How to fuse almost anything:

1D/2D robot’s localization as Maximum Aposteriori Estimate, measurement probability and motion model.

Karel Zimmermann

Prerequisites: Law of total probability

Specific population:

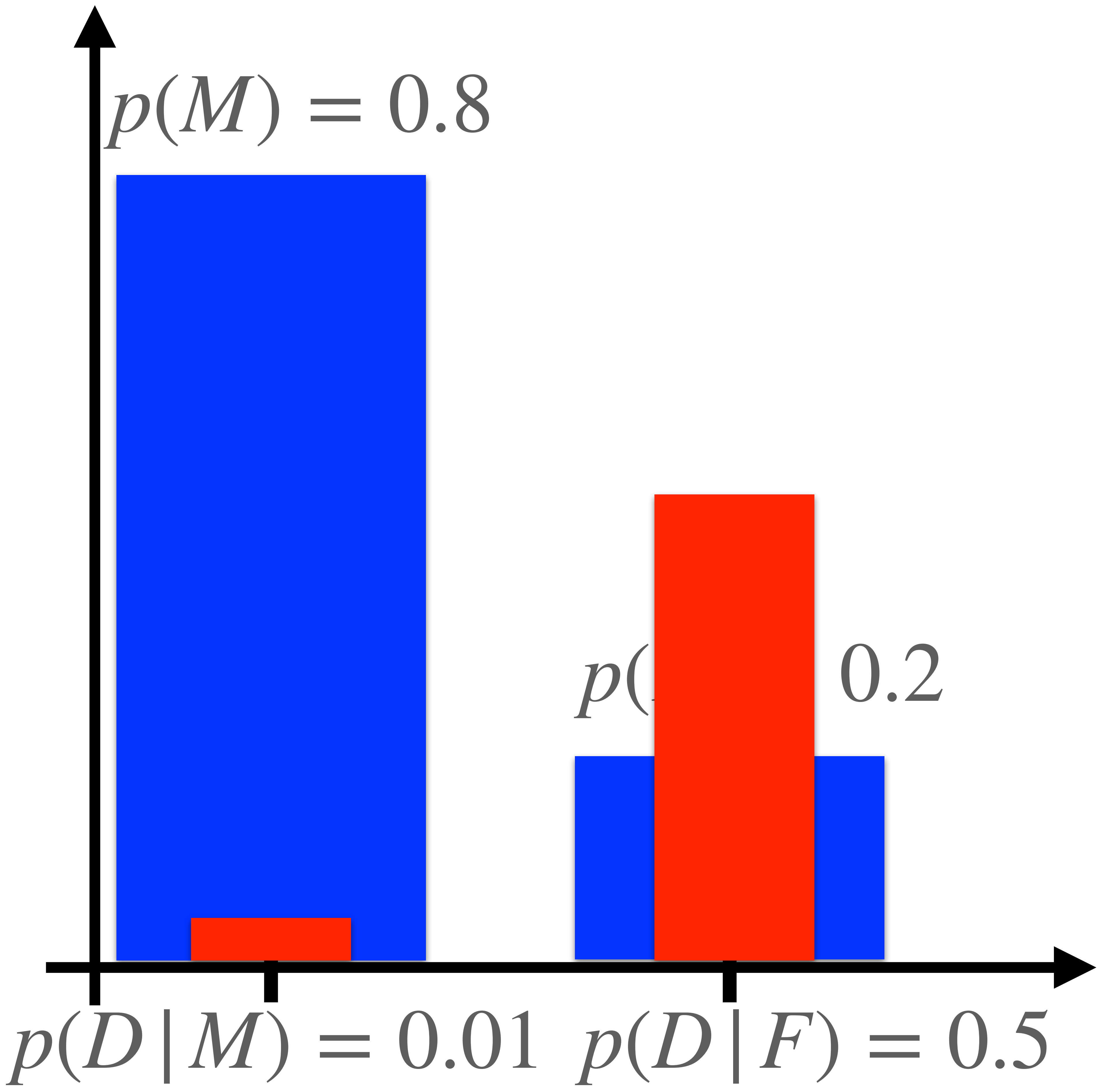
🟦 $p(M) = 0.8$ … 80% male

🟦 $p(F) = 0.2$ … 20% female

Gender-conditional probabilities to have a disease:

🟥 $p(D \mid M) = 0.01$ … 1% of males is ill

🟥 $p(D \mid F) = 0.5$ … 50% of females is ill



What is probability that a random sample has a disease?

$p(D) = p(D \mid M)p(M) + p(D \mid F)p(F)$ …it is mean of **red values** under **blue distribution**

$$= 0.01 \cdot 0.8 + 0.5 \cdot 0.2 = 0.108$$

A … have disease p … disease 0.1%

p … healthy 99.9%

B … positive test p … se tests always positive

p … eople, with positive test

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} \quad \frac{*\,0.001}{+\,0.999 * 0.05} \approx 1.9\,\%$$

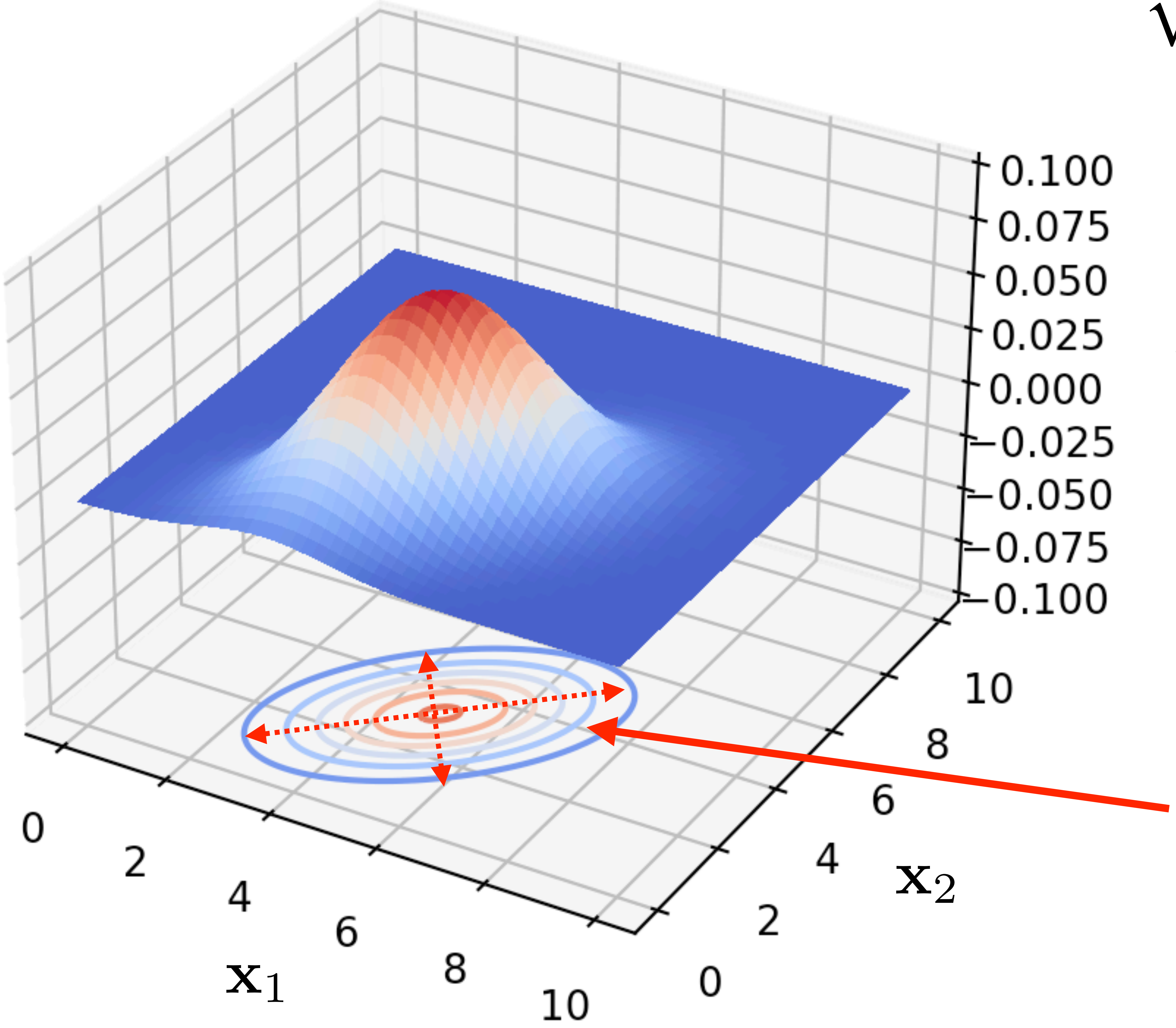
Only 18% of doctors+students from Harvard Medical school answered correctly.

Reason: think about people who tests positive (only 1.9% of them are actualy ill)

=> more likely to come from the healthy population

Prerequisites: Multivariate gaussian

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^{n}\det(\boldsymbol{\Sigma})}}$$



$\mathbf{x} \in \mathcal{R}^{n}$ … real n-dimensional random column vector

$\boldsymbol{\mu} \in \mathcal{R}^{n}$ … real n-dimensional mean

$\boldsymbol{\Sigma} \in \mathcal{R}^{n \times n}$ … symmetric positive definite covariance matrix

eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$ determine ellipse axes

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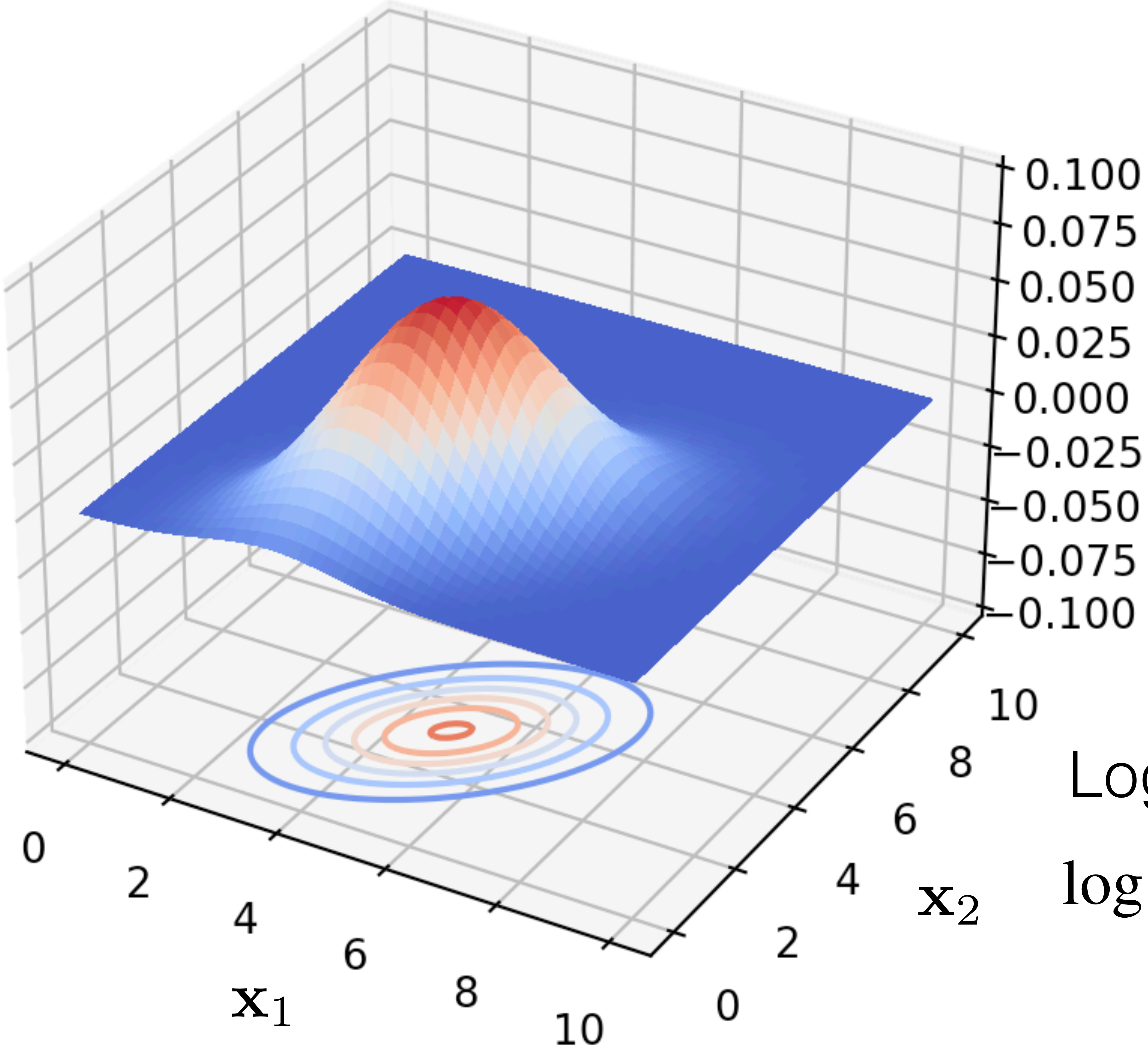

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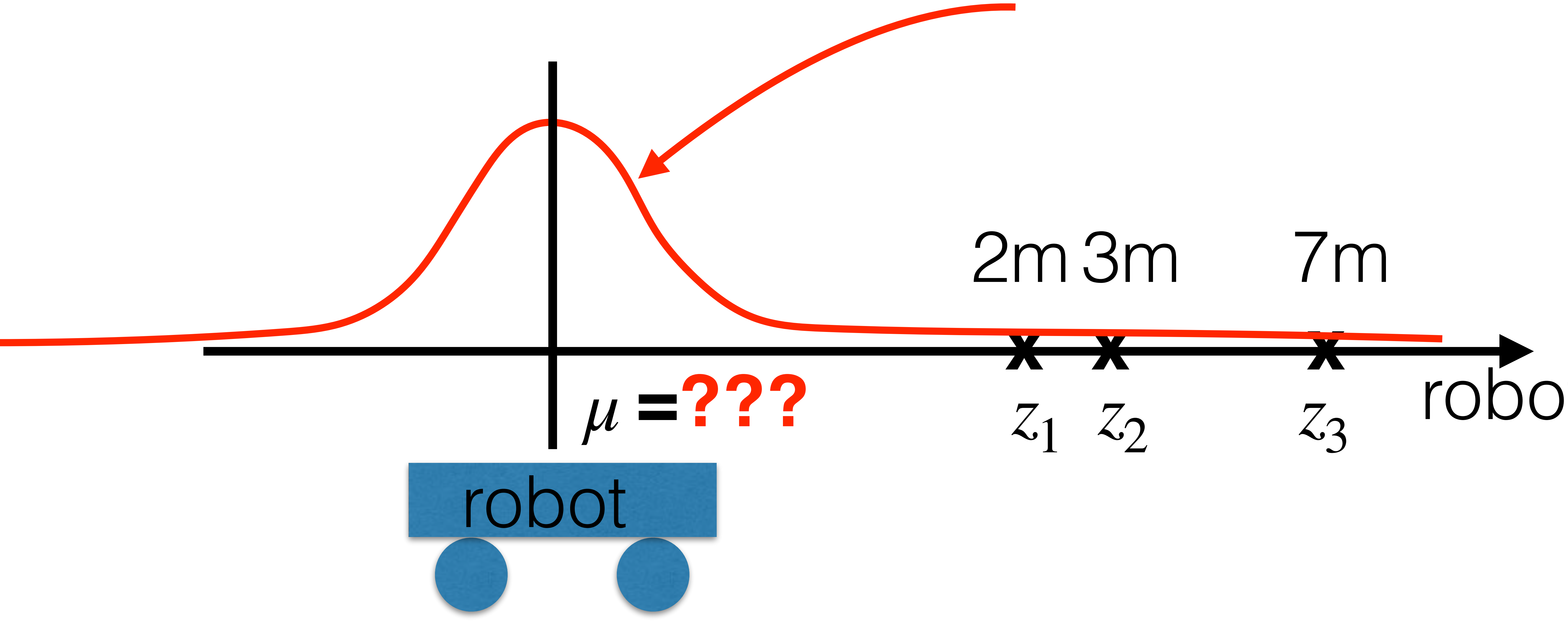
Logarithm of Gaussian is quadratic form:

$$\log\Big(\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\Big) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) + C$$

Motivation example: Absolute position measurements in wcf

Measurement probability: $p(z_i \,|\, \mu) = \mathcal{N}(z_i; \mu, \sigma^2)$



robot's position measurements in **wcf**

Where is the robot???

$$\mu^* = \arg\max_{\mu} p(\mu \,|\, z_1, z_2, z_3) = \arg\max_{\mu} \frac{p(z_1, z_2, z_3 \,|\, \mu) \cdot \cancel{p(\mu)}}{\cancel{p(z_1, z_2, z_3)}} \overset{\text{iid}}{=} \arg\max_{\mu} \left(\prod_i p(z_i \,|\, \mu) \right) =$$

$$= \arg\max_{\mu} \left(\prod_i \mathcal{N}(z_i; \mu, \sigma^2) \right) = \arg\max_{\mu} \prod_i K \cdot \exp\left(-\frac{\|z_i - \mu\|_2^2}{\sigma^2} \right) = \arg\min_{\mu} \sum_i \|z_i - \mu\|_2^2$$

what is this function?

