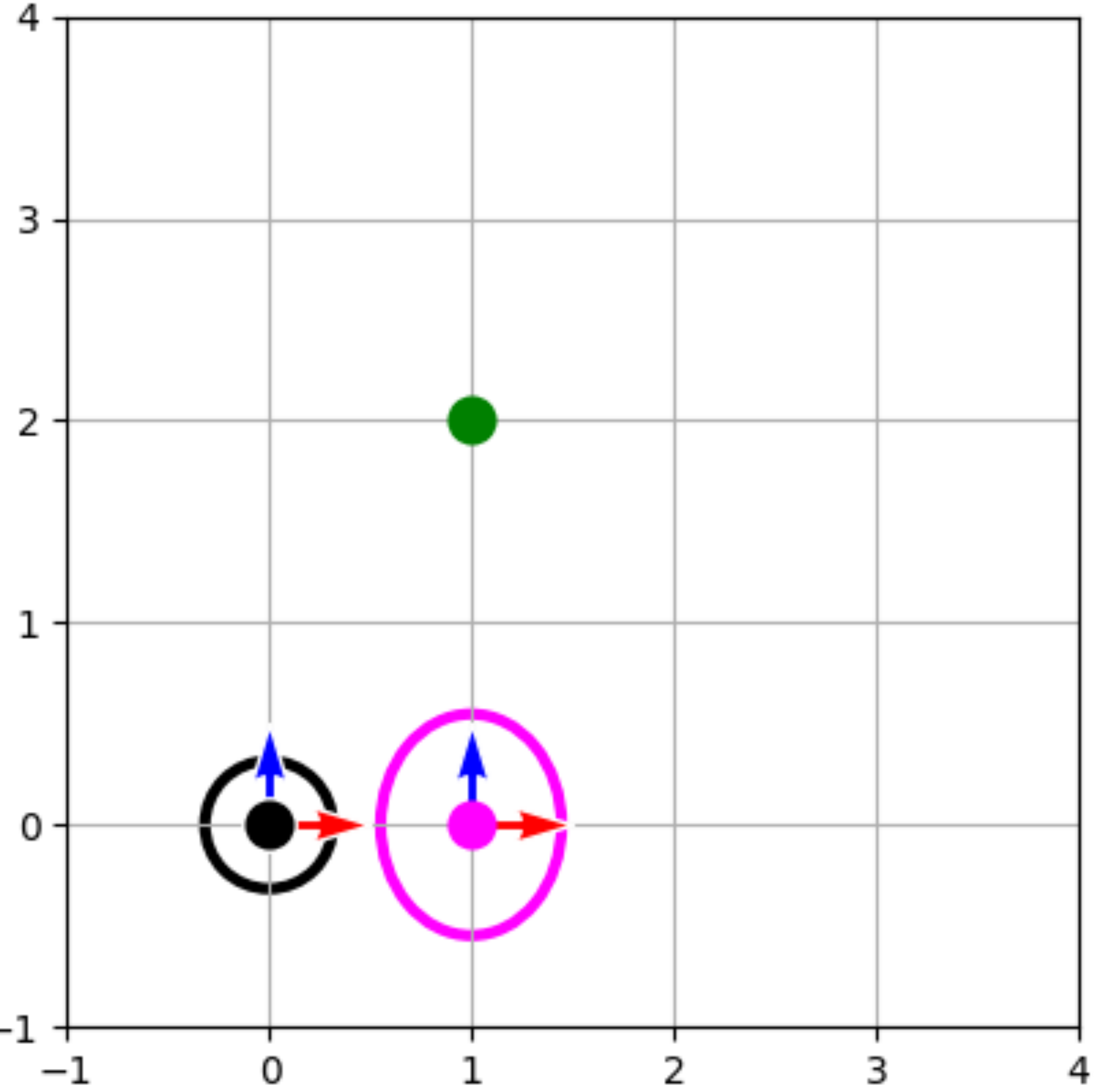

https://www.mrt.kit.edu/z/publ/download/LeLarge_Thesis_SLAM_FormulaStudent.pdf

# GraphSLAM vs Extended Kalman Filter (EKF)



Where does the EKF linearization fail????

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$




**Consistent** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$

$\mathrm{bel}(\mathbf{x}_0)$ … initial belief

$\overline{\mathrm{bel}}(\mathbf{x}_1)$ … prior bel. (prediction step)

$\mathrm{bel}(\mathbf{x}_1)$ … posterior bel. (meas. step)

$\mathbf{m}$ .. absolute marker pose

$\mathbf{z}_1^{\mathbf{m}}$ … marker measurement

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

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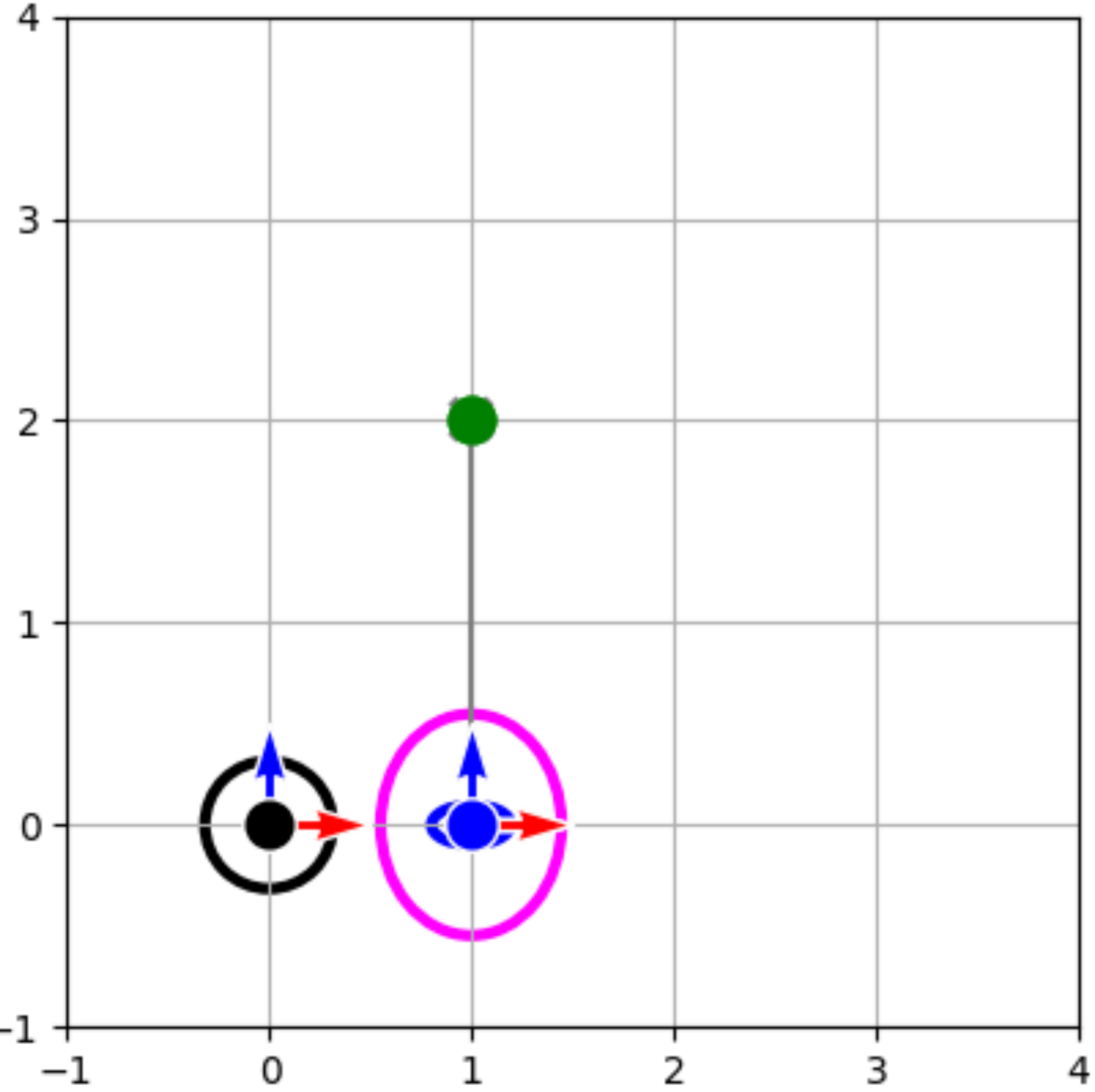


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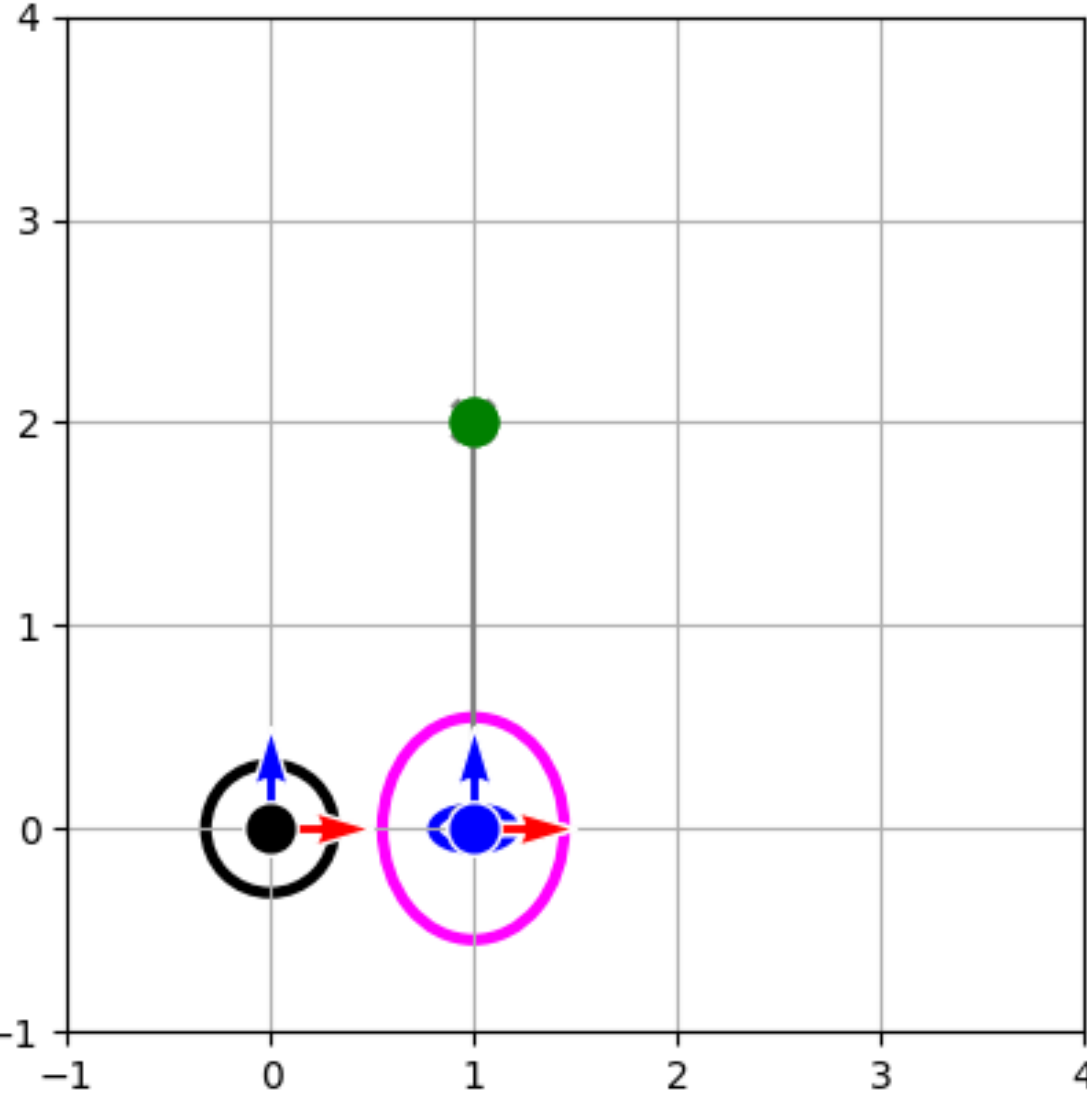


**Consistent** motion and measurement

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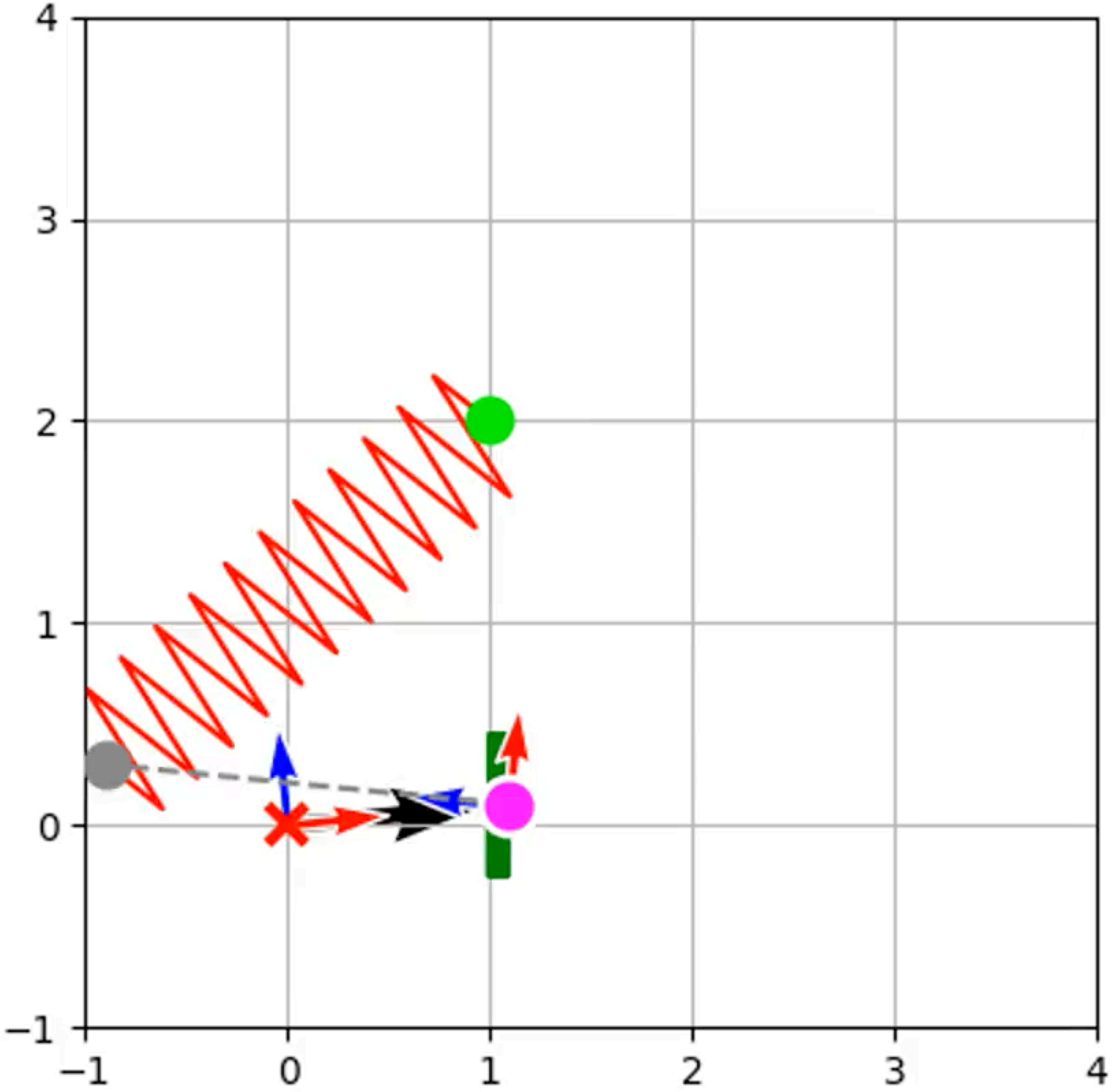
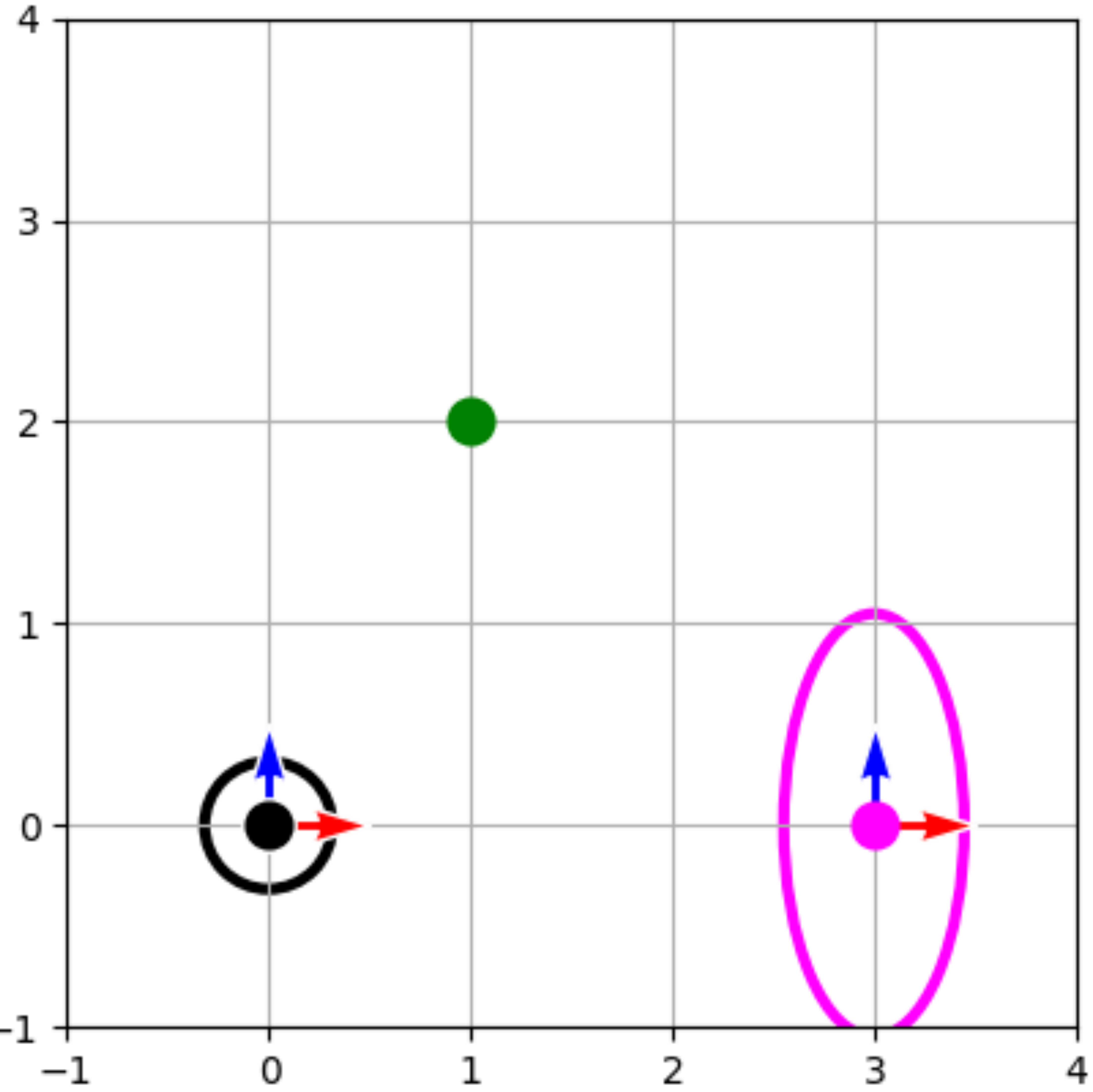
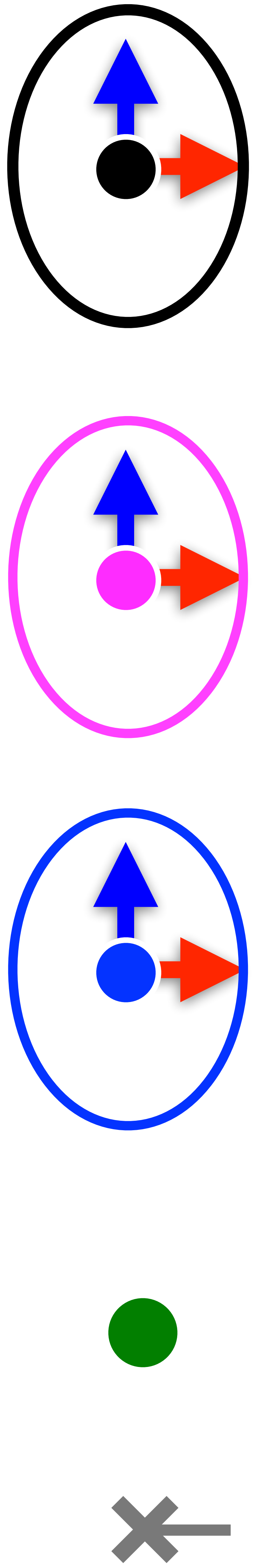