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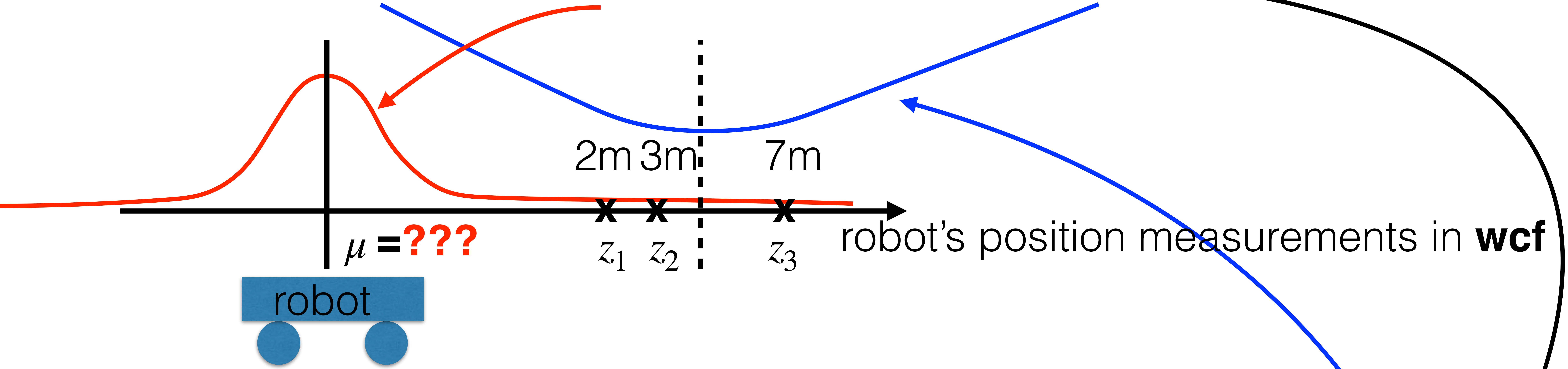

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What kind of assumptions have we used???

1. Uniform prior
2. Independence
3. Gaussian noise

Motivation example: Absolute position measurements in wcf

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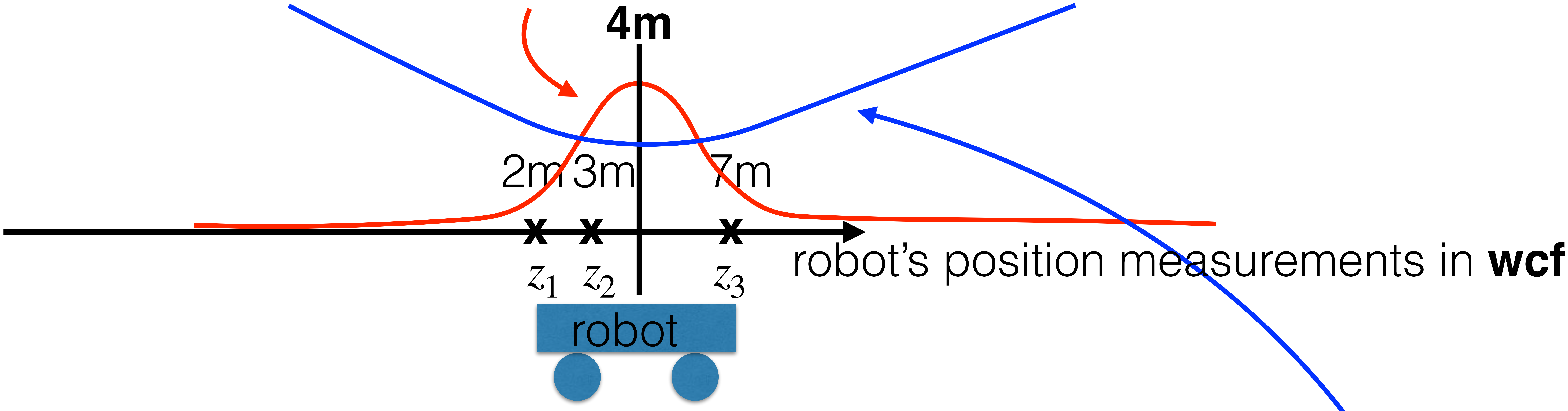

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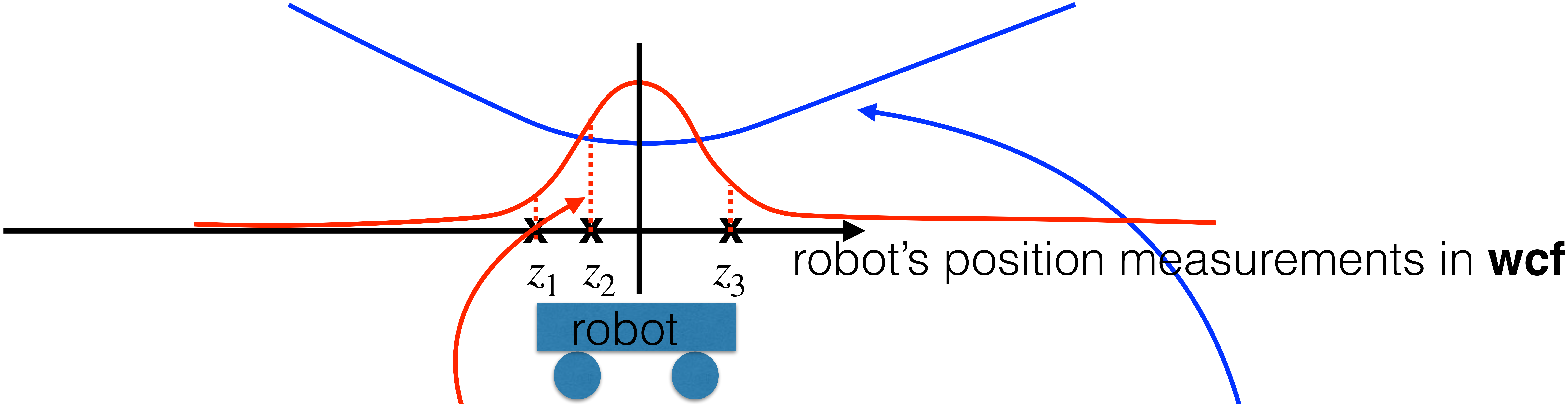
$$= \frac{\sum_i z_i}{N} = \mathbf{4m}$$

Motivation example: Absolute position measurements in wcf

Measurement probability: $p(z_i \mid \mu) = \mathcal{N}(z_i; \mu, \sigma^2)$



MAP MLE LS

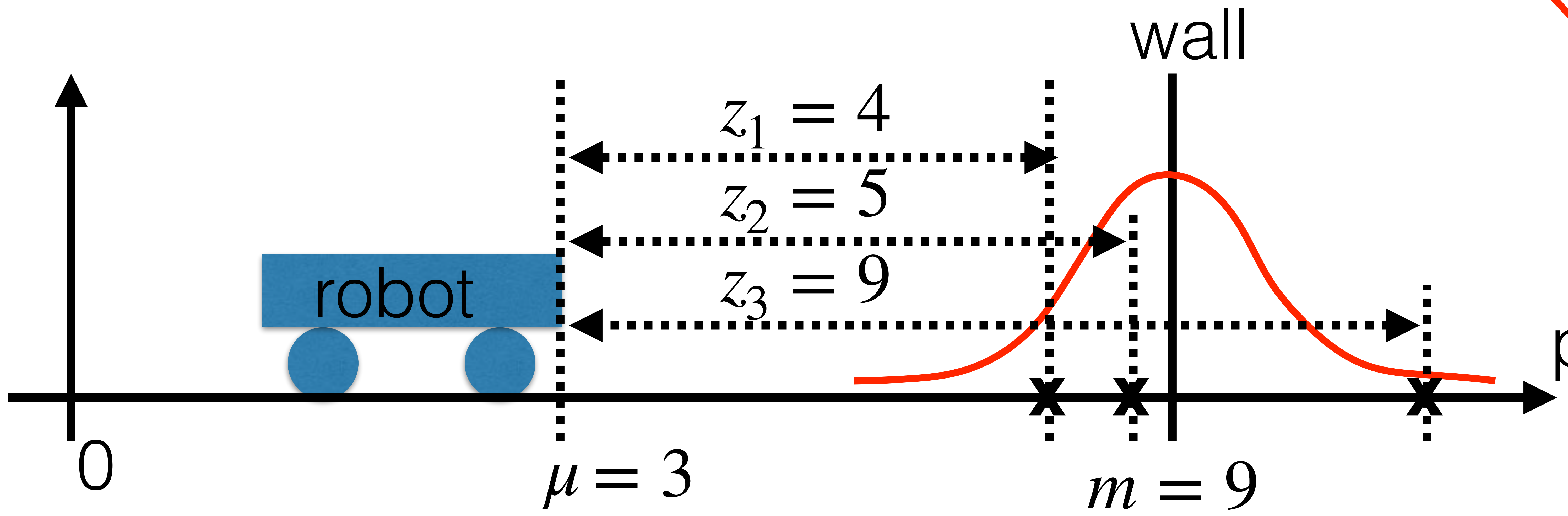
$$\mu^* = \arg\max_{\mu} p(\mu \mid z_1, z_2, z_3) = \arg\max_{\mu} \left(\prod_i p(z_i \mid \mu) \right) = \arg\min_{\mu} \sum_i \|z_i - \mu\|_2^2 = \frac{\sum_i z_i}{N}$$

maximizing product of gaussians <=> minimizing the sum of L2 differences.

Motivation example: Relative position measurements in rcf

Measurement probability: $p(z_i\,|\,\mu) = \mathcal{N}(z_i; m - \mu, \sigma^2)$

wall in rcf



position measurements in **rcf**

Where is the robot???

What is the measurement prob?

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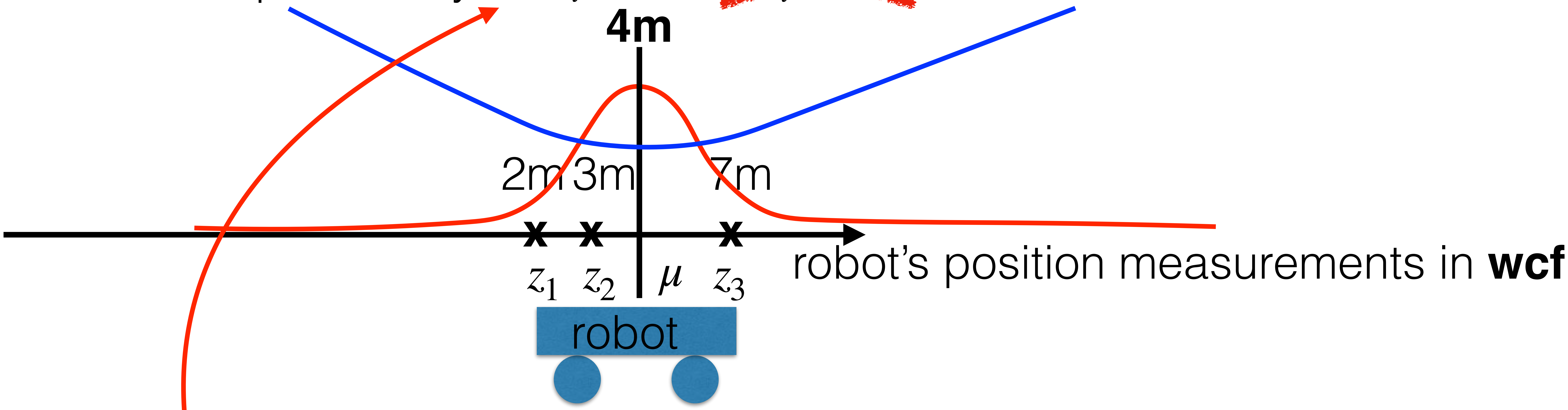
$$= \arg\min_{\mu} \sum_i \|m - z_i - \mu\|_2^2 \quad = \frac{\sum_i m - z_i}{N}$$ = ((9-4) + (9-5) + (9-9)) / 3 = **3**

Motivation example: Outlier rejection

Measurement probability: $p(z_i|\mu) = \cancel{\mathcal{N}(z_i;\mu,\sigma^2)}$



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How can I justify rejecting 7m as outlier and placing the robot between 2-3m?

By different measurement model

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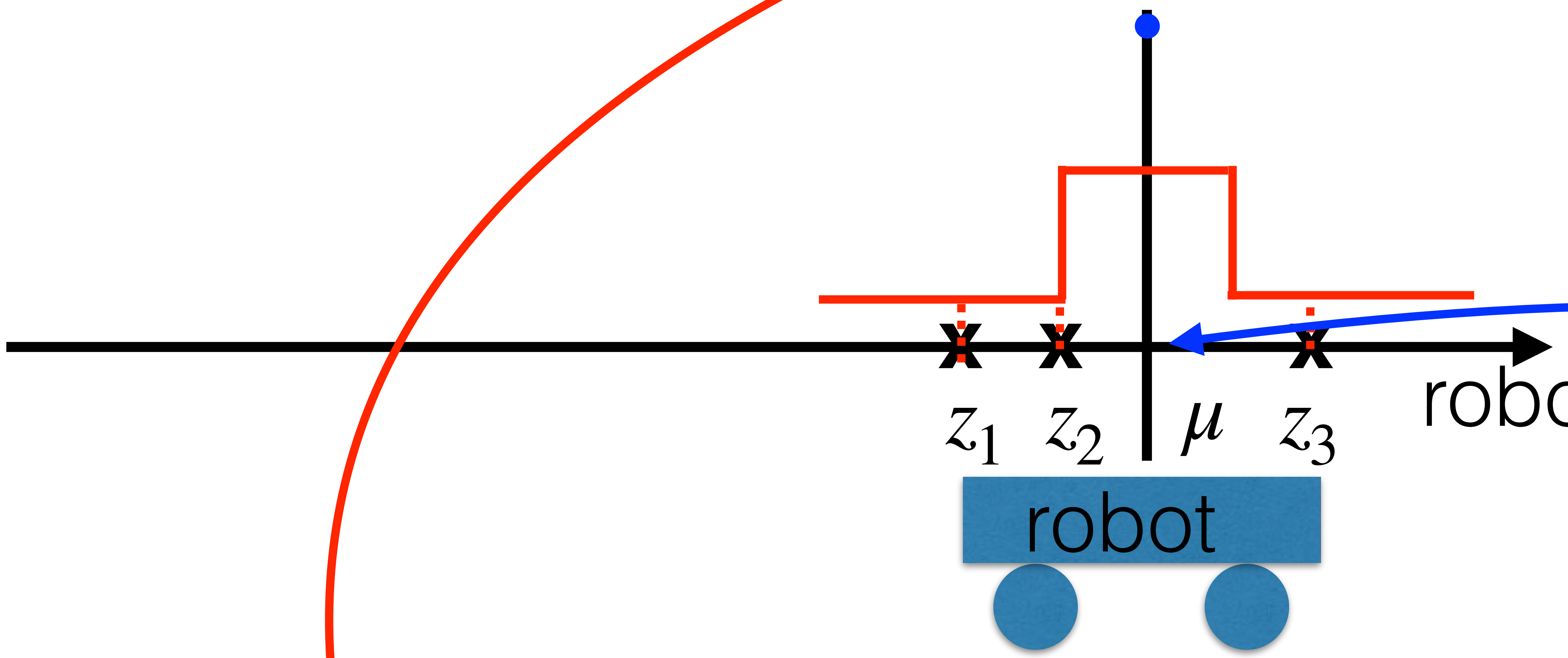

What is the value (high/low) of the criterion function at this point μ?

robot's position measurements in **wcf**

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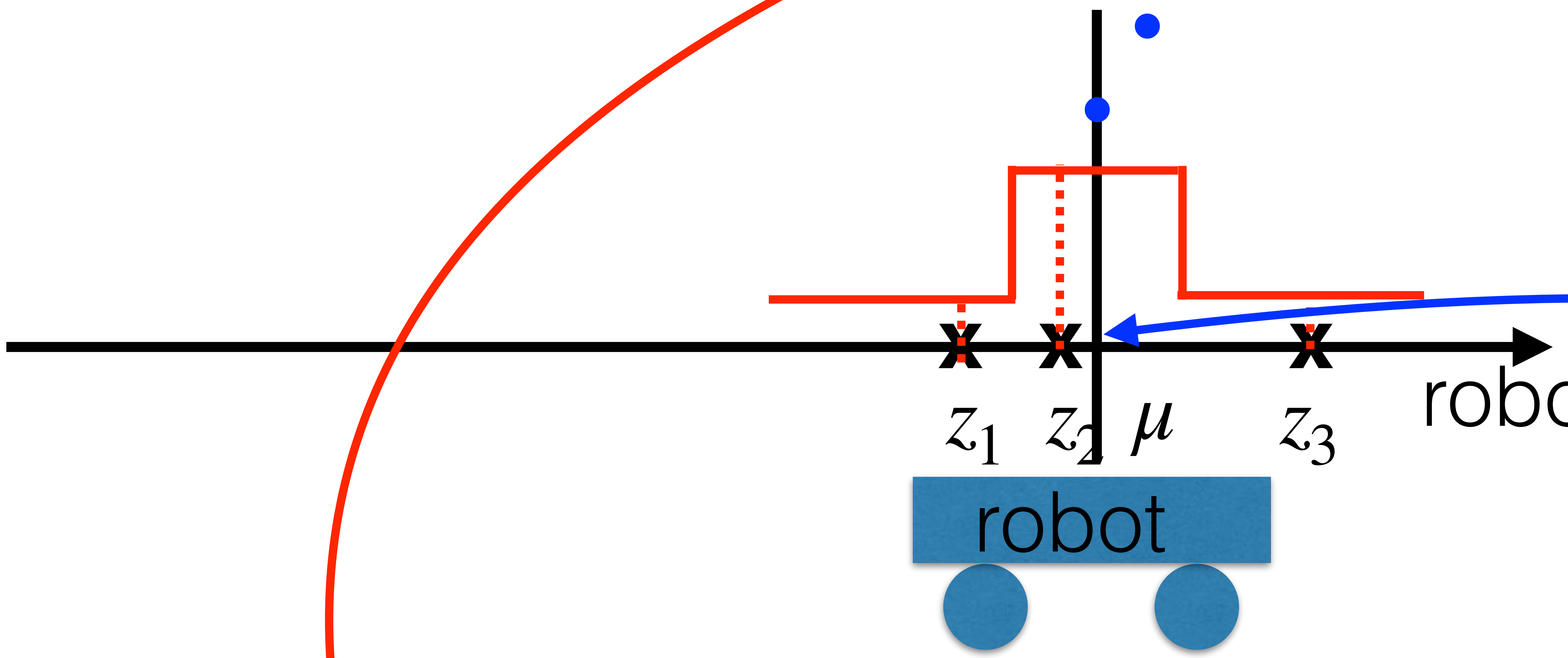