$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

**Inconsistent linear** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$



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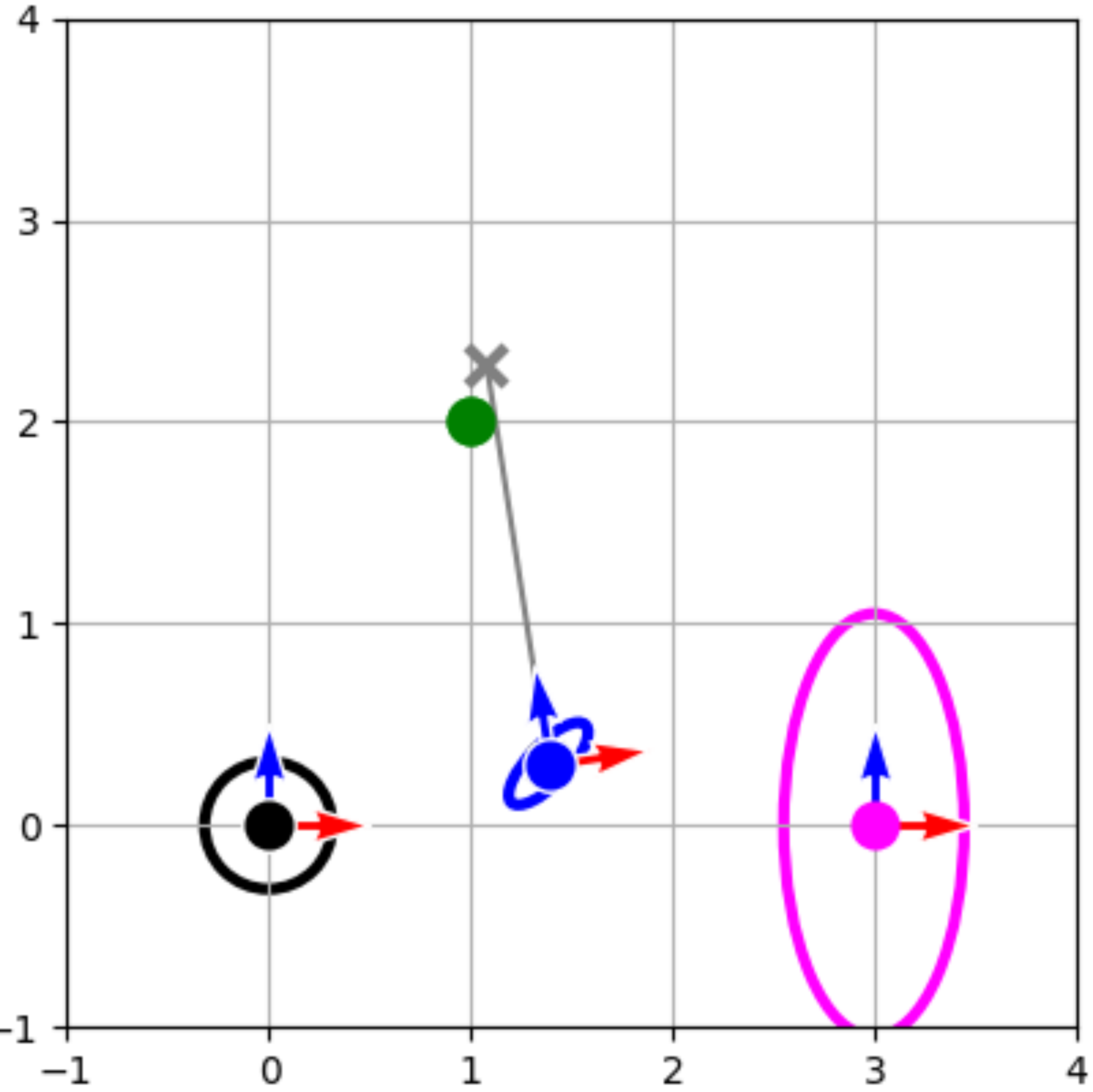
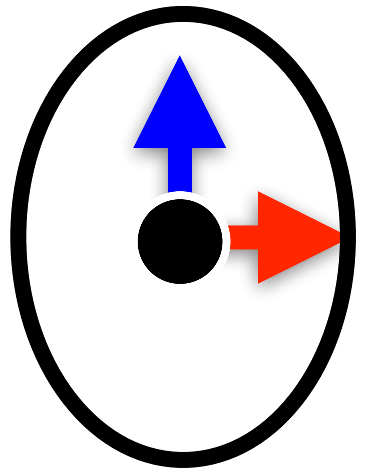
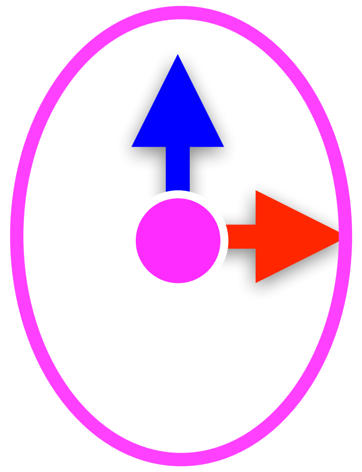
## **Inconsistent linear** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$

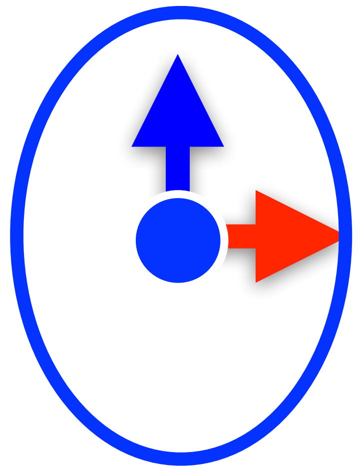


$\mathrm{bel}(\mathbf{x}_0)$ … initial belief



$\overline{\mathrm{bel}}(\mathbf{x}_1)$ … prior bel. (prediction step)



$\mathrm{bel}(\mathbf{x}_1)$ … posterior bel. (meas. step)

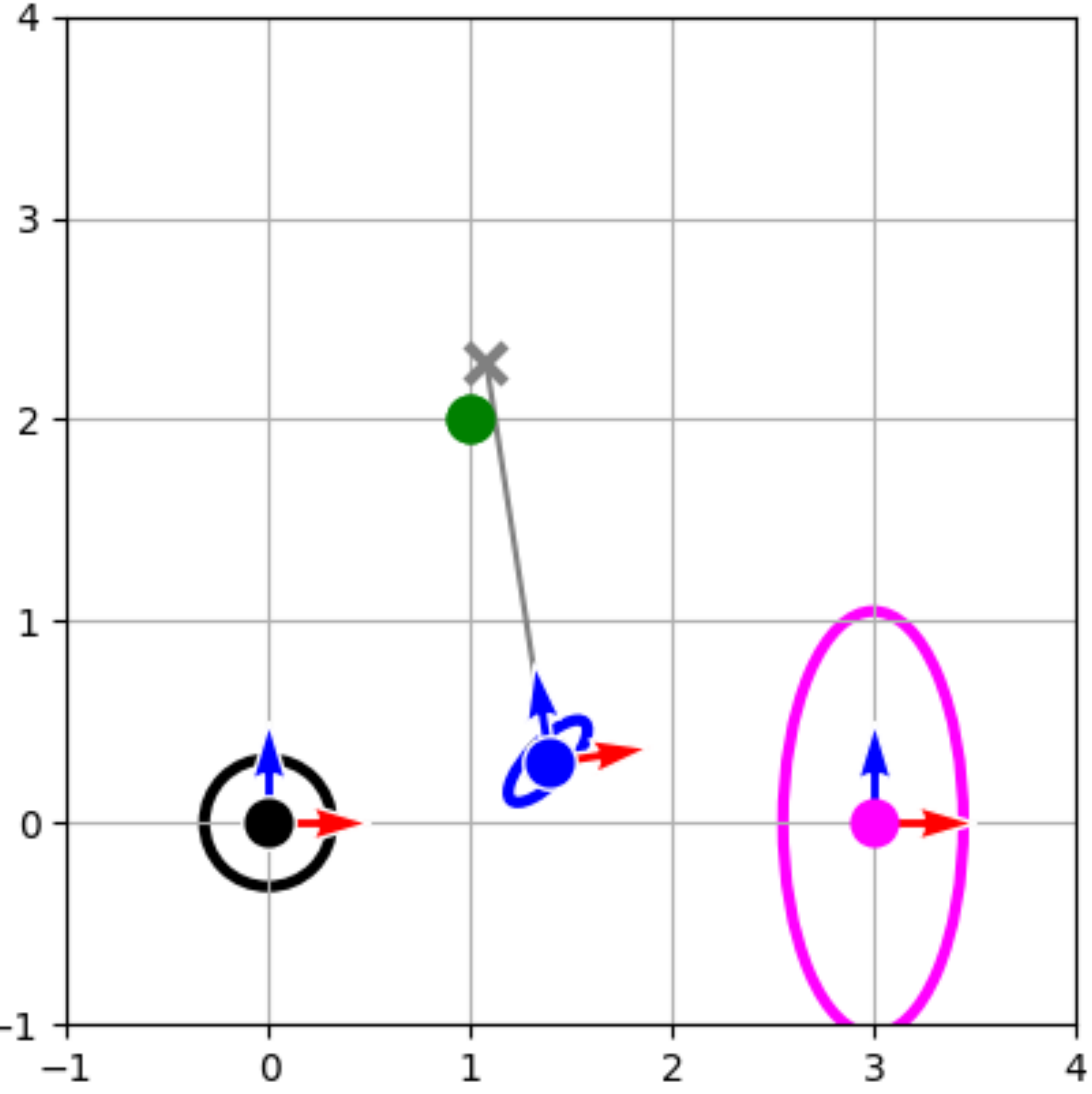
$\mathbf{m}$ .. absolute marker pose

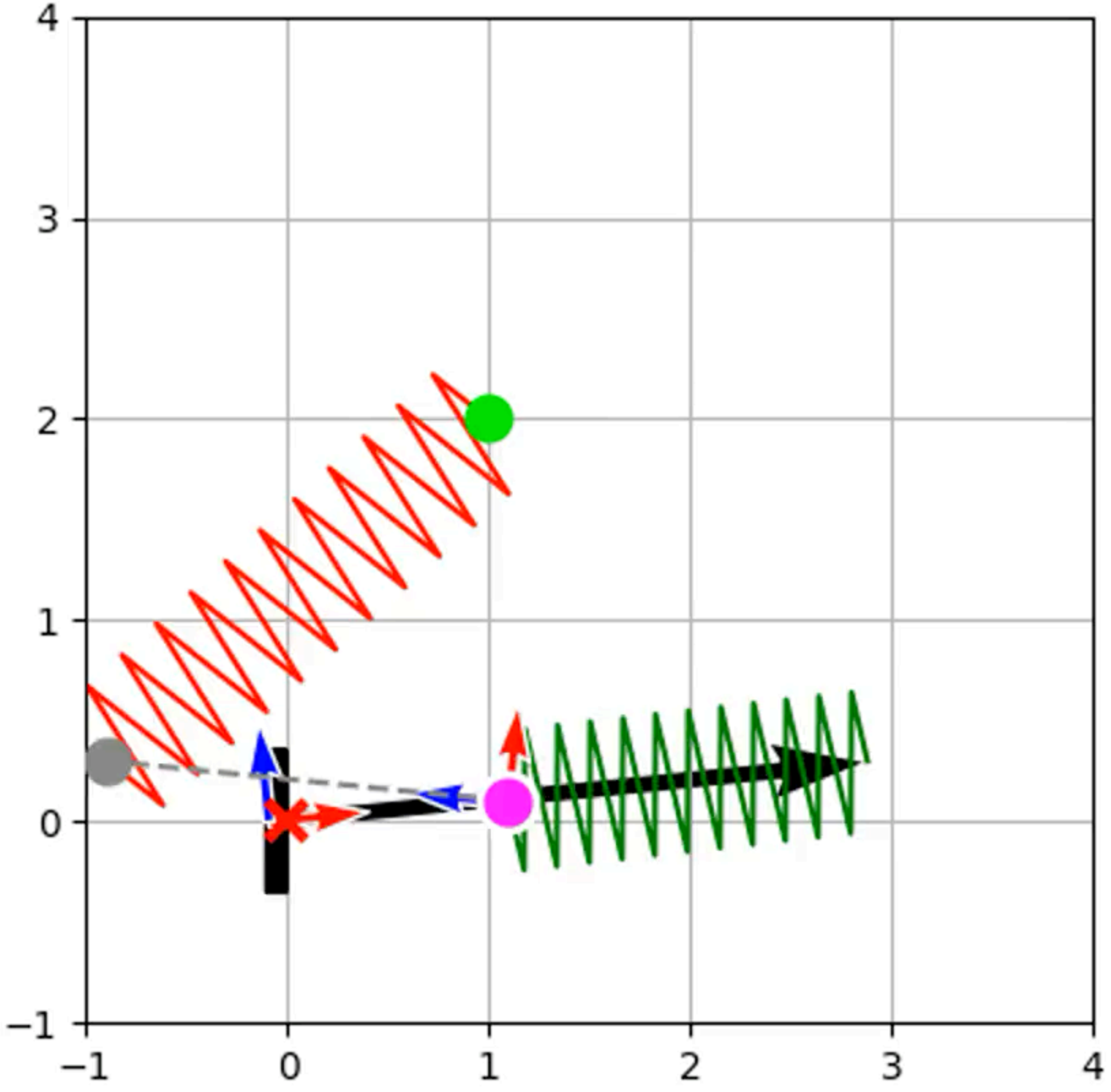
$\mathbf{z}_1^{\mathbf{m}}$ … marker measurement

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

**Inconsistent linear** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$

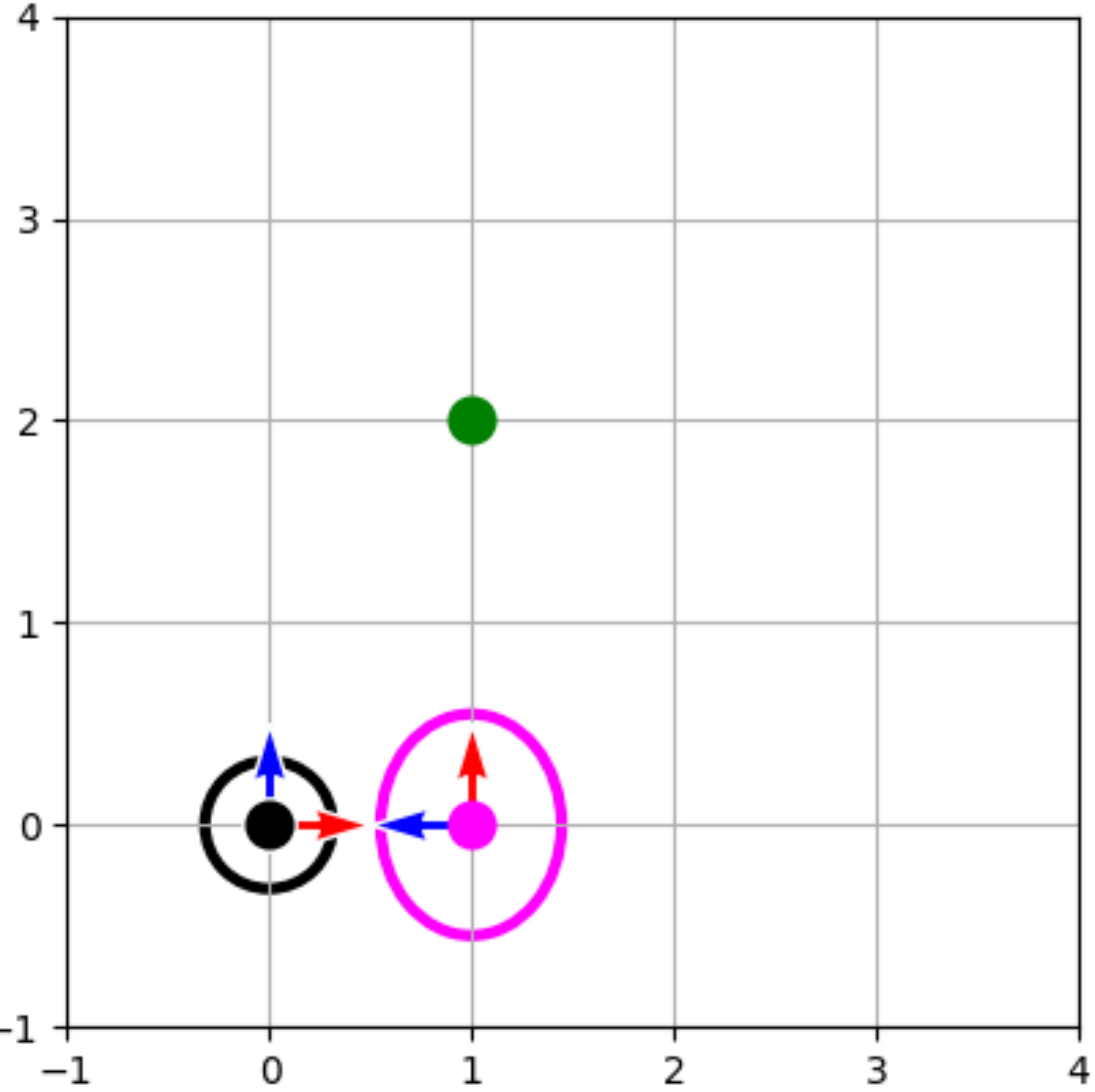


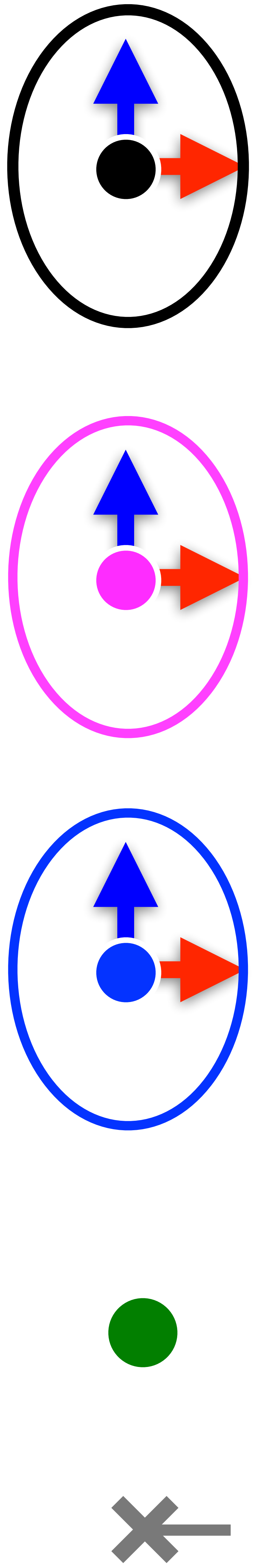



**Inconsistent angular** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$

$$\mathbf{u}_1 = \begin{bmatrix}1\\ \pi/2\end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix}0\\ 2\end{bmatrix}$$