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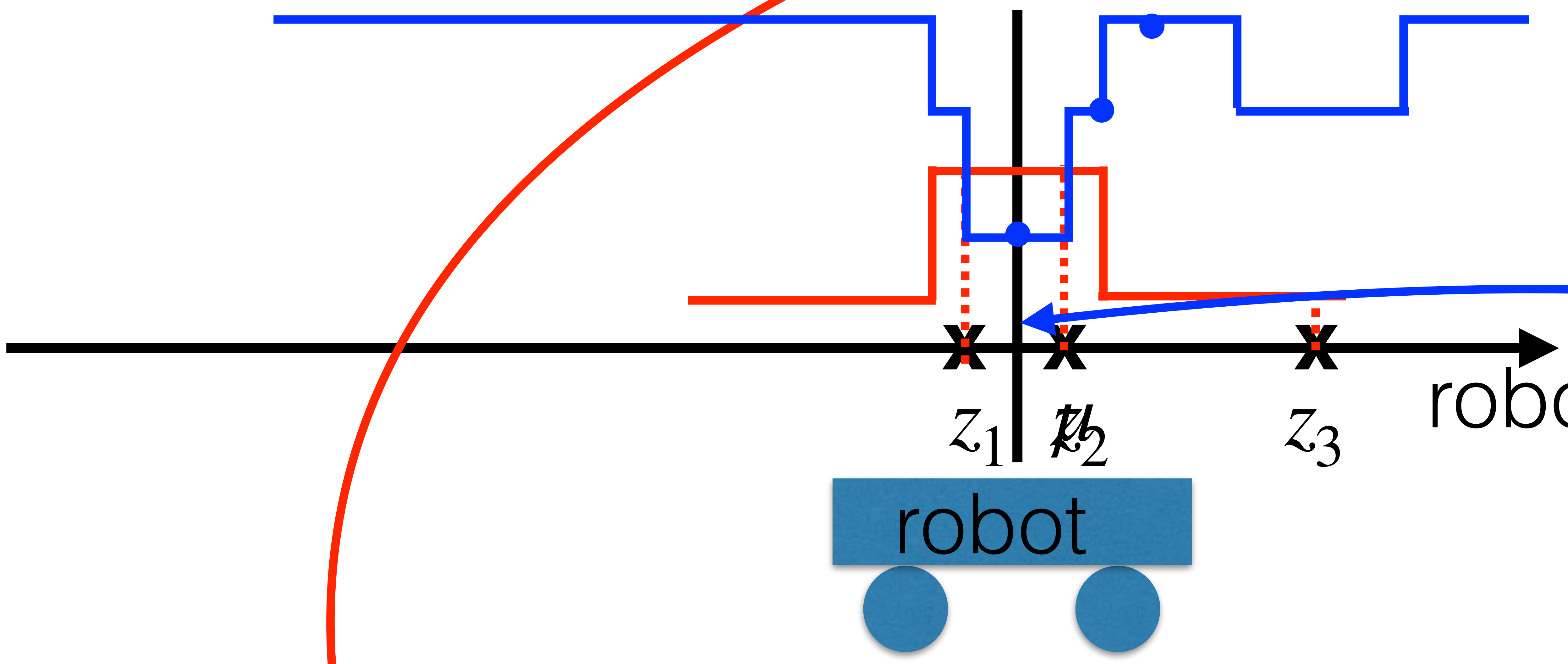

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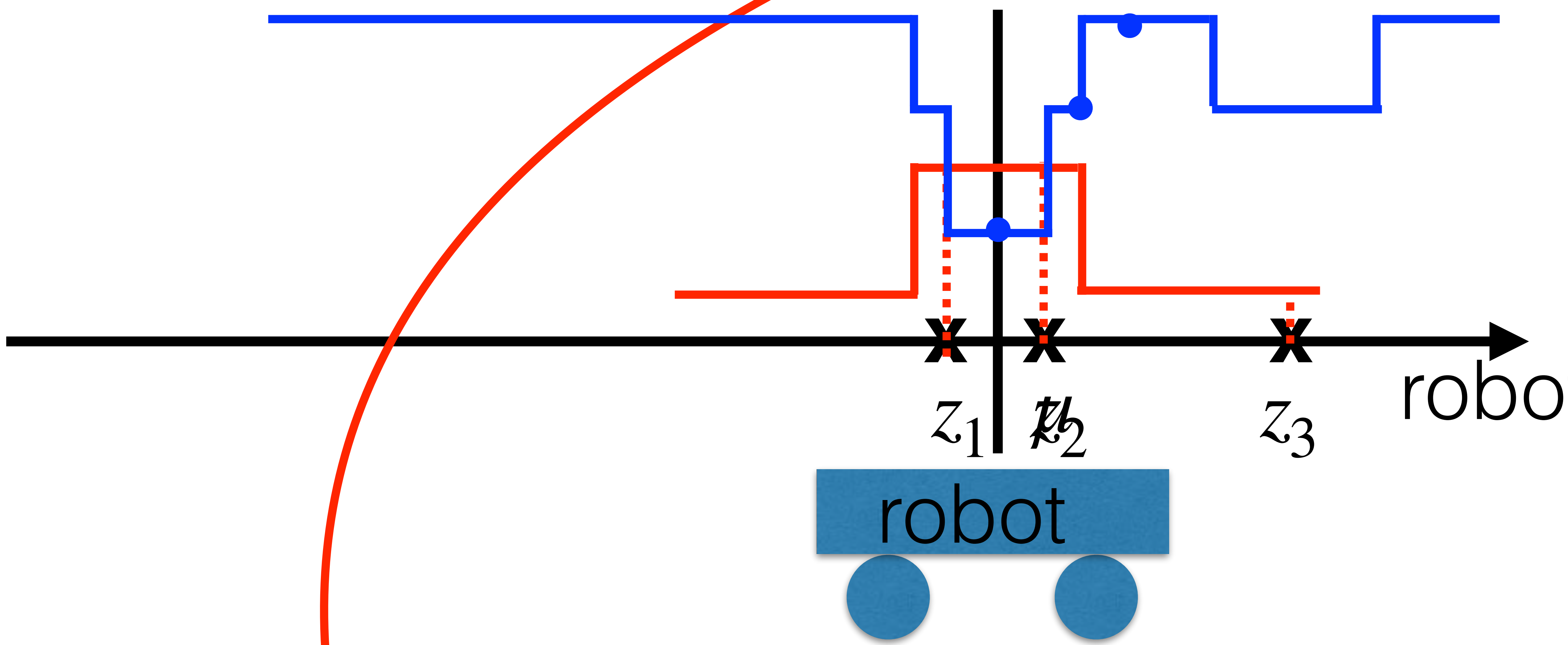


robot's position measurements in **wcf**

How can I find $\mu^\star$ that minimizes the blue function?

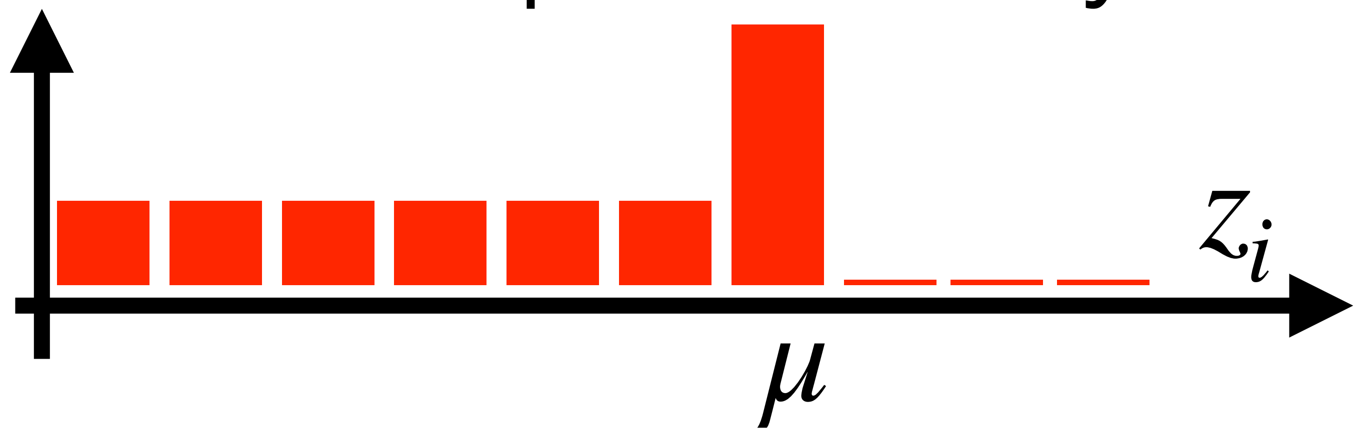
$$\mu^* = \arg\max_\mu p(\mu|z_1,z_2,z_3) = \arg\max_\mu \left(\prod_i p(z_i|\mu)\right) = \arg\min_\mu \sum_i -\log p(z_i|\mu)$$

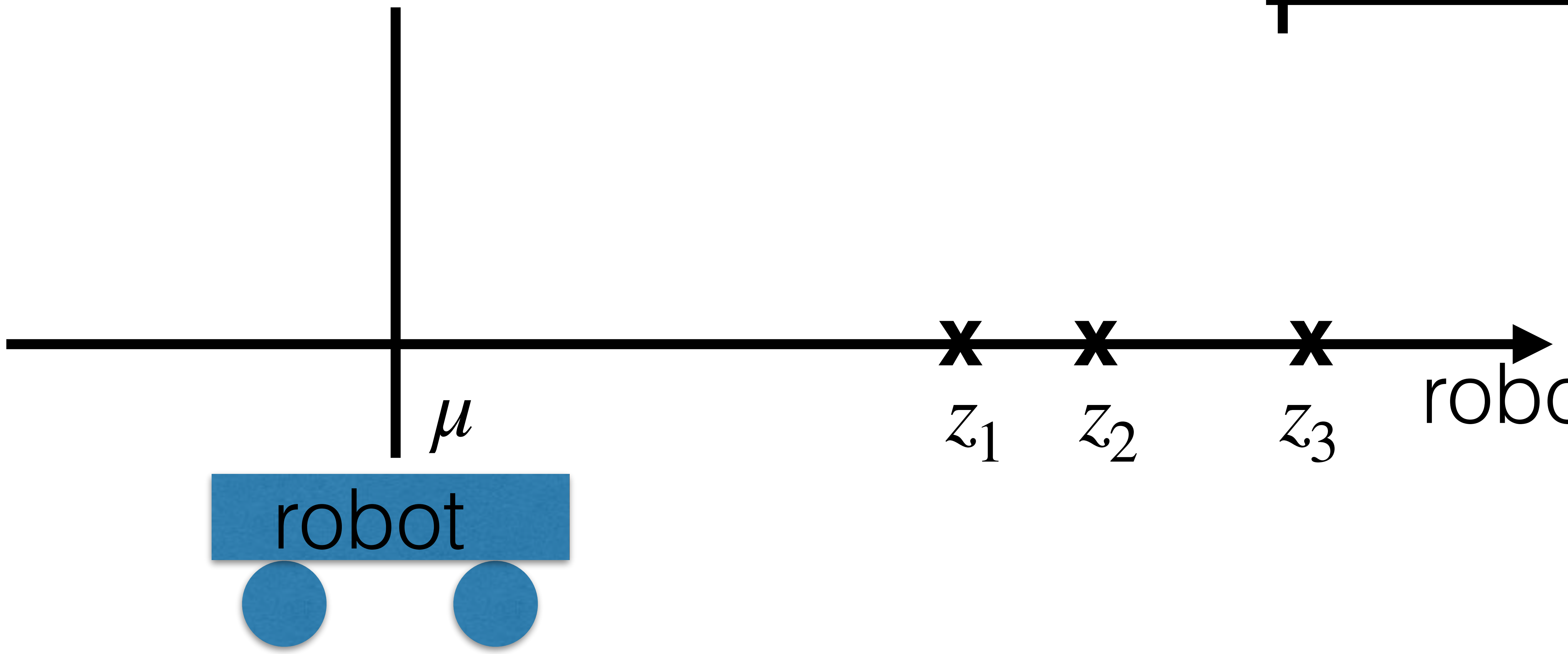
How can I justify rejecting 7m as outlier and placing the robot between 2-3m?

By different measurement model

Motivation example: discrete probability distribution

Measurement probability: $p(z_i|\mu) =$





robot's position measurements in **wcf**

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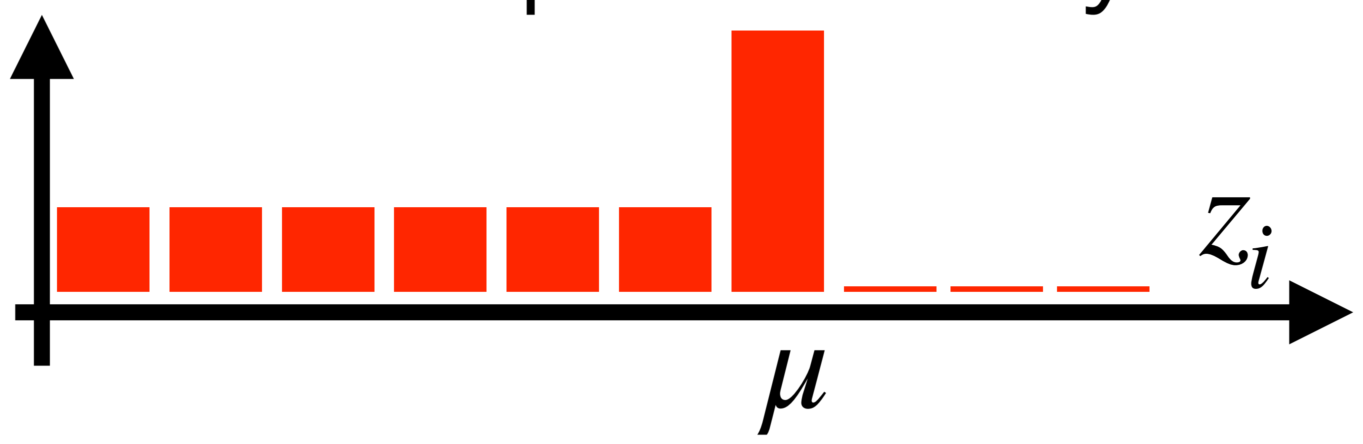

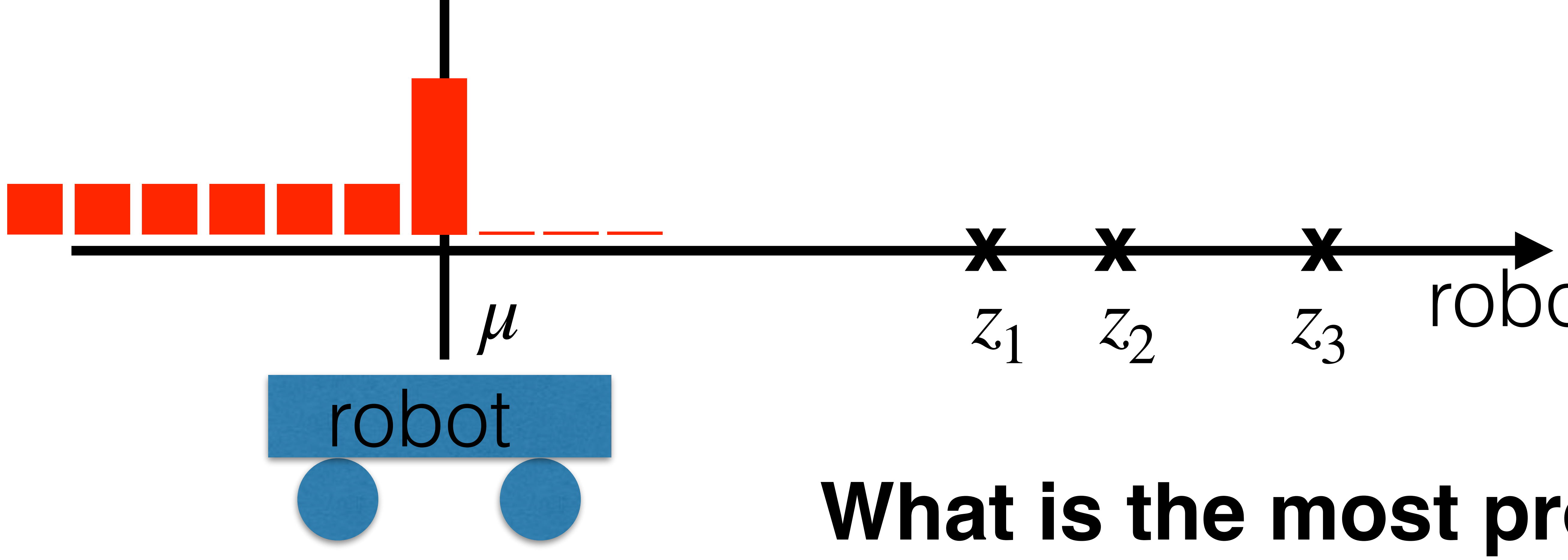

robot's position measurements in **wcf**

What is the most probable μ?

$$\mu^* = \arg\max_{\mu} p(\mu \,|\, z_1, z_2, z_3) = \arg\max_{\mu} \left(\prod_i p(z_i \,|\, \mu) \right) = \arg\min_{\mu} \sum_i - \log p(z_i \,|\, \mu)$$