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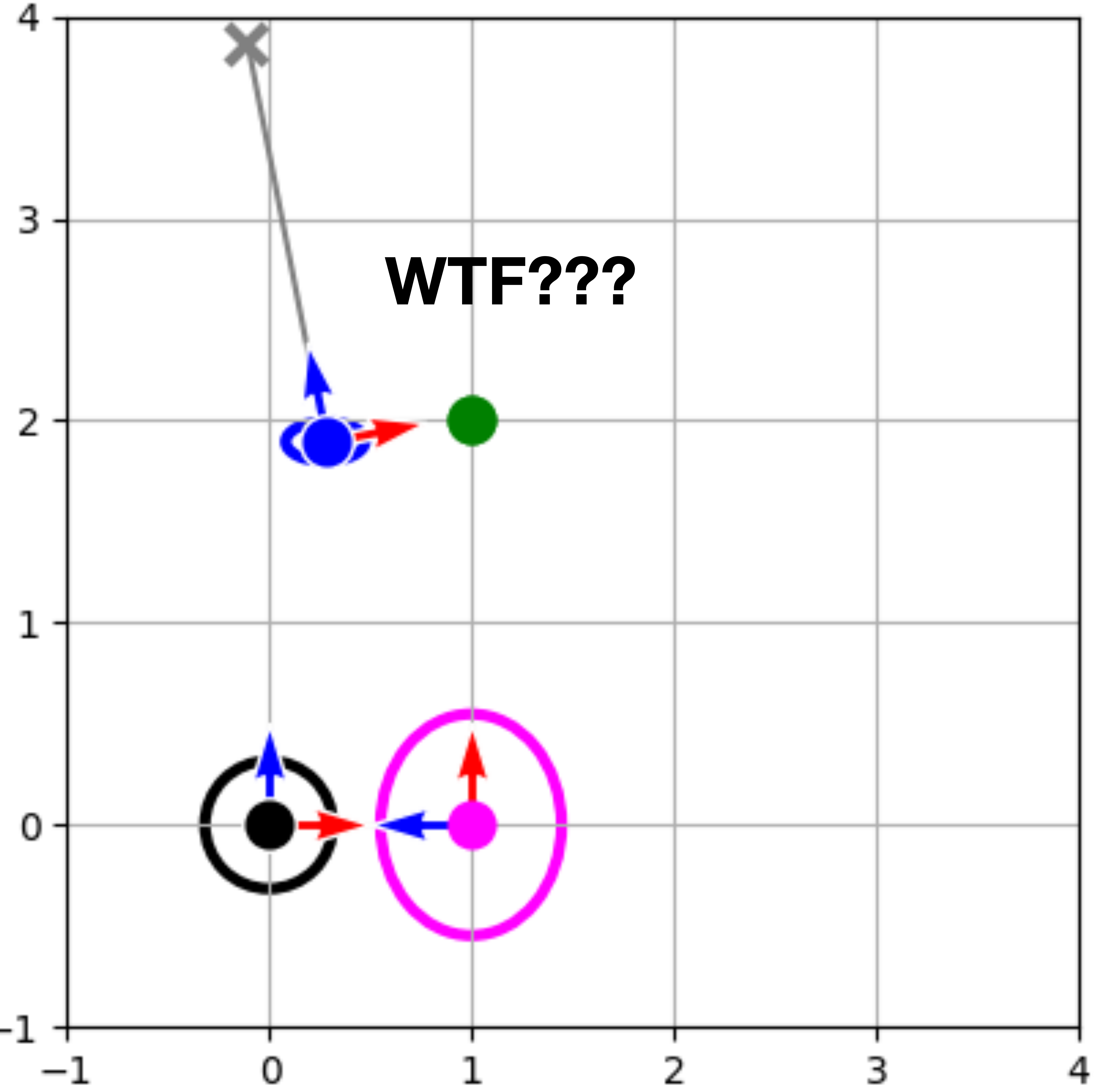
**Inconsistent angular** motion and measurement

$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ \pi/2 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

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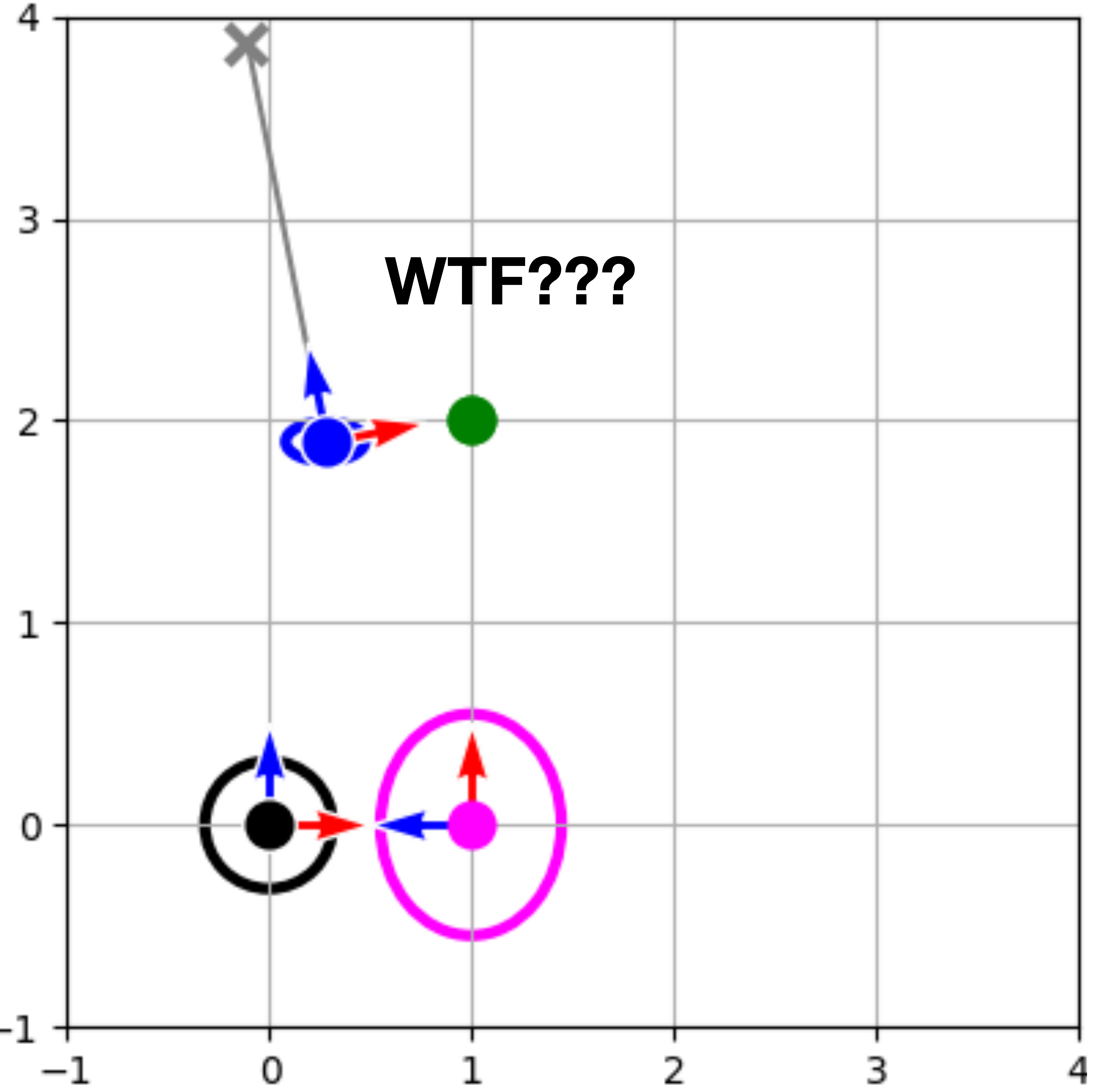
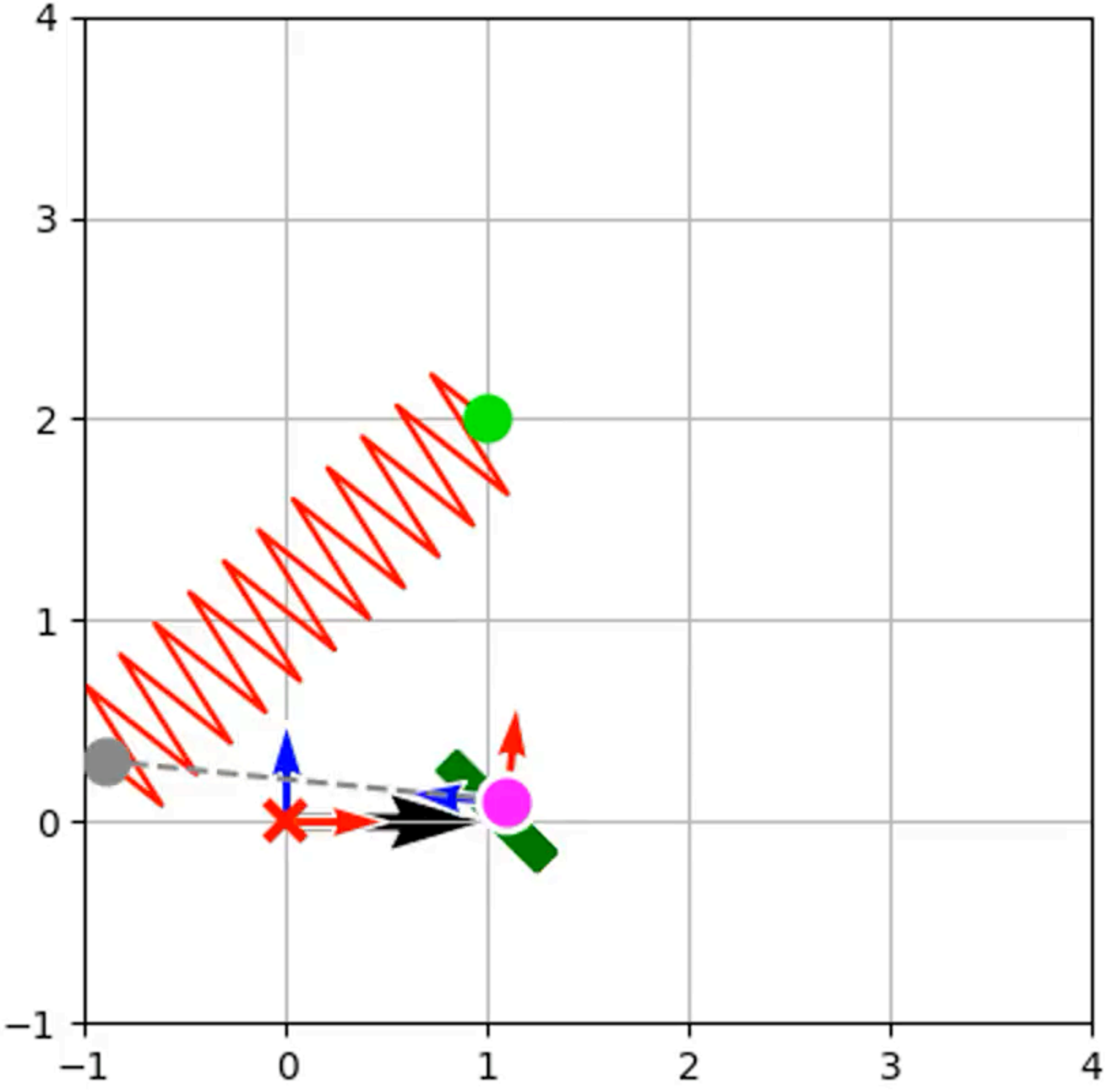
**Inconsistent angular** motion and measurement

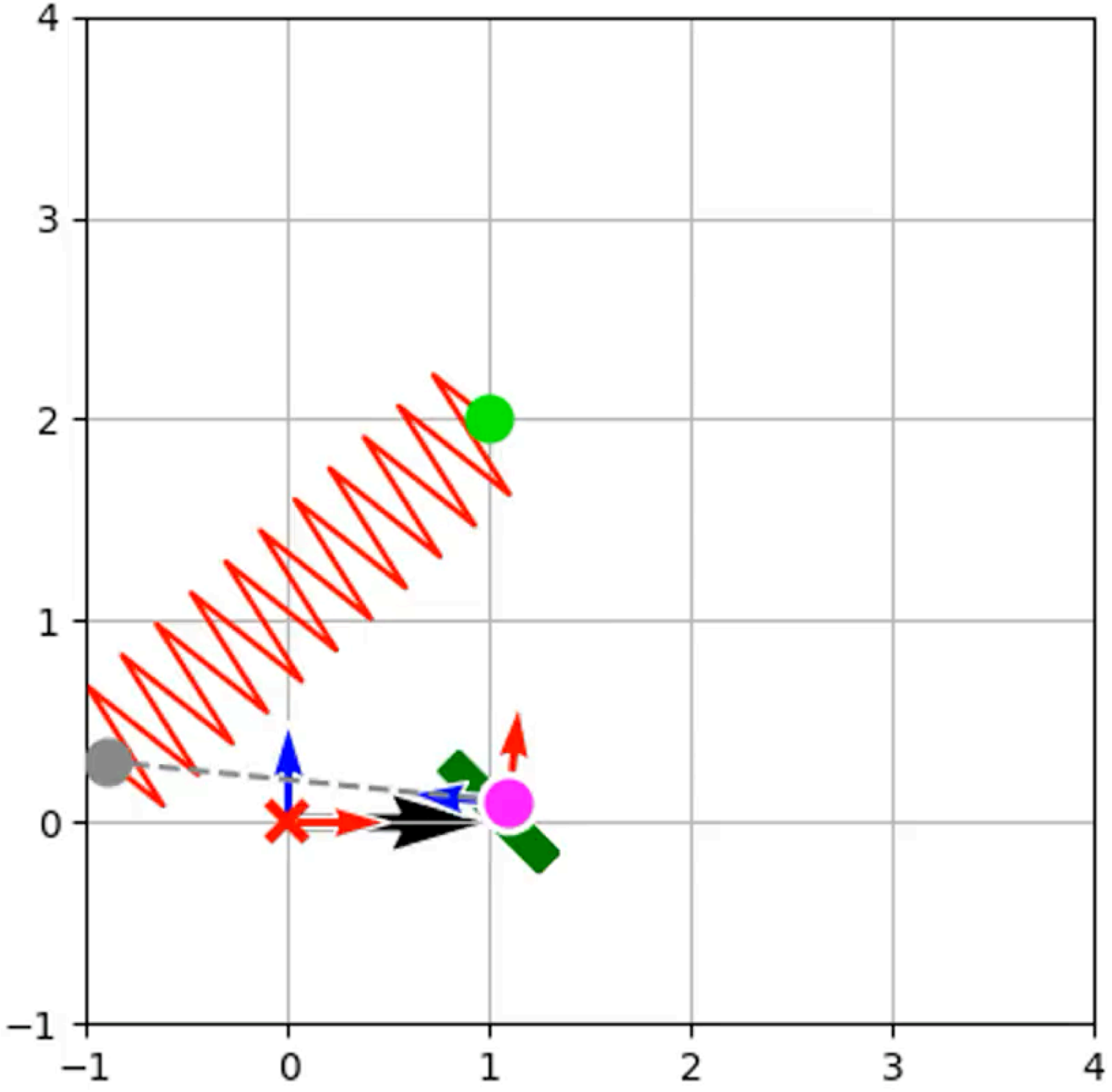
$$\|\mathbf{R}\|_F \ggg \|\mathbf{Q}\|_F$$



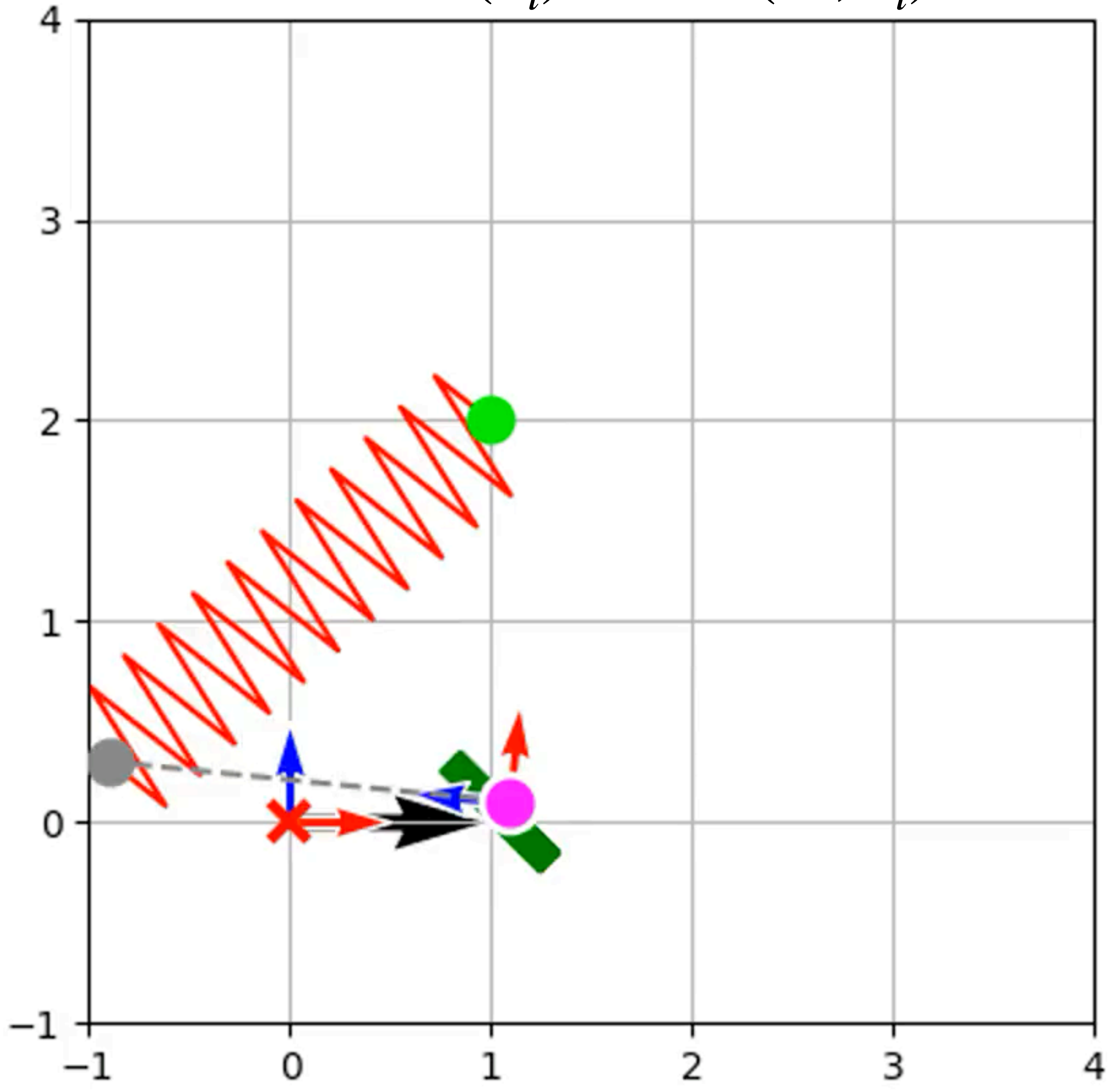
Linearized model

$h^{\mathbf{m}}(\mathbf{x}_t) \approx h(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t)$




Non-linear model
with true rotation

$h^{\mathbf{m}}(\mathbf{x}_t) = \mathrm{w2r}(\mathbf{m}, \mathbf{x}_t)$




**Inconsistent angular** motion and measurement

# Summary

## Motion model in EKF

Transition probability:

$$p\Bigg(\underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{\mathbf{x}_t} \Bigg| \underbrace{\begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix}}_{\mathbf{x}_{t-1}}, \underbrace{\begin{bmatrix} v_t \\ \omega_t \end{bmatrix}}_{\mathbf{u}_t}\Bigg) = \mathcal{N}\Bigg(\mathbf{x}_t;\ \underbrace{\begin{bmatrix} x_{t-1} + \frac{v_t}{\omega_t}\big( + \sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\big) \\ y_{t-1} + \frac{v_t}{\omega_t}\big( - \cos(\theta_{t-1} + \omega_t \Delta t) + \cos(\theta_{t-1})\big) \\ \theta_{t-1} + \omega \Delta t \end{bmatrix}}_{g(\mathbf{u}_t, \mathbf{x}_{t-1})},\ \mathbf{R}_t\Bigg)$$

$\approx \mathcal{N}(\mathbf{z}_t;\ g(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}), \mathbf{R}_t)$ around point $\boldsymbol{\mu}_{t-1} = \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix}$

$$\mathbf{G}_t = \frac{\partial g(\mathbf{u} = \mathbf{u}_t, \mathbf{x} = \theta_{t-1})}{\partial \mathbf{x}} = \begin{array}{c} \begin{array}{ccc} \dfrac{\partial}{\partial x_{t-1}} & \dfrac{\partial}{\partial y_{t-1}} & \dfrac{\partial}{\partial \theta_{t-1}} \end{array} \\ \begin{bmatrix} 1 & 0 & \frac{v_t}{\omega_t}\big( + \cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1})\big) \\ 0 & 1 & \frac{v_t}{\omega_t}\big( + \sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\big) \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

# Summary

## GPS measurement model in EKF

Measurement probability:

$$p\Bigg( \underbrace{\begin{bmatrix} z_t^x \\ z_t^y \end{bmatrix}}_{\mathbf{z}_t^{\text{gps}}} \;\Bigg|\; \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{\mathbf{x}_t} \Bigg) = \mathcal{N}\Bigg( \mathbf{z}_t^{\text{gps}}; \; \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{h^{\text{gps}}(\mathbf{x}_t)} \;, \; Q_t^{\text{gps}} \Bigg)$$

$$\mathbf{H}_t = \mathbf{C}_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