Motivation example: discrete probability distribution

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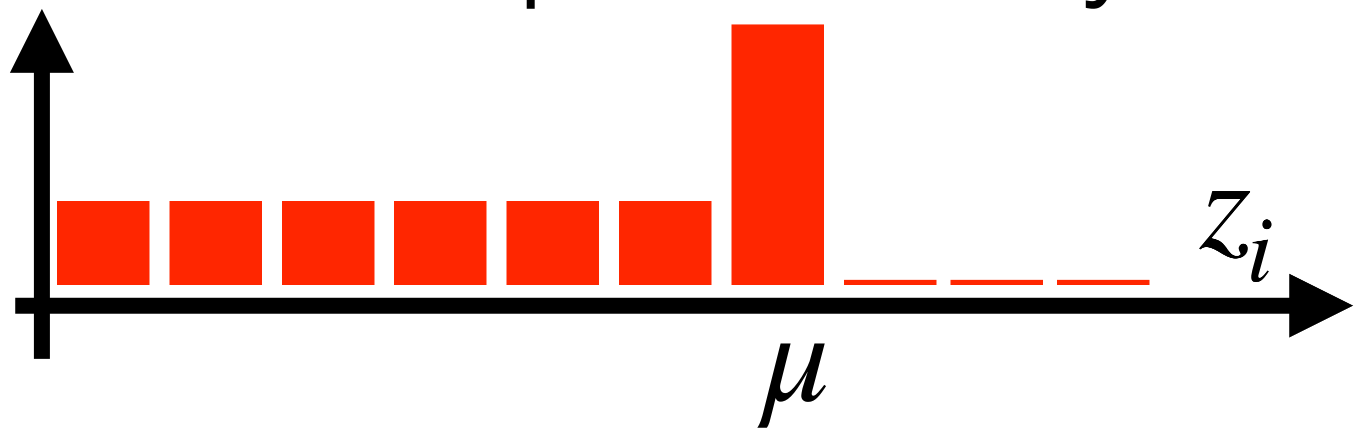

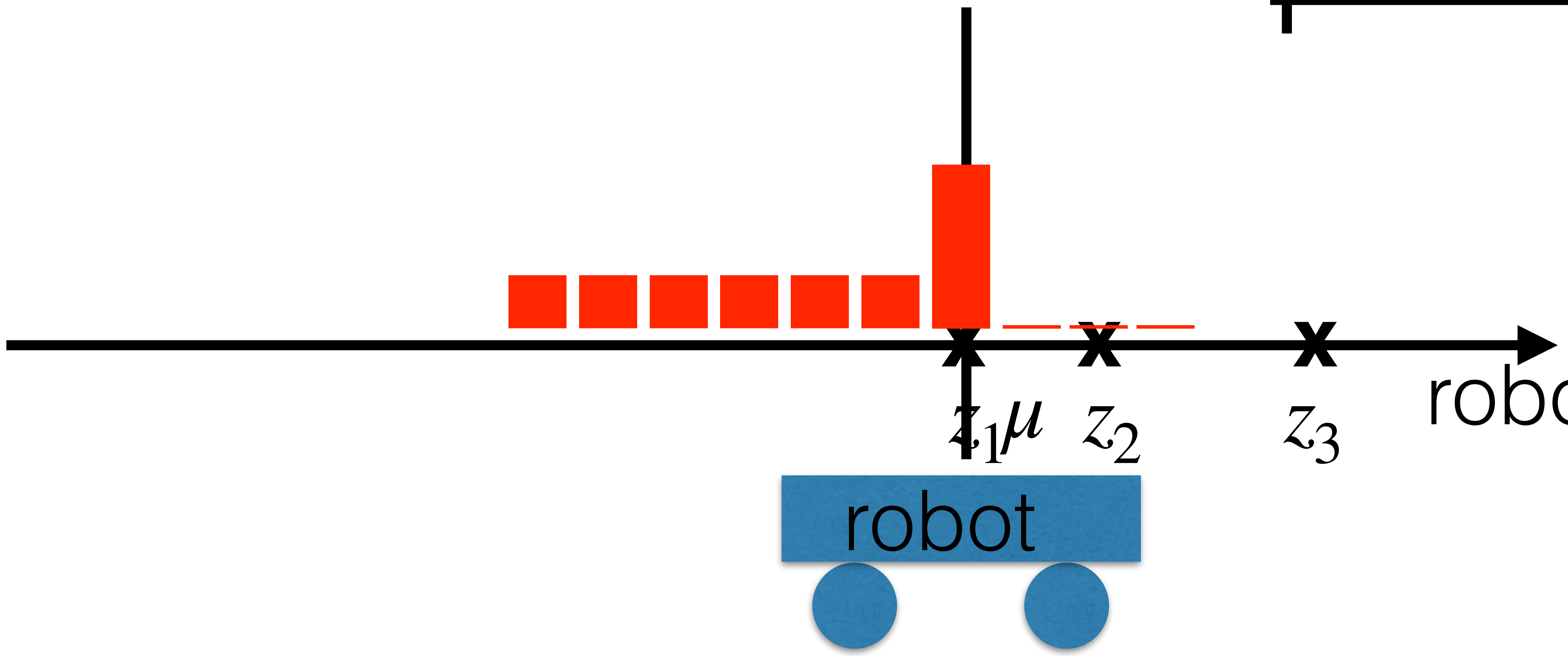

robot's position measurements in **wcf**

= **???** = **???**

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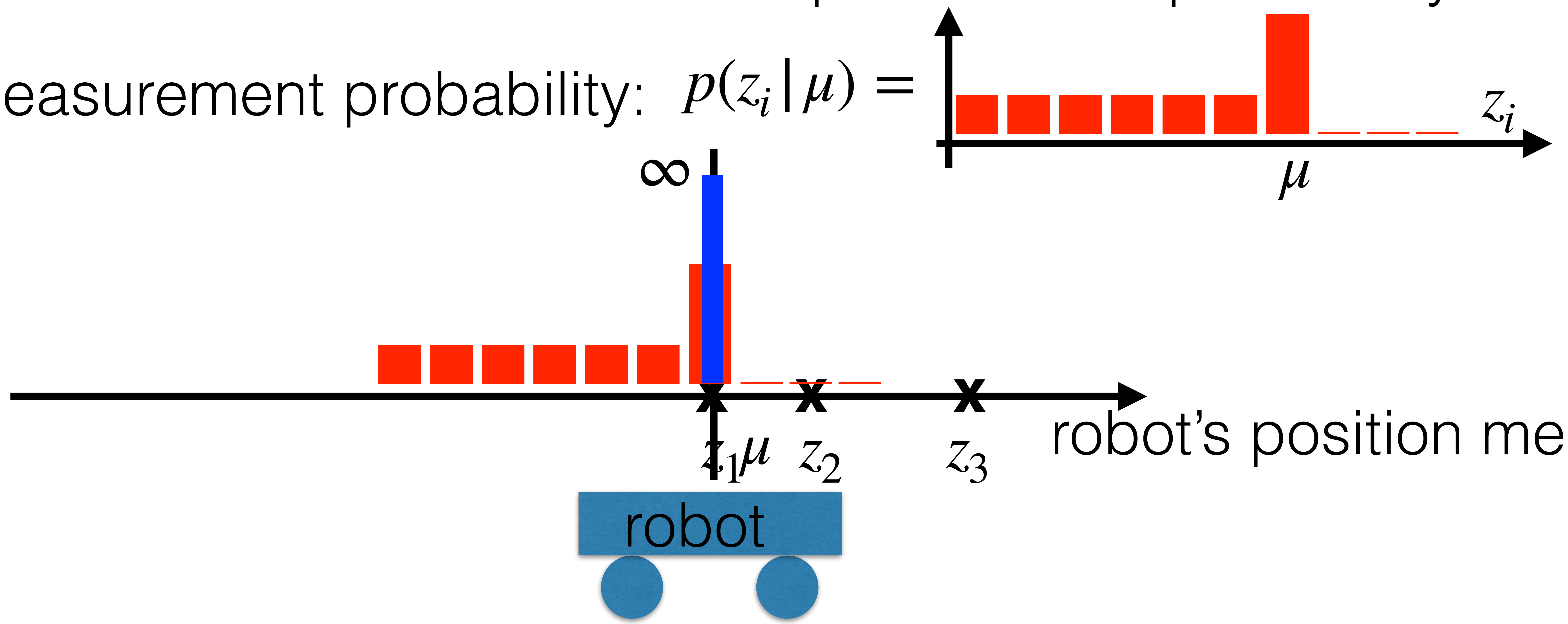
Motivation example: discrete probability distribution

Measurement probability: $p(z_i|\mu) =$



robot's position measurements in **wcf**

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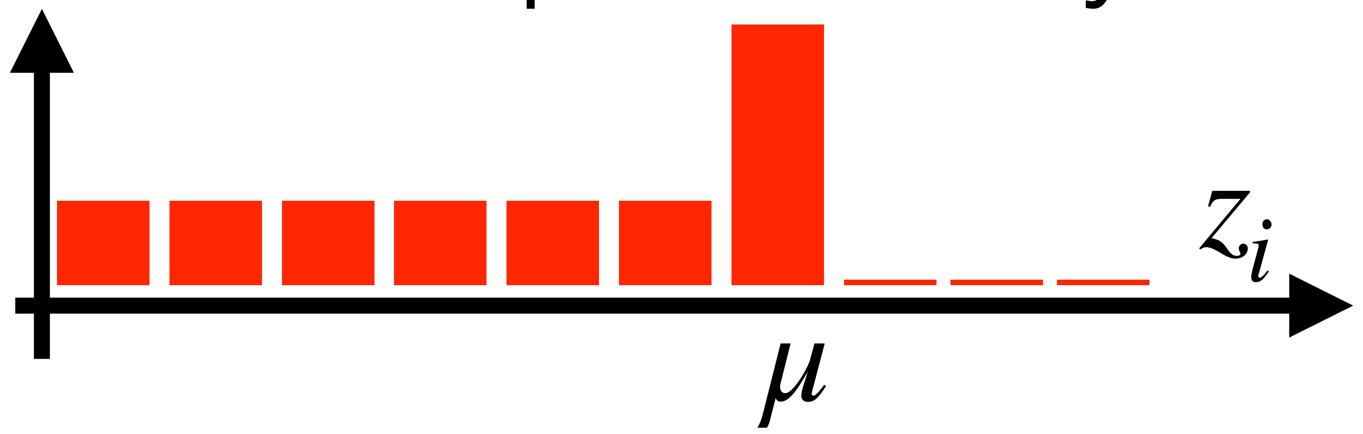
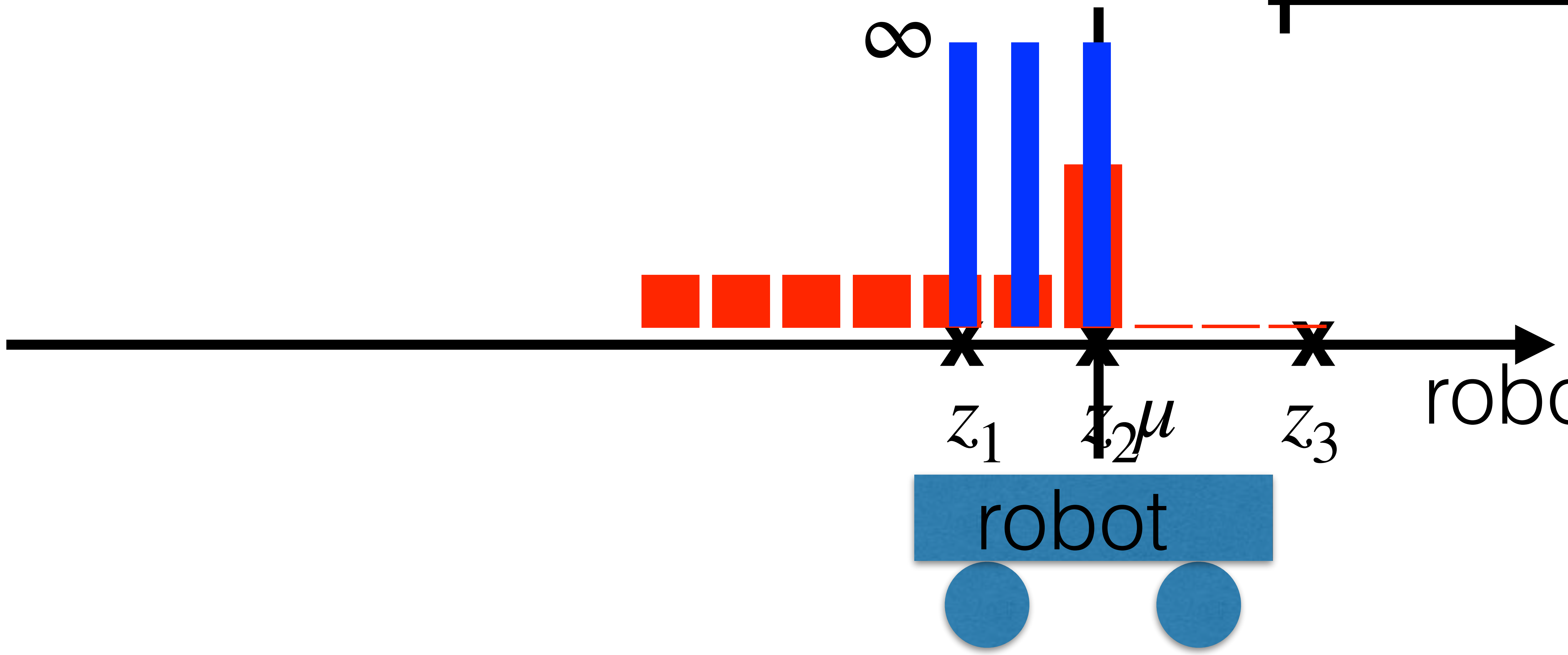
Motivation example: discrete probability distribution

Measurement probability: $p(z_i|\mu) =$





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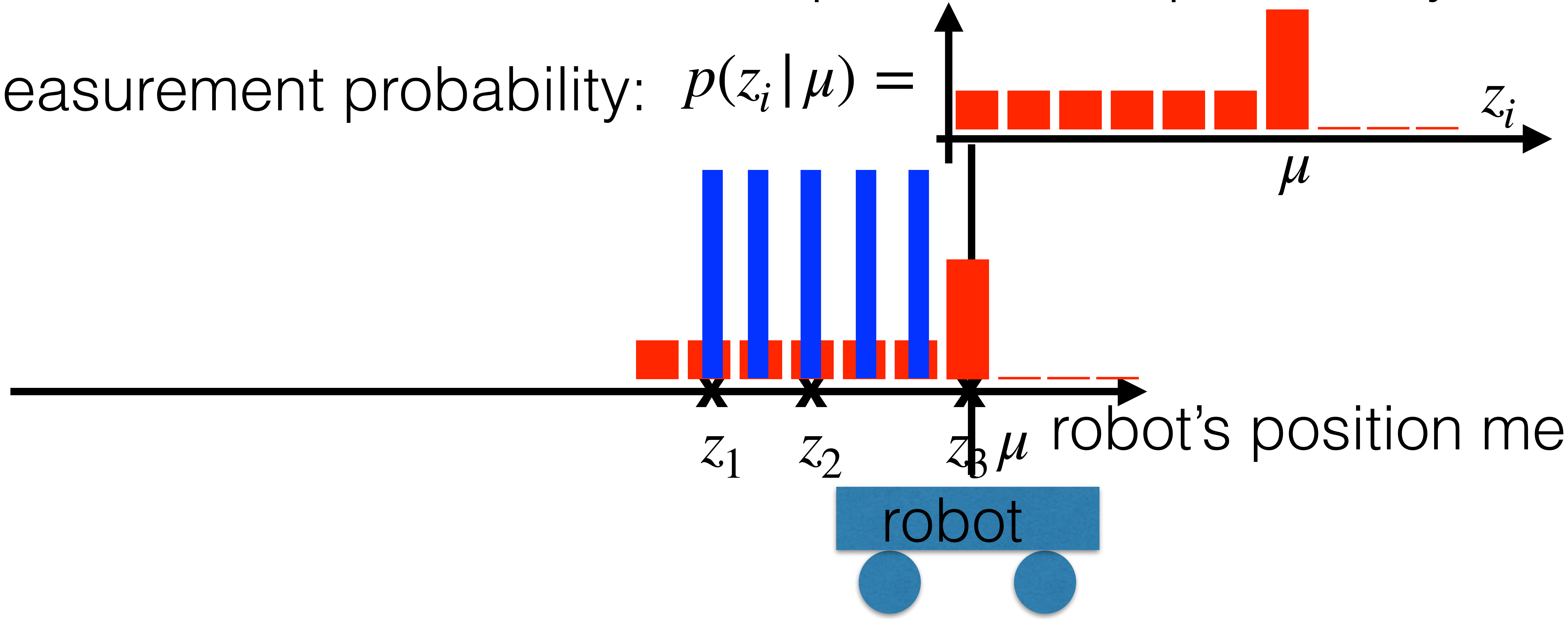