# Summary

## Marker measurement model in EKF localization


$$p\left(\underbrace{\begin{bmatrix} z_t^x \\ z_t^y \\ z_t^\theta \end{bmatrix}}_{\mathbf{z}_t^{\mathbf{m}}} \mid \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{\mathbf{x}_t}\right) = \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\mathrm{w2r}(\mathbf{m}, \mathbf{x_t})}_{h^{\mathbf{m}}(\mathbf{x}_t)}\ ,\ Q_t^{\mathbf{m}}\right)$$

$$= \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\begin{bmatrix} +\cos\theta_t \cdot (m^x - x_t) + \sin\theta_t \cdot (m^y - y_t) \\ -\sin\theta_t \cdot (m^x - x_t) + \cos\theta_t \cdot (m^y - y_t) \\ m^\theta - \theta_t \end{bmatrix}}_{h^{\mathbf{m}}(\mathbf{x}_t)}\ ,\ Q_t^{\mathbf{m}}\right)$$

$$\approx \mathcal{N}(\mathbf{z}_t;\ h^{\mathbf{m}}(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t), \mathbf{Q}_t)$$

around point $\bar{\boldsymbol{\mu}}_t = \begin{bmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \end{bmatrix}$

$$\mathbf{H}_t = \begin{array}{c} \begin{array}{ccc} \dfrac{\partial}{\partial x_t} & \dfrac{\partial}{\partial y_t} & \dfrac{\partial}{\partial \theta_t} \end{array} \\ \begin{bmatrix} -\cos\bar{\theta}_t & -\sin\bar{\theta}_t & -\sin\bar{\theta}_t \cdot (m^x - \bar{x}_t) + \cos\bar{\theta}_t \cdot (m^y - \bar{y}_t) \\ +\sin\bar{\theta}_t & -\cos\bar{\theta}_t & -\cos\bar{\theta}_t \cdot (m^x - \bar{x}_t) - \sin\bar{\theta}_t \cdot (m^y - \bar{y}_t) \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

# Marker measurement model in EKF SLAM


$$p\left(\underbrace{\begin{bmatrix} z_t^x \\ z_t^y \\ z_t^\theta \end{bmatrix}}_{\mathbf{z}_t^{\mathbf{m}}} \,\middle|\, \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ m^x \\ m^y \\ m^\theta \end{bmatrix}}_{\mathbf{x}_t}\right) = \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\mathrm{w2r}(\mathbf{m}, \mathbf{x_t})}_{h^{\mathbf{m}}(\mathbf{x}_t)}\ ,\ Q_t^{\mathbf{m}}\right)$$

$$= \mathcal{N}\left(\mathbf{z}_t^{\mathbf{m}};\ \underbrace{\begin{bmatrix} +\cos\theta_t \cdot (m^x - x_t) + \sin\theta_t \cdot (m^y - y_t) \\ -\sin\theta_t \cdot (m^x - x_t) + \cos\theta_t \cdot (m^y - y_t) \\ m^\theta - \theta_t \end{bmatrix}}_{h^{\mathbf{m}}(\mathbf{x}_t)}\ ,\ Q_t^{\mathbf{m}}\right)$$

$$\approx \mathcal{N}(\mathbf{z}_t;\ h(\bar{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

around point $\bar{\boldsymbol{\mu}}_t = \begin{bmatrix} \bar{x}_t \\ \bar{y}_t \\ \bar{\theta}_t \\ \bar{m}_t^x \\ \bar{m}_t^y \\ \bar{m}_t^\theta \end{bmatrix}$

$$\mathbf{H}_t = \begin{array}{c} \begin{array}{cccccc} \dfrac{\partial}{\partial x_t} & \dfrac{\partial}{\partial y_t} & \dfrac{\partial}{\partial \theta_t} & \dfrac{\partial}{\partial m^x} & \dfrac{\partial}{\partial m^y} & \dfrac{\partial}{\partial m^\theta} \end{array} \\ \begin{bmatrix} -\cos\bar{\theta}_t & -\sin\bar{\theta}_t & -\sin\bar{\theta}_t \cdot (\bar{m}^x - \bar{x}_t) + \cos\bar{\theta}_t \cdot (\bar{m}^y - \bar{y}_t) & \cos\bar{\theta}_t & \sin\bar{\theta}_t & 0 \\ +\sin\bar{\theta}_t & -\cos\bar{\theta}_t & -\cos\bar{\theta}_t \cdot (\bar{m}^x - \bar{x}_t) - \sin\bar{\theta}_t \cdot (\bar{m}^y - \bar{y}_t) & -\sin\bar{\theta}_t & +\cos\bar{\theta}_t & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

# Extended Kalman Filter

Non-linear system with Gaussian noise:

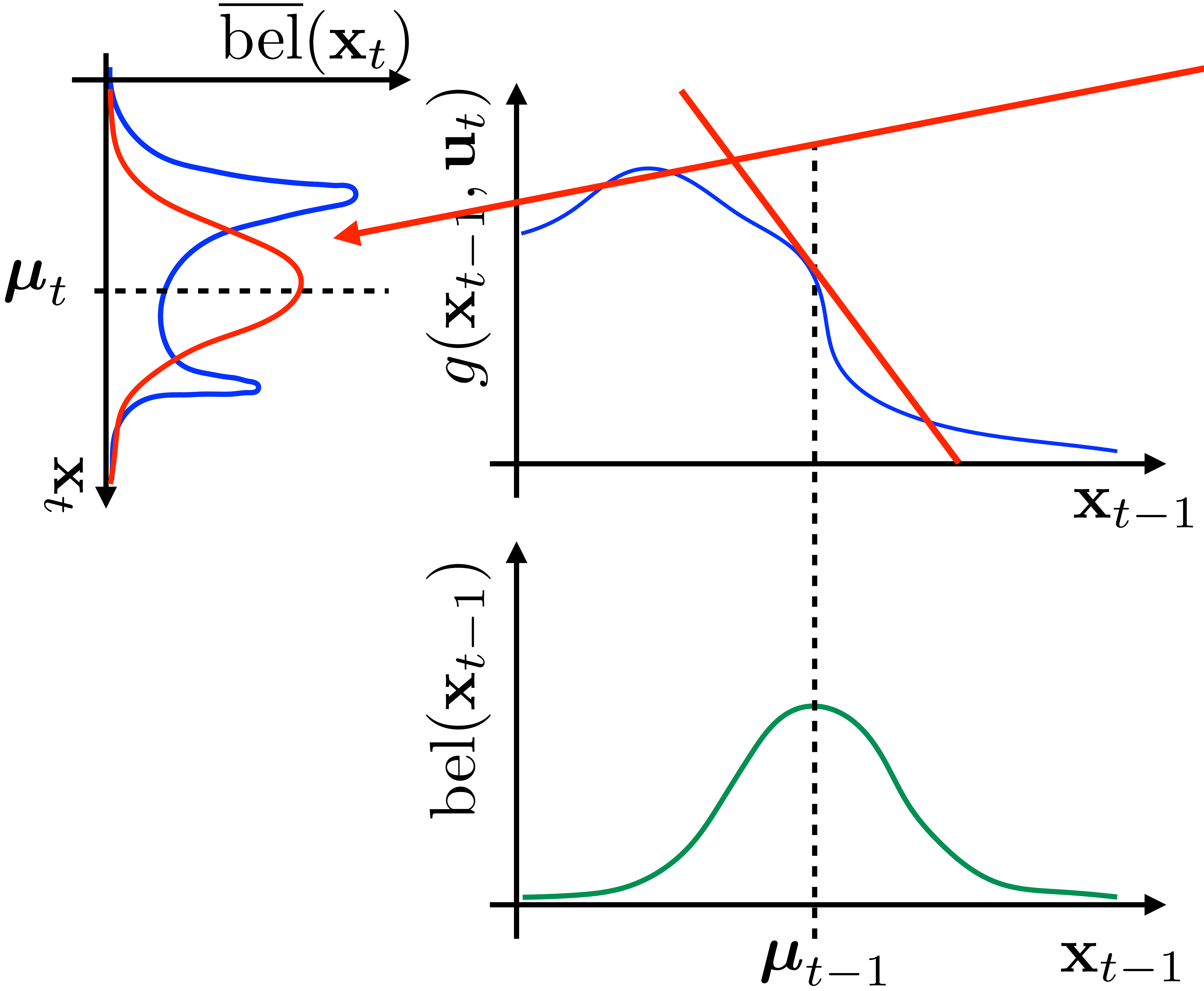
$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{x}_{t-1},\mathbf{u}_t),\ \mathbf{R}_t)$$

$$p(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}_{\mathbf{z}_t}(h(\mathbf{x}_t),\ \mathbf{Q}_t)$$

Linearized system with Gaussian noise:

$$\approx \mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1}-\boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

$$\approx \mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t) + \mathbf{H}_t(\mathbf{x}_t-\overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$




Is it a big issue?

# Extended Kalman Filter

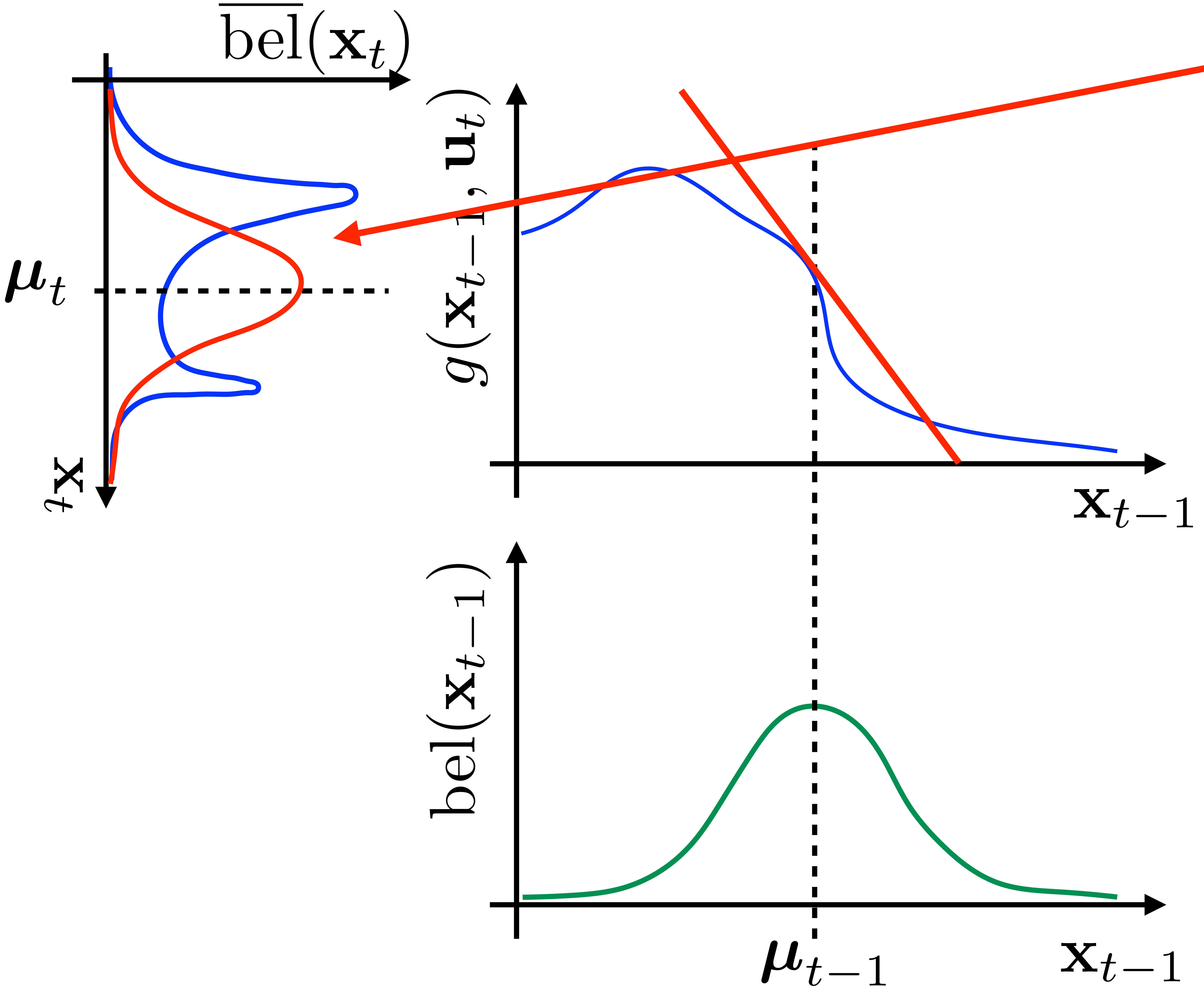
**Non-linear system with Gaussian noise:**