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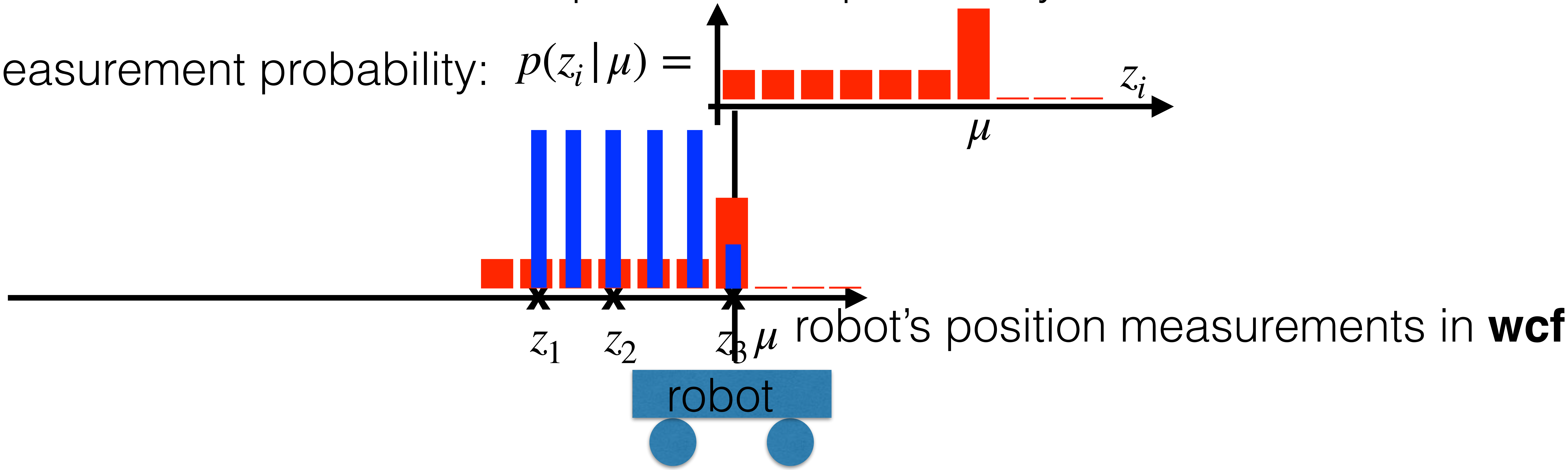

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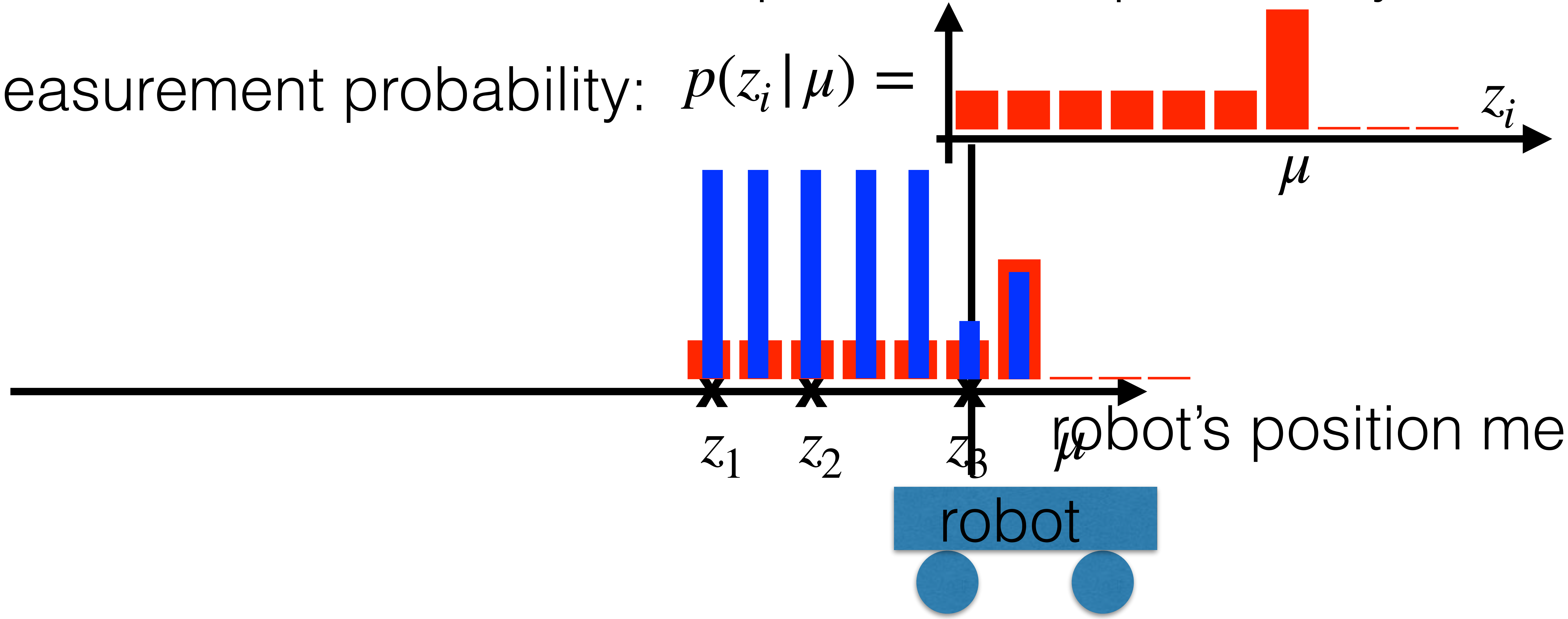

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robot's position measurements in **wcf**

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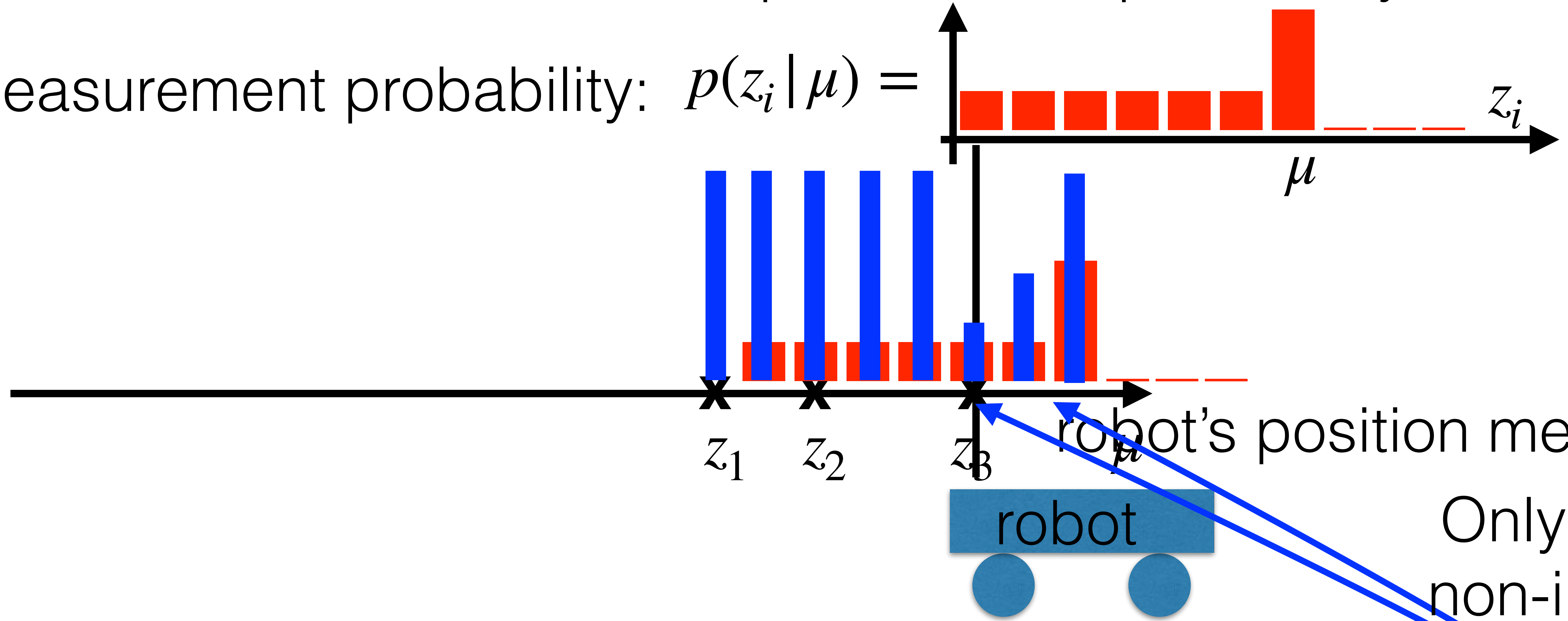


robot's position measurements in **wcf**

Only two robot poses with non-infinite criterion values

$$\mu^* = \arg\max_{\mu} p(\mu|z_1, z_2, z_3) = \arg\max_{\mu} \left(\overset{= 0}{\prod_i p(z_i|\mu)} \right) = \arg\min_{\mu} \sum_i \overset{= \infty}{- \log p(z_i|\mu)}$$

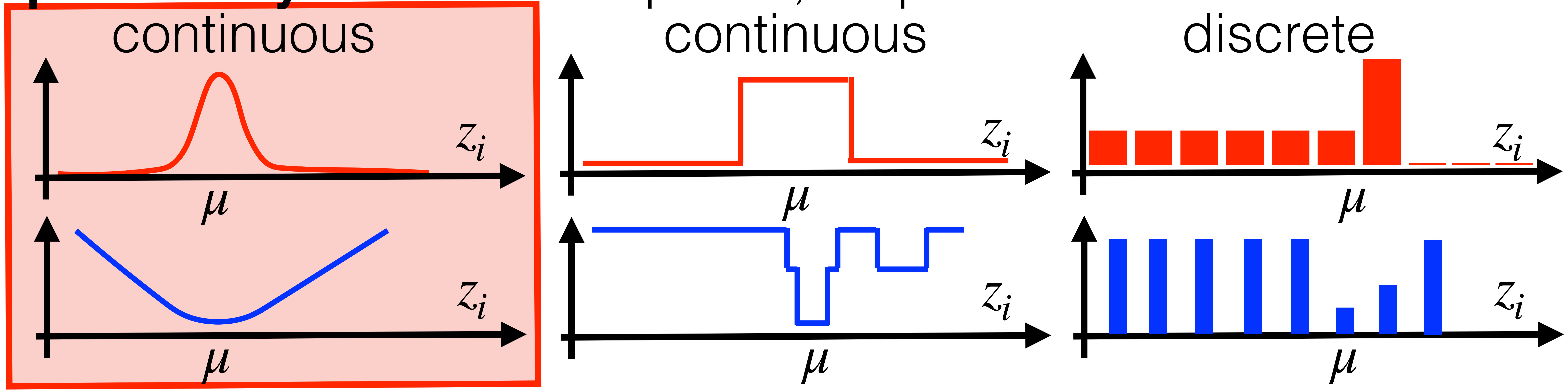
Zeros in measurement probability means “never ever” => very restrictive

Lesson learned from motivation examples

- **Robot's localization** = MAP estimate of its pose given measurements
- **Measurements probability** binds robot's poses, map and measurements

$$p(z_i\,|\,\mu) =$$

$$\sum_i -\log p(z_i\,|\,\mu) =$$

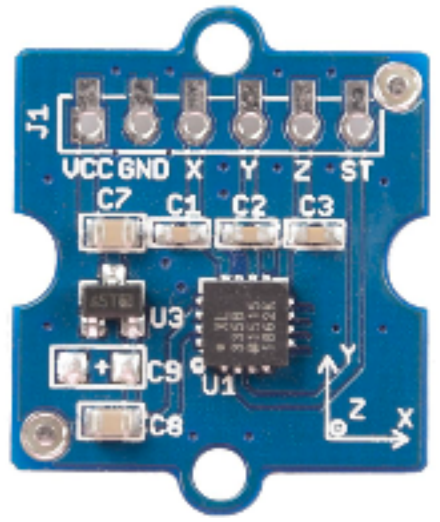


- **Criterion** optimization friendly vs optimization unfriendly
- **Assumptions Gaussian** $p(z_i\,|\,\mu) = \mathcal{N}(z_i; \boxed{h(\mu, m)}, \sigma^2)$ **+ iid** reduces the MAP estimate to LS problem (next 3 lectures)
  - $h(\mu, m)$ transfers the state (e.g. pose) into measurement space
  - $h(\mu, m)$ linear vs non-linear => linear / non-linear LS
  - two optimisation approches: filters (KF, EKF, UKF), GraphSLAM
  - zero in $p(z_i\,|\,\mu)$ means that given the pose the measurement is impossible

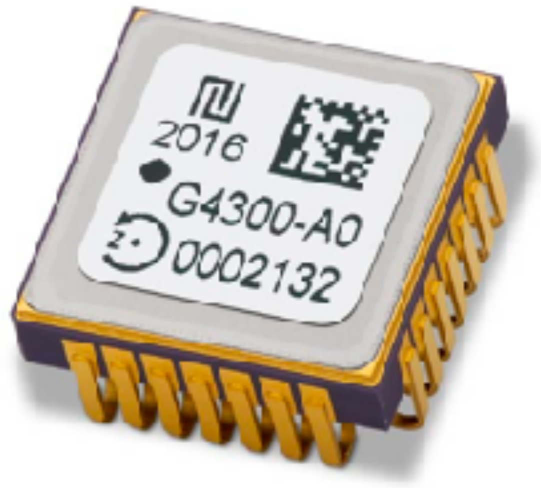
- **What sensors can we use for the localization?**

Sensors for localisation (odometry)

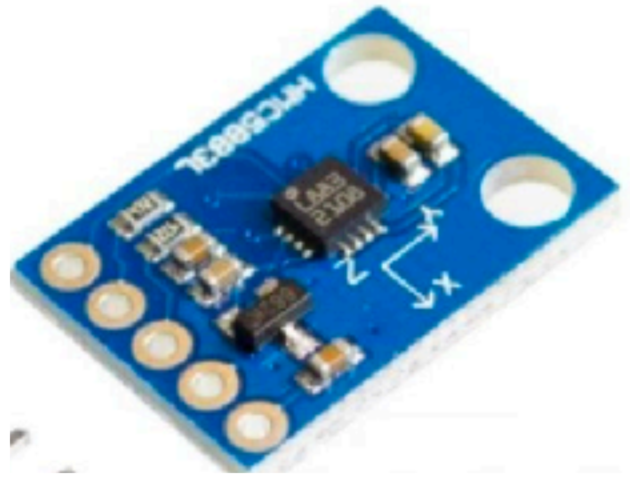
Motor encoders (wheel/joint position/velocity)



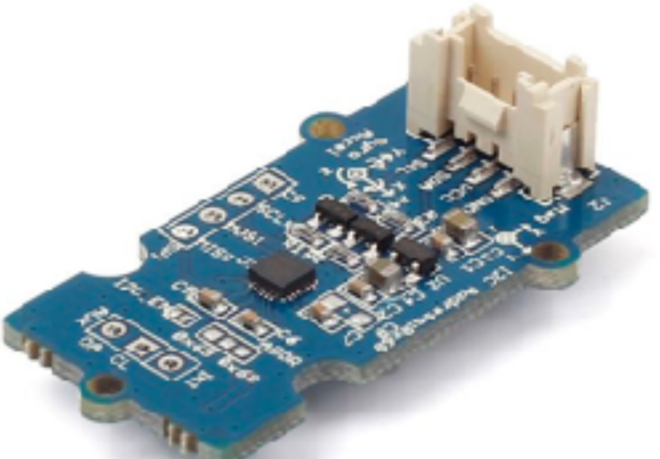
Accelerometer (linear acceleration)



Gyroscope (angular velocity)



Magnetometer (angle to magnetic north)



IMU: Accelerometer+Gyroscope+Magnetometer (9DOF measurements

Sensors for localisation (exteroceptive)

Camera (RGB images - spectral responses projected on image plane)

Stereo camera

RGBD camera (kinect, real sense, …

Lidar

Sonar

Radar

Satelite navigation (GPS/GNSS)

SONARDYNE beacons

UWB (Ultra Wideband Radio)

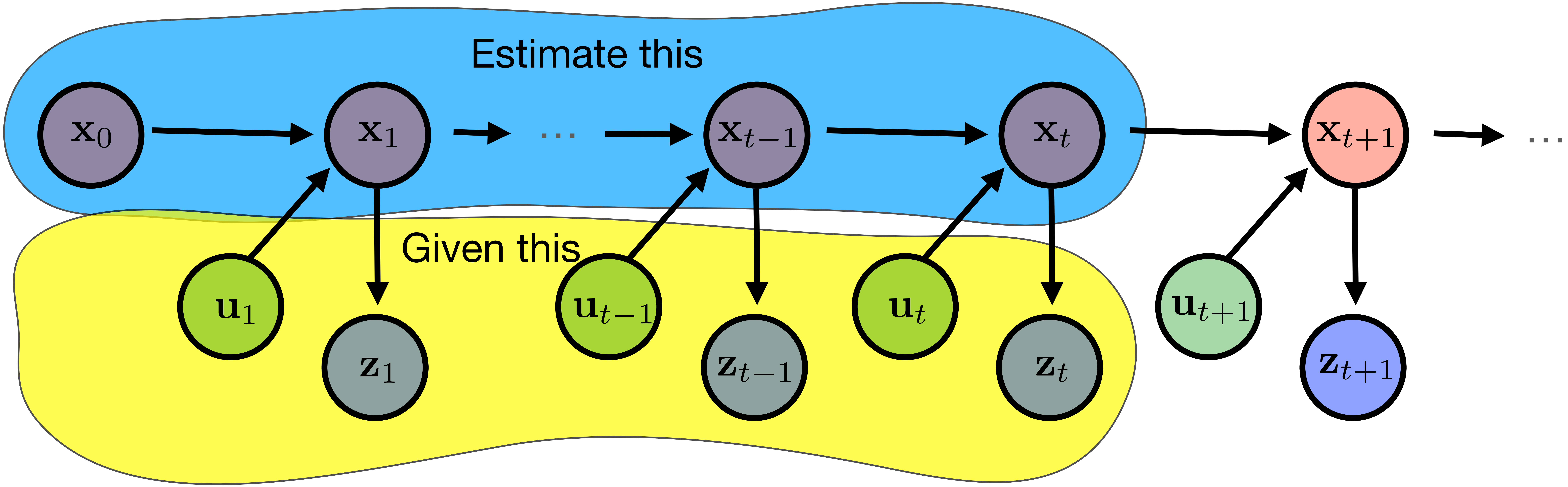
Localisation problem definition

Today only 1D/2D translations (no rotations)