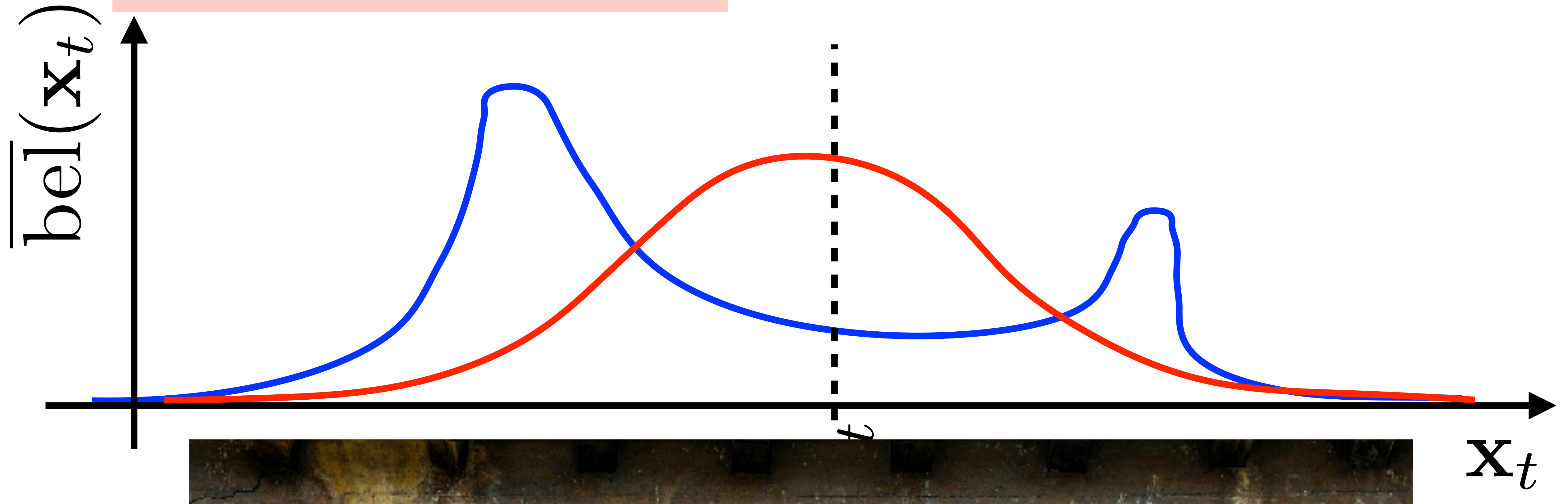
**Linearized system with Gaussian noise:**

$$p(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t)=\mathcal{N}_{\mathbf{x}_t}(g(\mathbf{x}_{t-1},\mathbf{u}_t),\ \mathbf{R}_t)\ \approx\mathcal{N}_{\mathbf{x}_t}(g(\mathbf{u}_t,\boldsymbol{\mu}_{t-1})+\mathbf{G}_t(\mathbf{x}_{t-1}-\boldsymbol{\mu}_{t-1}),\ \mathbf{R}_t)$$

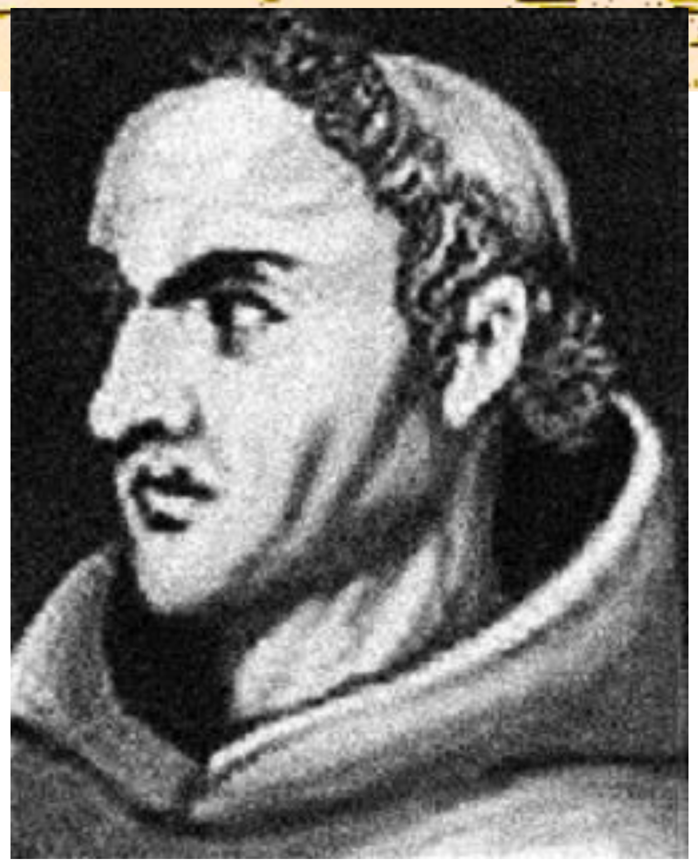
$$p(\mathbf{z}_t|\mathbf{x}_t)=\mathcal{N}_{\mathbf{z}_t}(h(\mathbf{x}_t),\ \mathbf{Q}_t)\ \approx\mathcal{N}_{\mathbf{z}_t}(h(\overline{\boldsymbol{\mu}}_t)+\mathbf{H}_t(\mathbf{x}_t-\overline{\boldsymbol{\mu}}_t),\ \mathbf{Q}_t)$$

Is it a big issue?



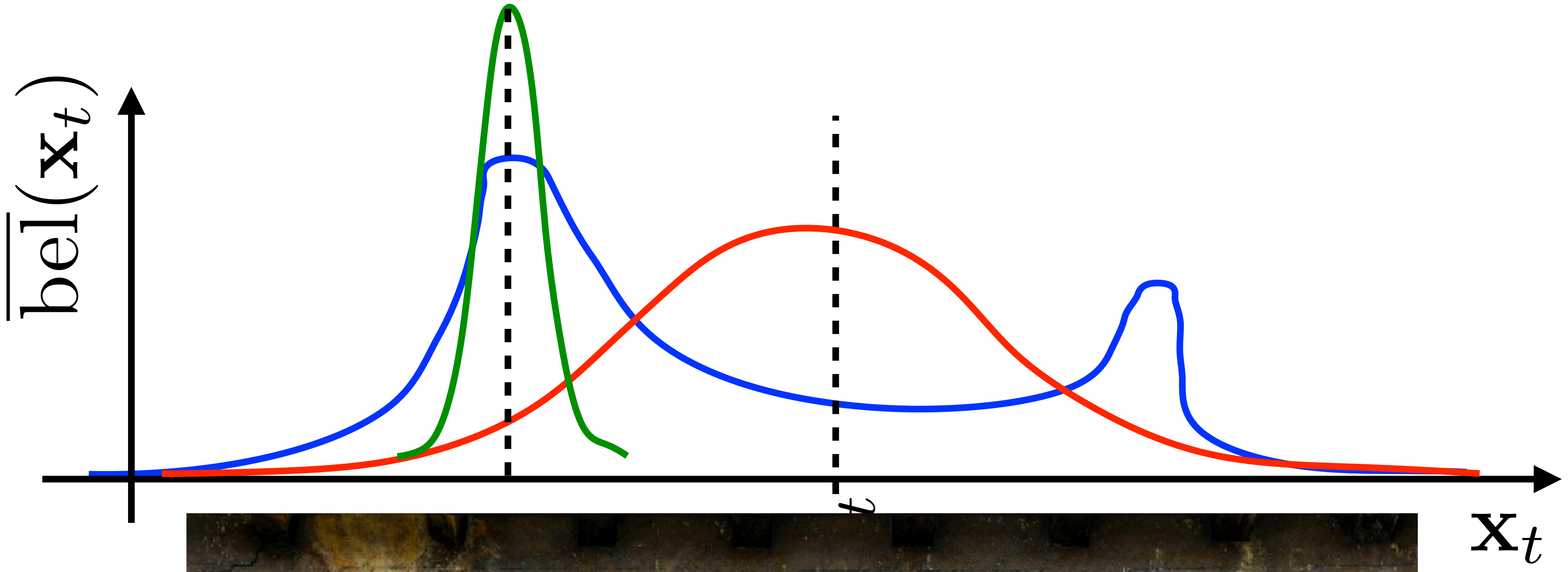

# Extended Kalman Filter

Jean Buridan

Beware of ending up as the mythical dunkey!

Sometimes better to use
the most dominant mode only




# Summary Extended Kalman Filter (EKF)

- EKF is **suboptimal** observer of the current state for non-linear systems under Gaussian noise
- EKF is KF with transition and measurement probabilities approximated by the first order Taylor expansion around current state.
- It cannot **relinearize** in contrast to GN/LM/TR !!!
- It cannot **represent other than Gaussian distr.** in contrast to factorgraphs !!
- It nicely **scales to higher dimension** and does not grow to infinity.
- It has been used for onboard guidance and navigation system for the Apollo Spacecraft Mission https://en.wikipedia.org/wiki/Apollo_(spacecraft)
- There are other ways of non-linearity approximation such as Assumed Density Filter (ADF) or Unscented Kalman Filter (UKF).