States:	$\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t \in \mathcal{R}^n$	…. ~~6DOF~~ robot's poses (no map for now)
Actions:	$\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathcal{R}^m$	…. generated by external source
Measurements:	$\mathbf{z}_1, \ldots, \mathbf{z}_t \in \mathcal{R}^k$	…. comes from variety of sensors

Unknown

MAP: $\mathbf{x}^* = \arg\max_{\mathbf{x}} p(\mathbf{x}\,|\,\mathbf{z},\mathbf{u}) = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{x}_0\ldots\mathbf{x}_t\,|\,\mathbf{z}_1\ldots\mathbf{z}_t, \mathbf{u}_1\ldots\mathbf{u}_t)$



Localisation problem definition

Today only 1D/2D translations (no rotations)

States: $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t \in \mathcal{R}^n$ …. ~~6DOF~~ robot's poses (no map for now)

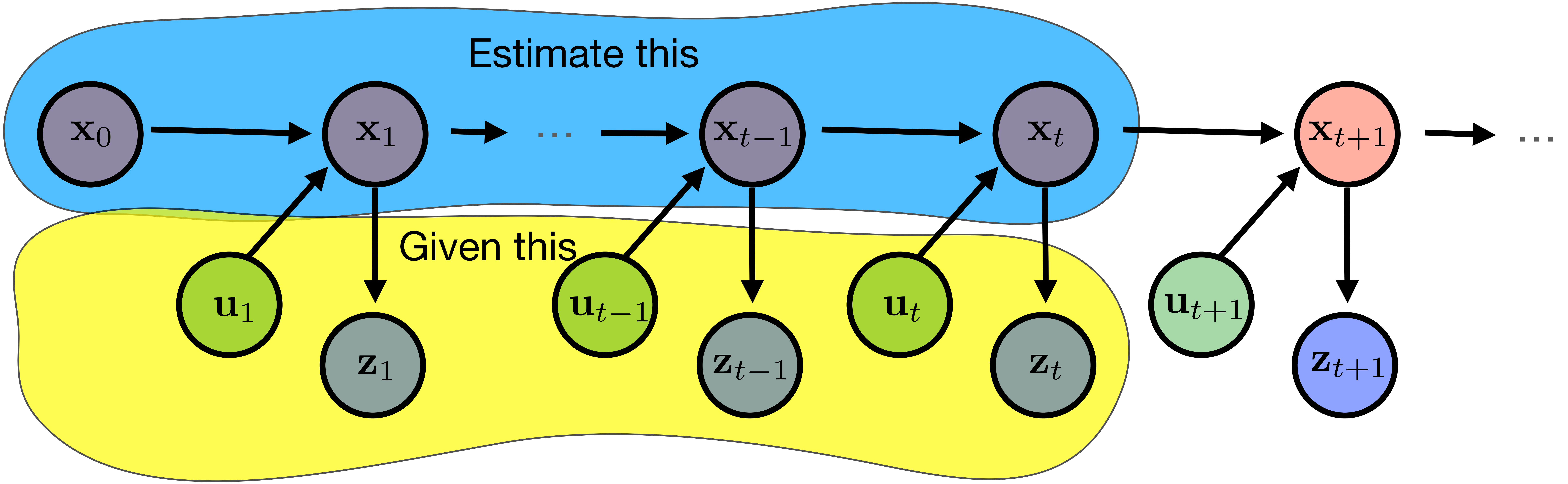
Actions: $\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathcal{R}^m$ …. generated by external source

Measurements: $\mathbf{z}_1, \ldots, \mathbf{z}_t \in \mathcal{R}^k$ …. comes from variety of sensors

Unknown

MAP: $$\mathbf{x}^* = \arg\max_{\mathbf{x}} p(\mathbf{x}\,|\,\mathbf{z},\mathbf{u}) = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{x}_0\ldots\mathbf{x}_t\,|\,\mathbf{z}_1\ldots\mathbf{z}_t,\mathbf{u}_1\ldots\mathbf{u}_t)$$

1. Construct p(x|z)
2. Optimize poses



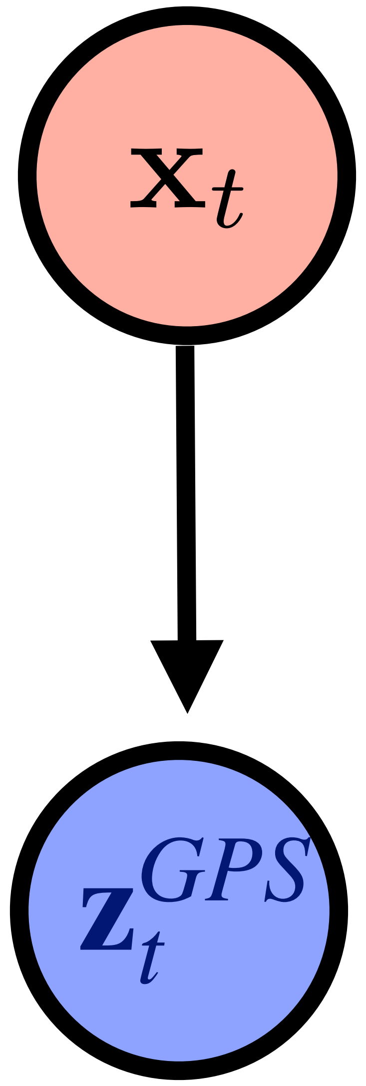
Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

Aposterior pdf

1. Construct p(x|z)

MAP: $$\mathbf{x}_t^* = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t) \downarrow = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t \,|\, \mathbf{z}_t^{GPS})$$



.... ???

....

Satelite navigation (GPS/GNSS)

Assumption: $$p(\mathbf{z}_t^{GPS} \,|\, \mathbf{x}_t) = \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})$$

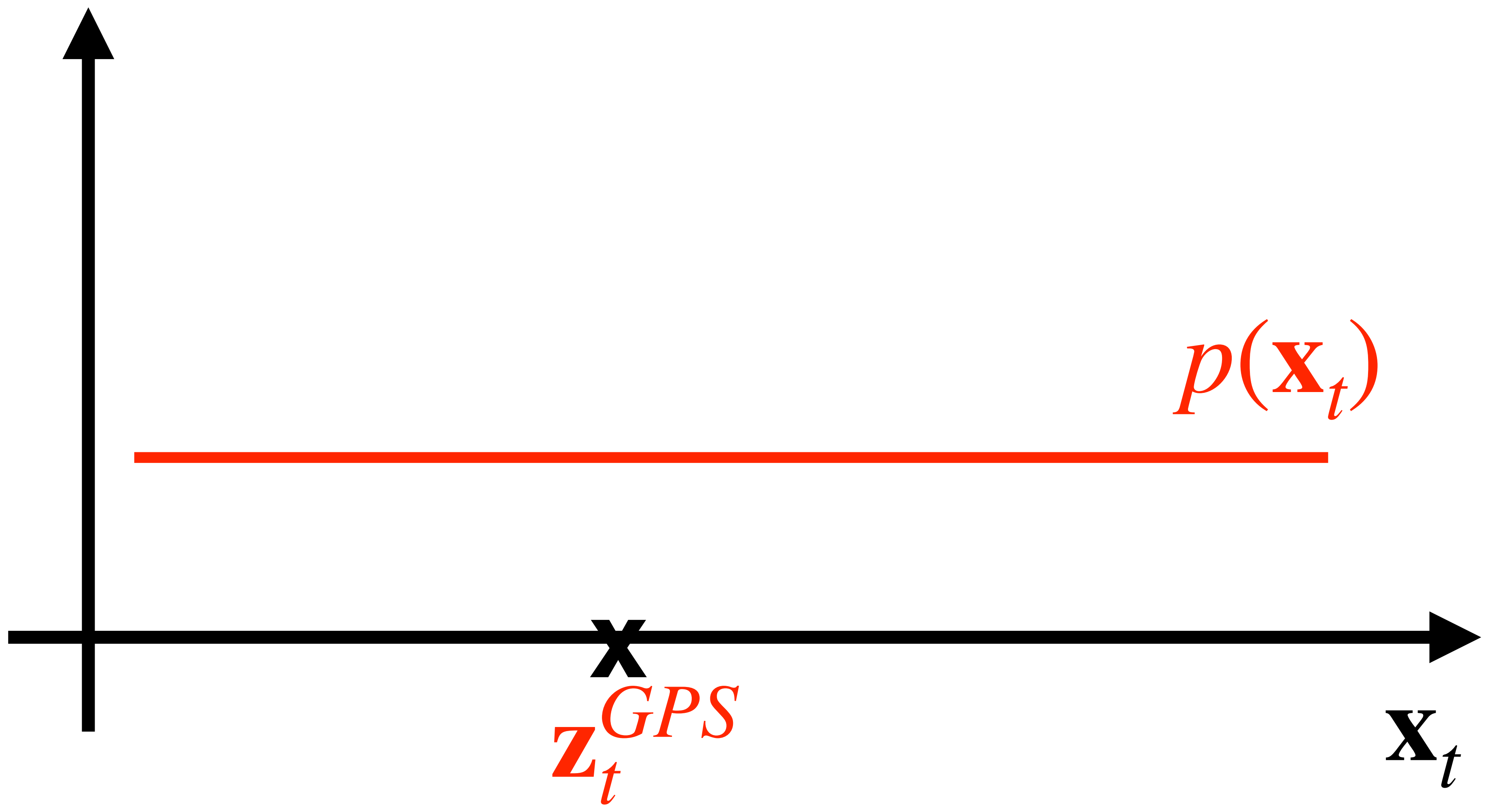
Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

Aposterior pdf

MAP: $\mathbf{x}_t^* = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t) \overset{\text{Assume only gps measurement in time t is known}}{=} \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t^{GPS})$

$$\overset{\text{Bayes theorem}}{=} \arg\max_{\mathbf{x}_t} \frac{\overset{\text{likelihood}}{p(\mathbf{z}_t^{GPS} | \mathbf{x}_t)}\ \overset{\text{prior}}{p(\mathbf{x}_t)}}{\underset{\text{normalization}}{p(\mathbf{z}_t^{GPS})}} \overset{\text{Uniform prior}}{=} \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS} | \mathbf{x}_t)$$



Assumption: $p(\mathbf{z}_t^{GPS} | \mathbf{x}_t) = \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})$

Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

Aposterior pdf

MAP: $\mathbf{x}_t^* = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t) = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t^{GPS})$

Bayes theorem

Uniform prior

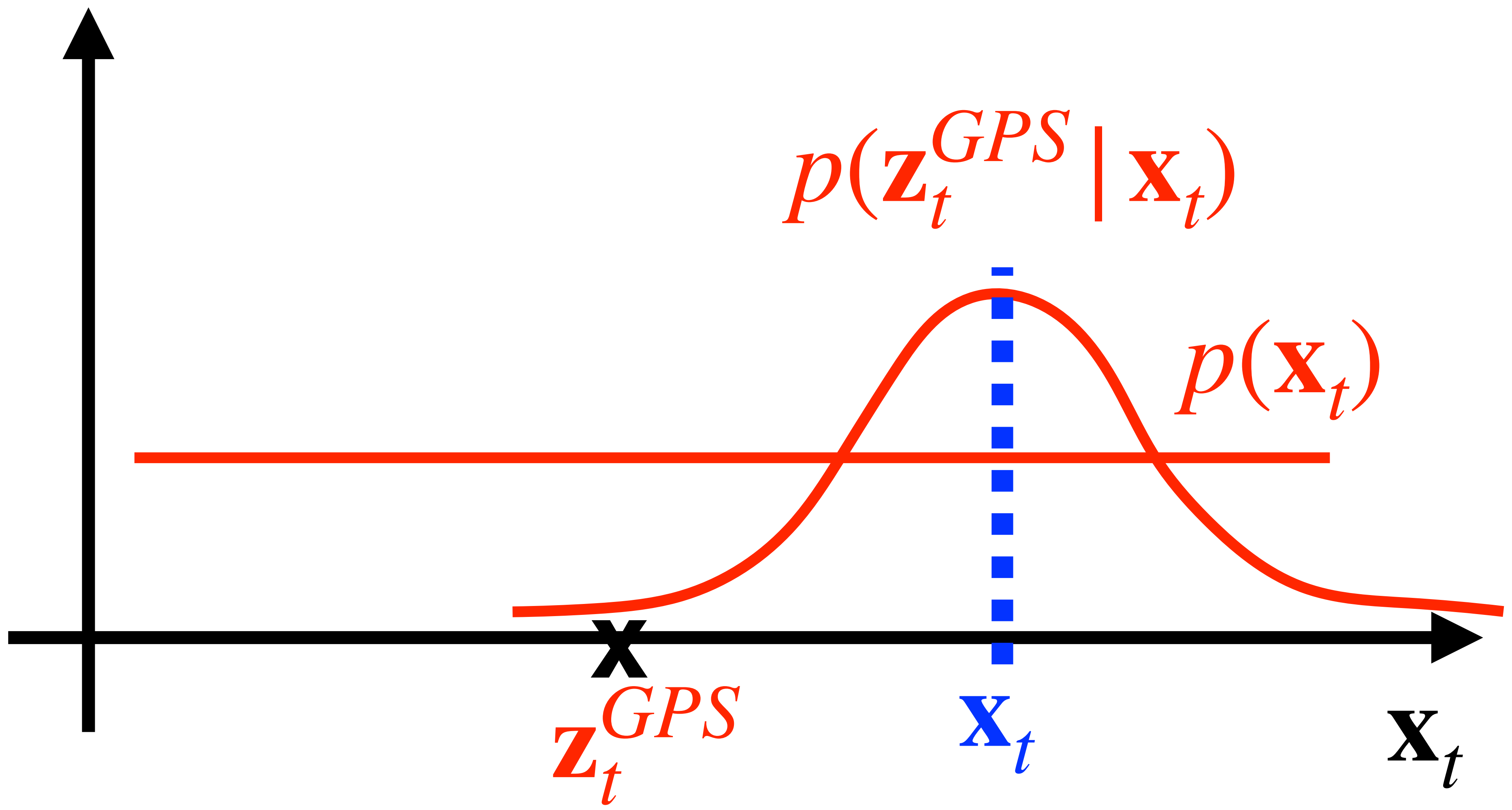
Normal measurement prob. (likelihood)

likelihood prior

$$= \arg\max_{\mathbf{x}_t} \frac{p(\mathbf{z}_t^{GPS} | \mathbf{x}_t)\, p(\mathbf{x}_t)}{p(\mathbf{z}_t^{GPS})} = \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS} | \mathbf{x}_t) = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})$$

normalization

What is the most probable $\mathbf{x}^\star$?



Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

Aposterior pdf

$$\text{MAP: } \mathbf{x}_t^* = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t|\mathbf{z}_t) = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t|\mathbf{z}_t^{GPS})$$

Bayes theorem

Uniform prior

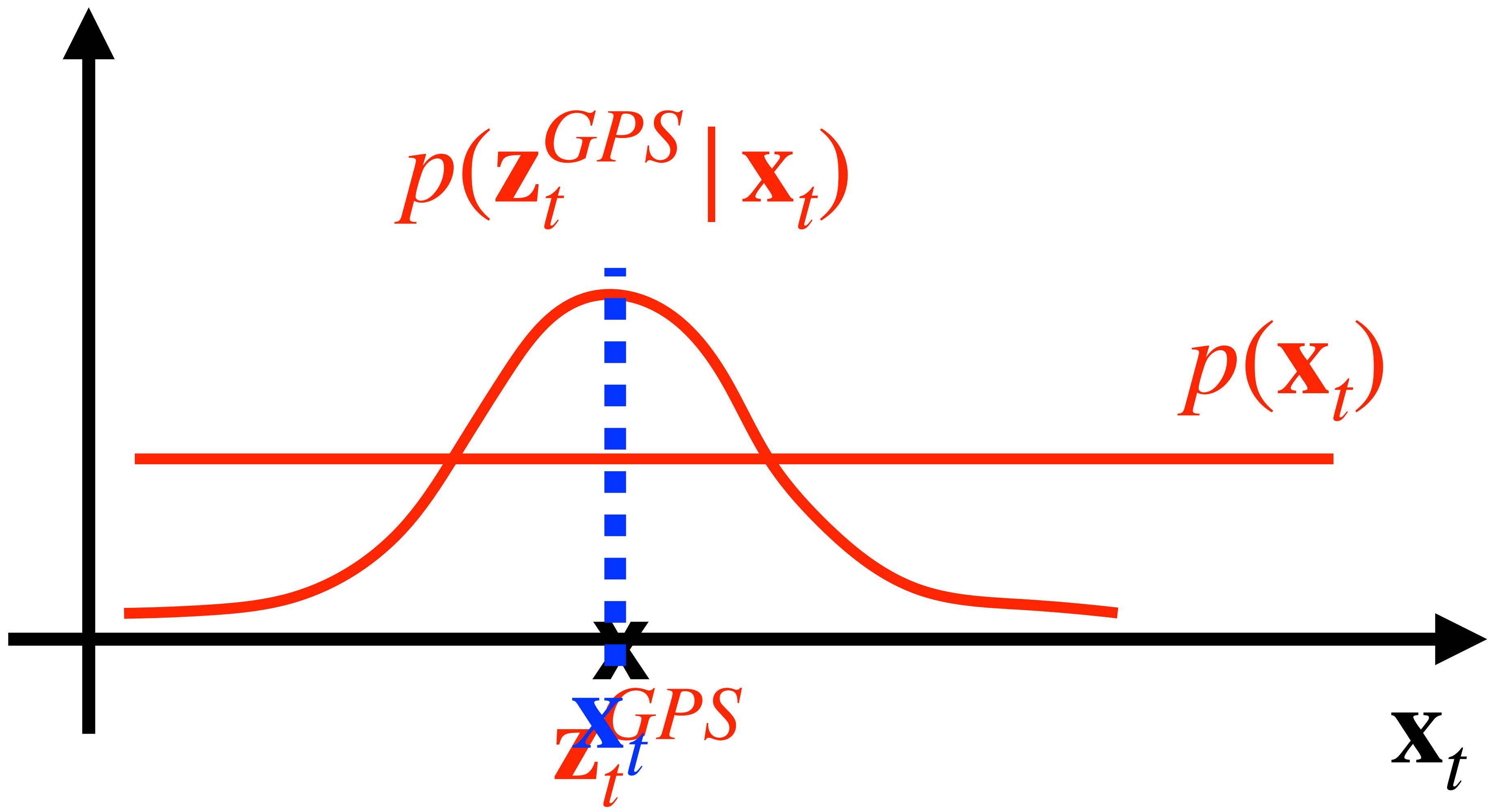
Normal measurement prob. (likelihood)

$$= \arg\max_{\mathbf{x}_t} \frac{\overset{\text{likelihood}}{p(\mathbf{z}_t^{GPS}|\mathbf{x}_t)}\ \overset{\text{prior}}{p(\mathbf{x}_t)}}{\underset{\text{normalization}}{p(\mathbf{z}_t^{GPS})}} = \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS}|\mathbf{x}_t) = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})$$

Measurements $\mathbf{z}_t^{GPS}$ are normally distributed around the true position $\mathbf{x}_t$

Correct way

What is the most probable $\mathbf{x}^\star$?



$$= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{x}_t; \mathbf{z}_t^{GPS}, \Sigma_t^{GPS})$$

True positions $\mathbf{x}_t$ are normally distributed around measurement $\mathbf{z}_t^{GPS}$

Incorrect but visualization friendly way

Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

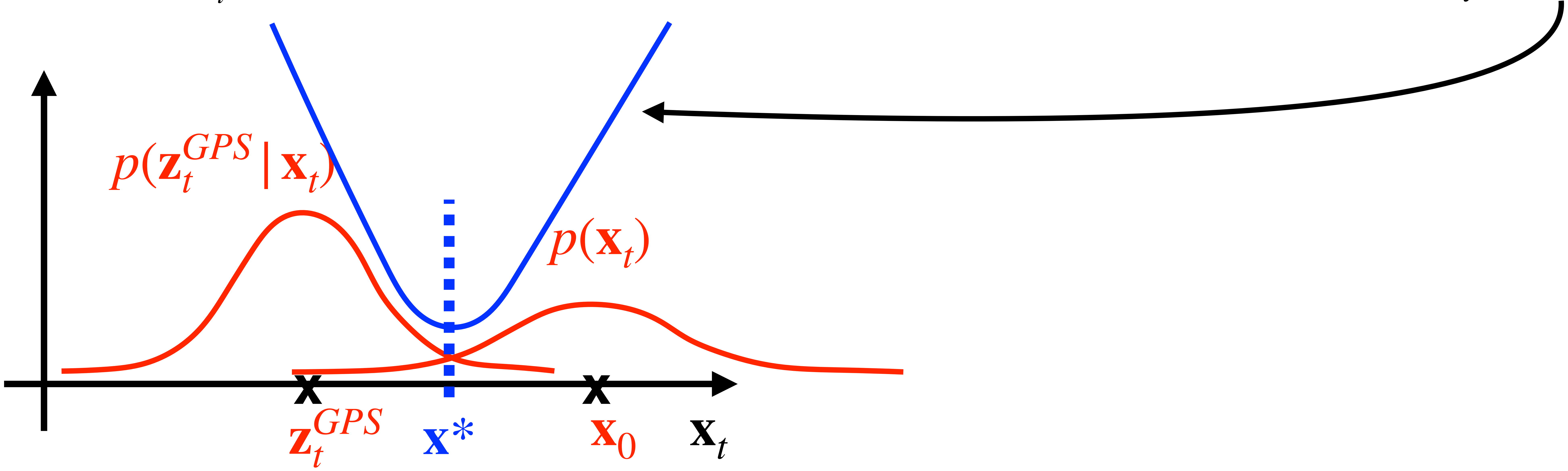
$$\text{MAP:}\quad \mathbf{x}_t^* = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t) \quad = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t | \mathbf{z}_t^{GPS})$$

Bayes theorem

Normal prior and likelihood

$$= \arg\max_{\mathbf{x}_t} \frac{p(\mathbf{z}_t^{GPS} | \mathbf{x}_t)\, p(\mathbf{x}_t)}{p(\mathbf{z}_t^{GPS})} \quad = \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS} | \mathbf{x}_t) p(\mathbf{x}_t)$$

$$= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS}) \mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \quad = \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^2 \frac{1}{\Sigma_t^{GPS}} + (\mathbf{x}_t - \mathbf{x}_0)^2 \frac{1}{\Sigma_0}$$

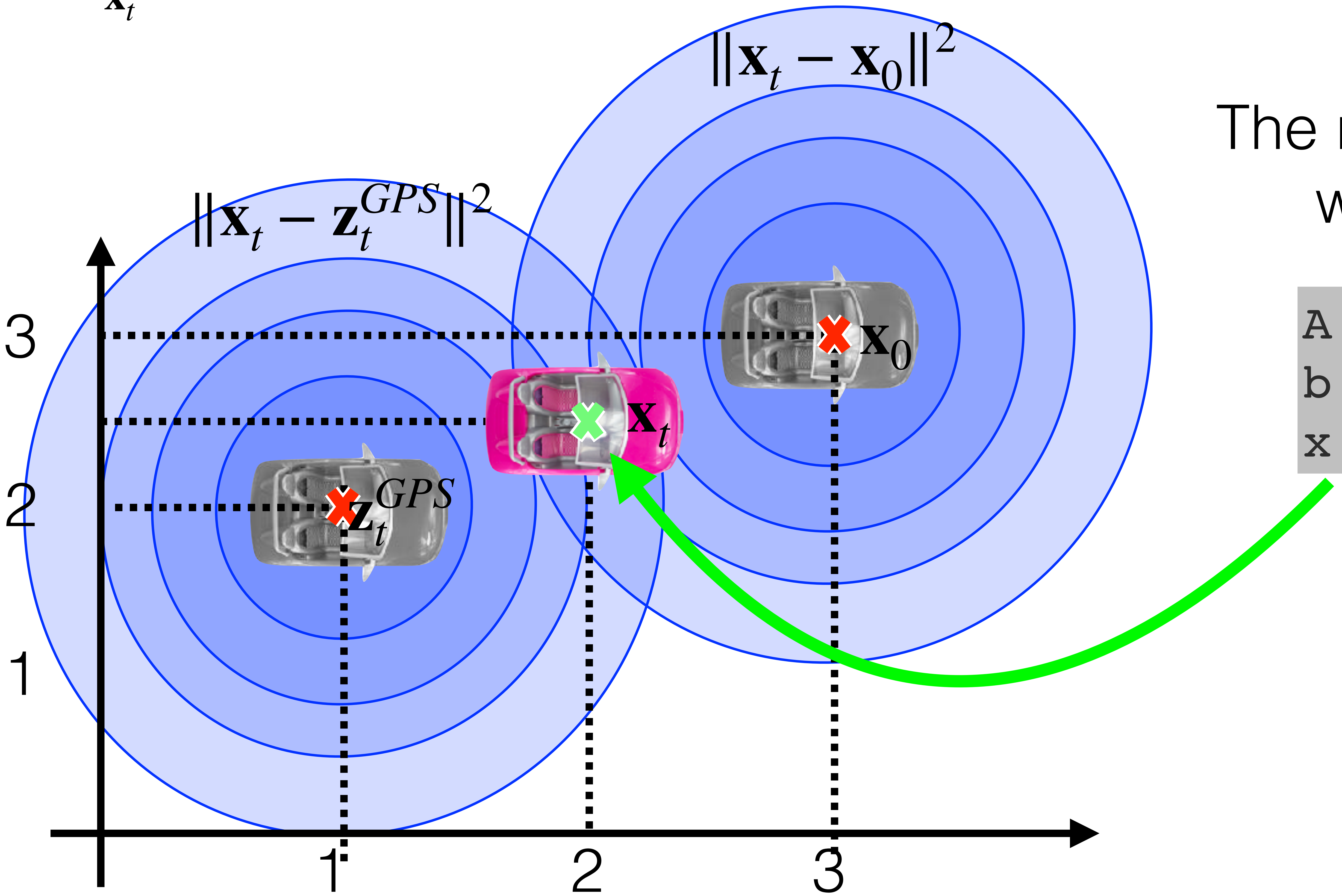


Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$
\begin{aligned}
\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \\
&= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}} (\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1} (\mathbf{x}_t - \mathbf{x}_0) \\
&= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2
\end{aligned}
$$

$$\mathbf{z}_t^{GPS} = [1,2]^\top, \ \ \mathbf{x}_0 = [3,3]^\top$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



The result is linear least squares with closed-form solution

```
A = [1,0; 0,1; 1,0; 0,1]
b = [1; 2; 3; 3]
x = pinv(A)*b
```

Example: 2D Localisation from **GPS** measurements only in **single time instance**

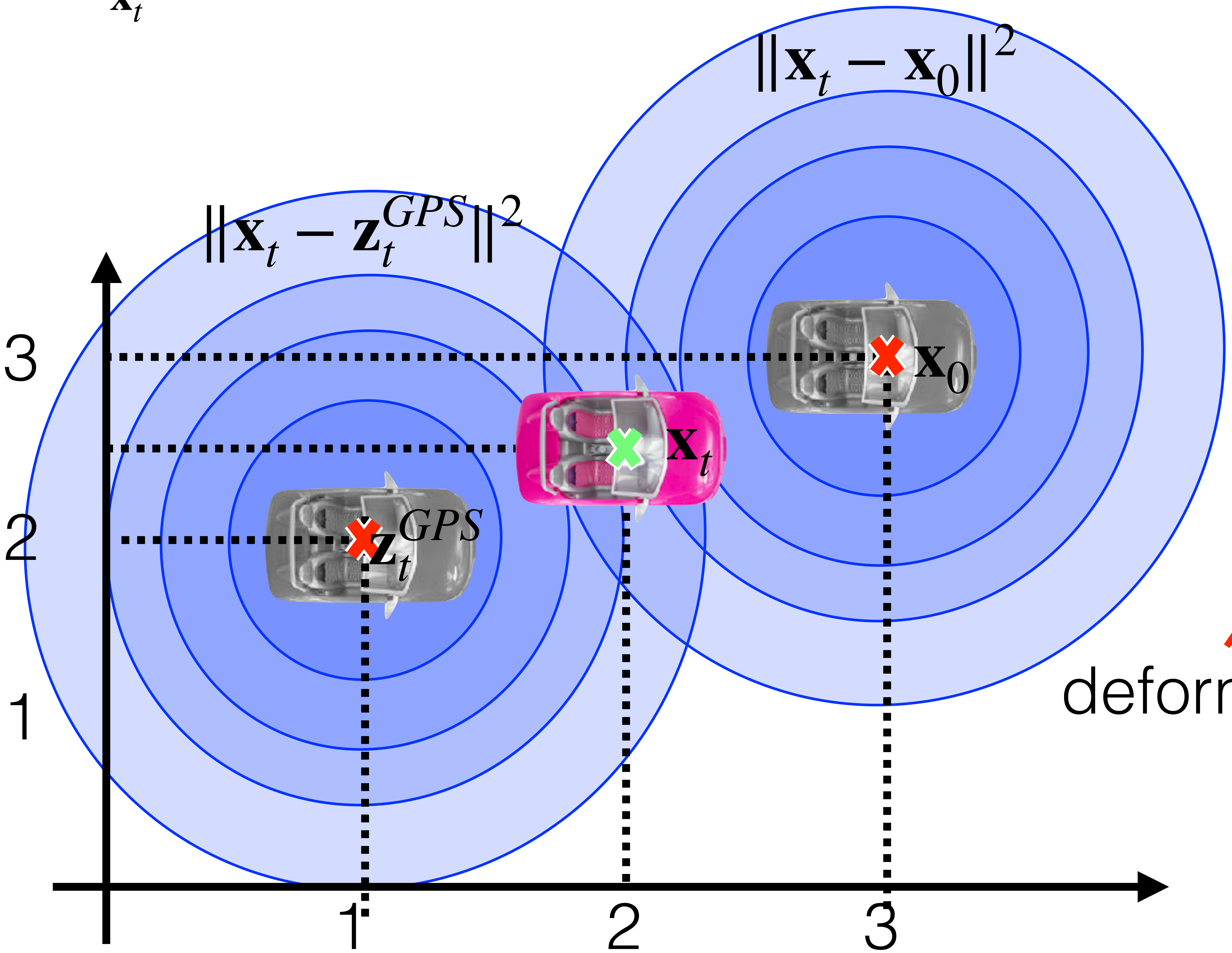
$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0)$$

$$= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}}(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1}(\mathbf{x}_t - \mathbf{x}_0)$$

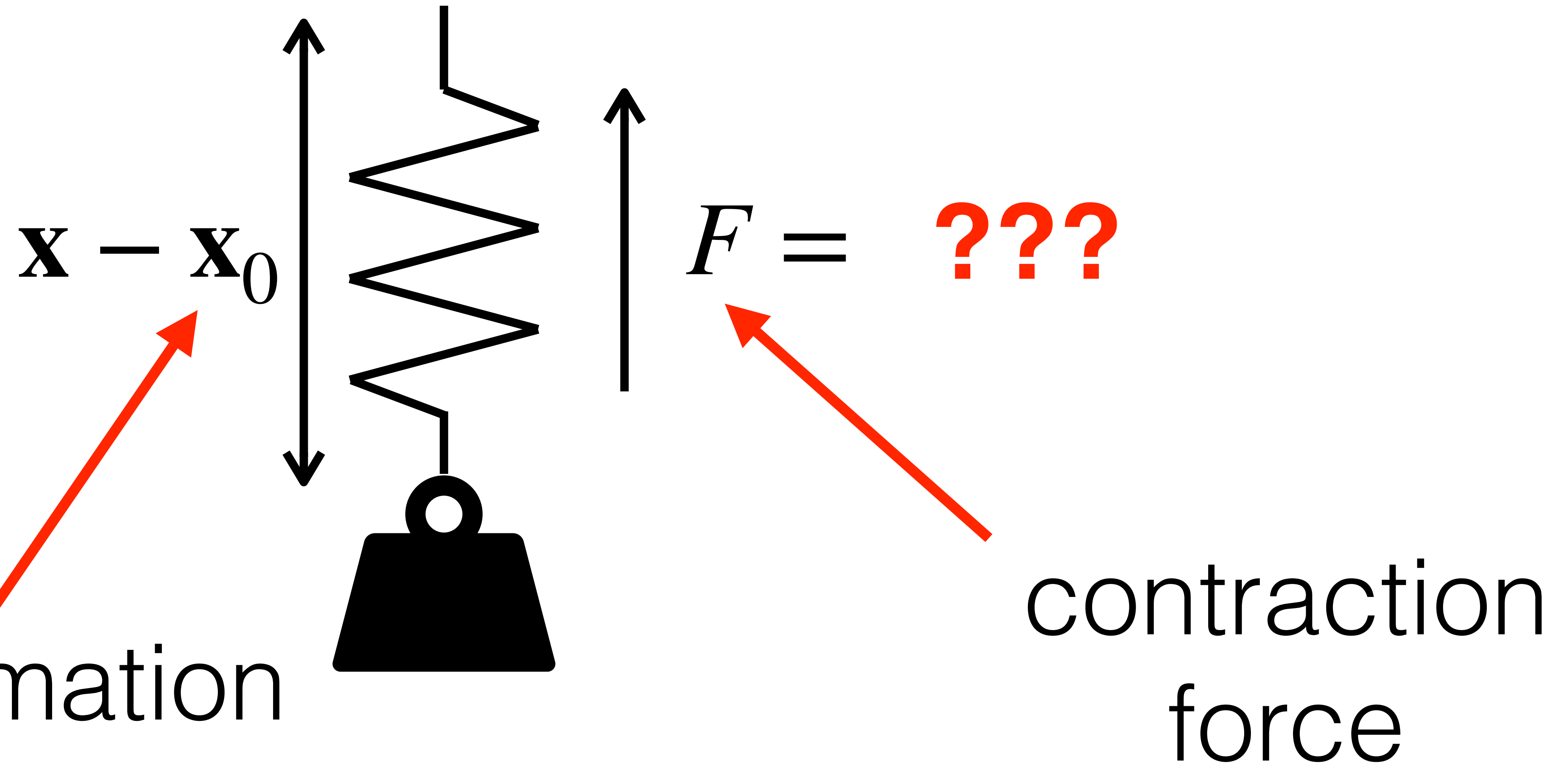
$$= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2$$

$$\mathbf{z}_t^{GPS} = [1,2]^\top, \quad \mathbf{x}_0 = [3,3]^\top$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Who remembers Hook's law of an ideal spring?



Example: 2D Localisation from **GPS** measurements only in **single time instance**

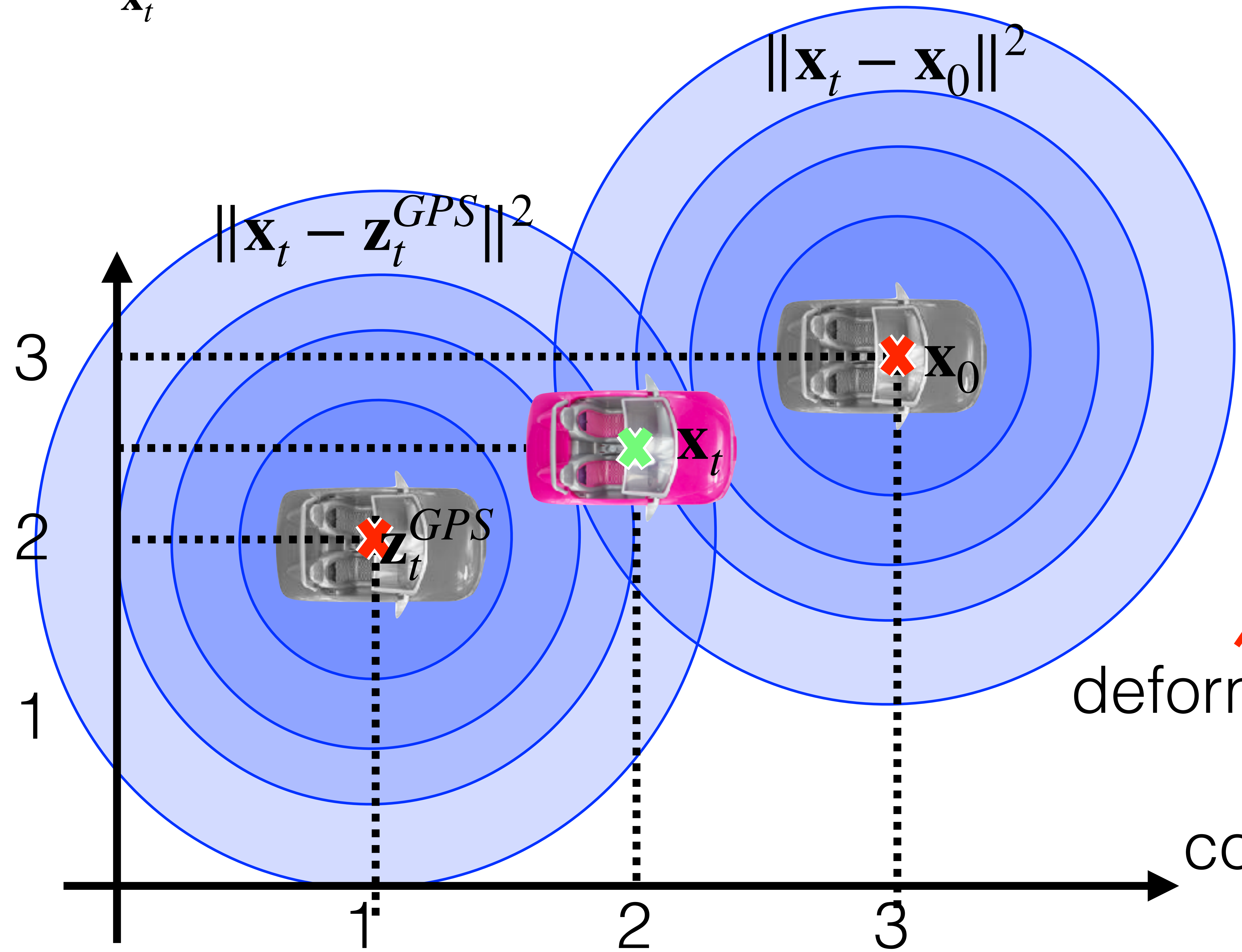
$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0)$$

$$= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}}(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1}(\mathbf{x}_t - \mathbf{x}_0)$$

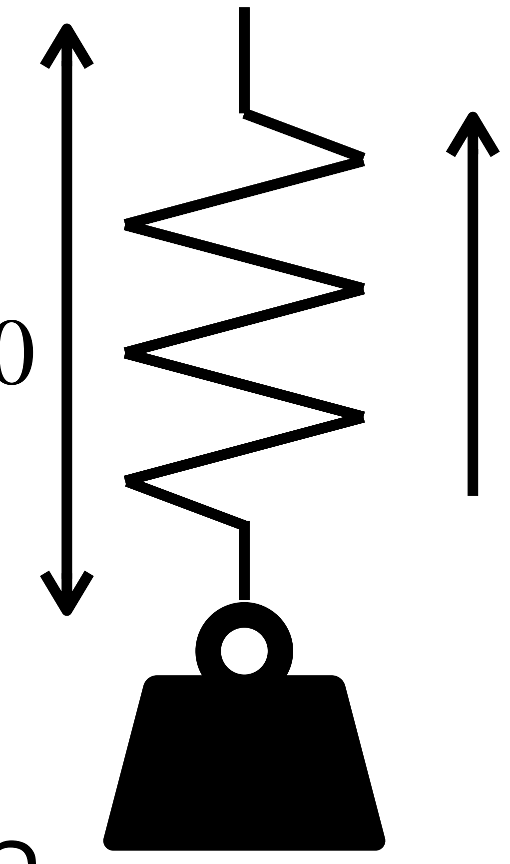
$$= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2$$

$$\mathbf{z}_t^{GPS} = [1,2]^\top, \quad \mathbf{x}_0 = [3,3]$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Who remembers Hook's law of an ideal spring?



$\mathbf{x} - \mathbf{x}_0$

$F = k \cdot (\mathbf{x} - \mathbf{x}_0)$

deformation

contraction force

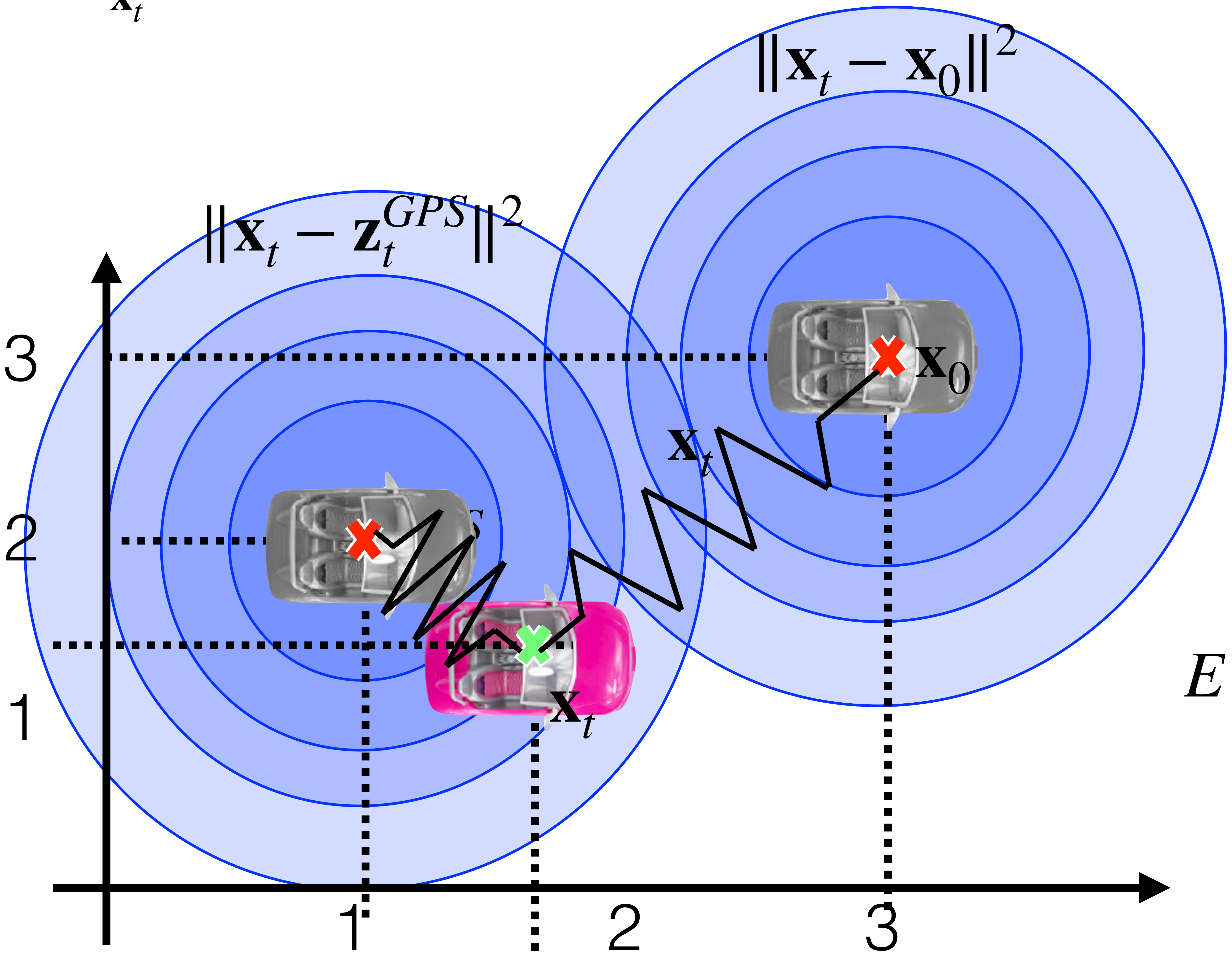
conserved energy: $E = \frac{1}{2} \cdot k \cdot (\mathbf{x} - \mathbf{x}_0)^2$

Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\begin{aligned}\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \\ &= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}}(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1}(\mathbf{x}_t - \mathbf{x}_0) \\ &= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2\end{aligned}$$

$$\mathbf{z}_t^{GPS} = [1,2]^\top, \quad \mathbf{x}_0 = [3,3]^\top$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Least squares solution

Equilibrium of mechanical machine
(i.e. state with minimum energy)

conserved energy:

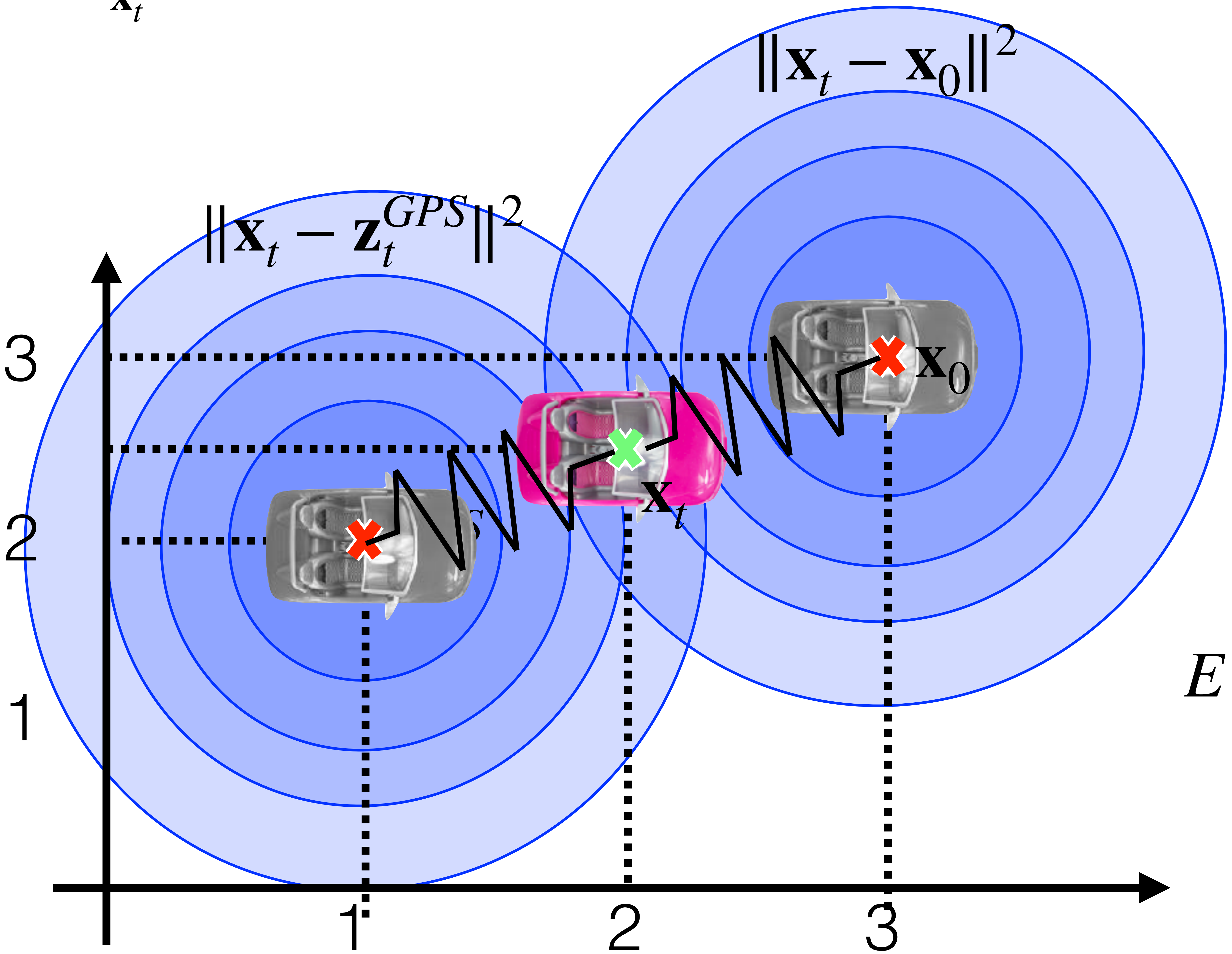
$$E = \frac{1}{2} \cdot k \cdot \|\mathbf{x}_t - \mathbf{x}_0\|^2 + \frac{1}{2} \cdot k \cdot \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2$$

Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$
\begin{aligned}
\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \\
&= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}} (\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1} (\mathbf{x}_t - \mathbf{x}_0) \\
&= \arg\min_{\mathbf{x}_t} \; \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2
\end{aligned}
$$

$$\mathbf{z}_t^{GPS} = [1{,}2]^\top, \;\; \mathbf{x}_0 = [3{,}3]^\top$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Least squares solution

⇕

Equilibrium of mechanical machine
(i.e. state with minimum energy)

conserved energy:

$$E = \frac{1}{2} \cdot k \cdot \|\mathbf{x}_t - \mathbf{x}_0\|^2 + \frac{1}{2} \cdot k \cdot \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2$$

Example: 2D Localisation from **GPS** measurements only in **single time instance**

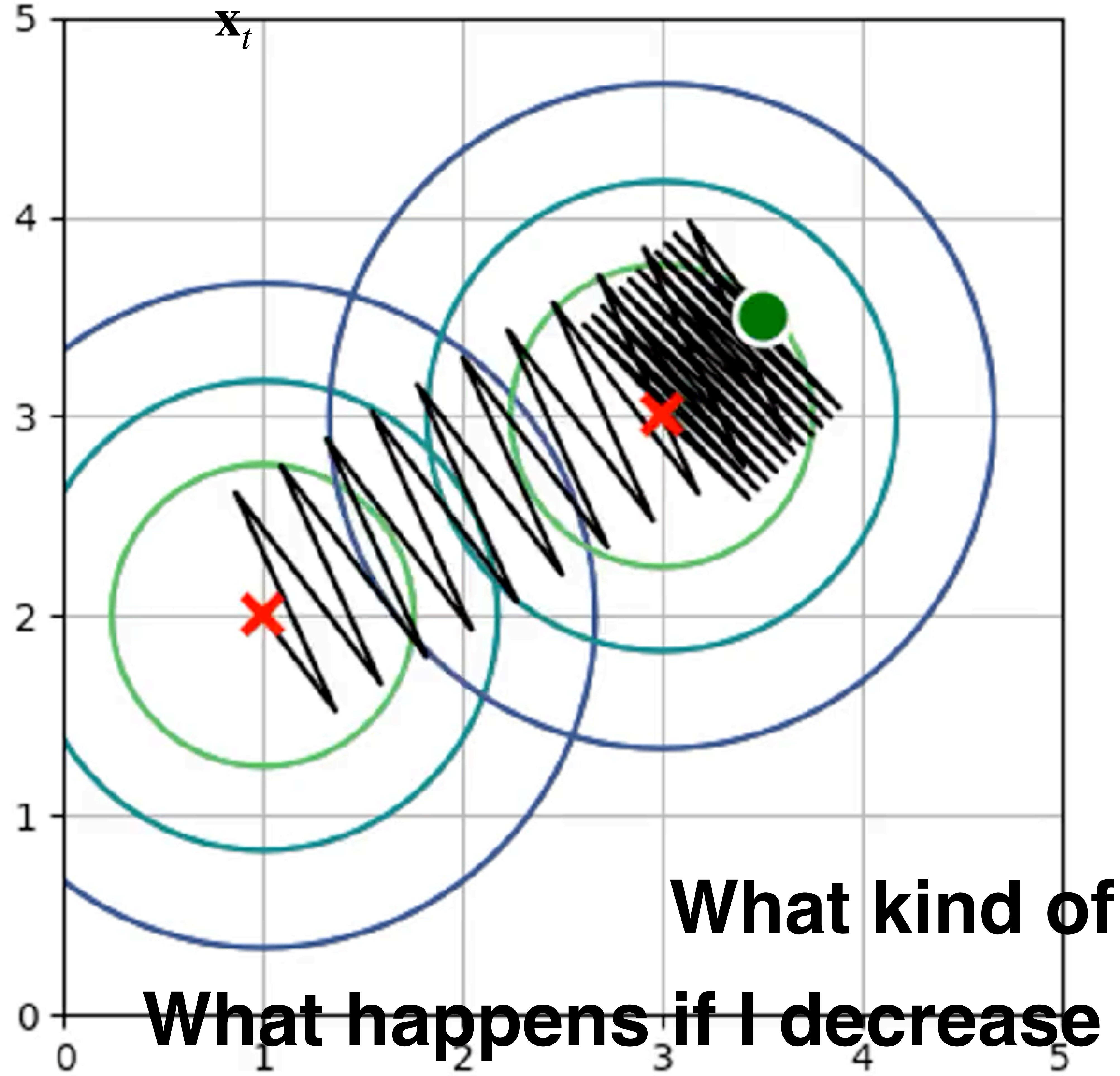
$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0)$$

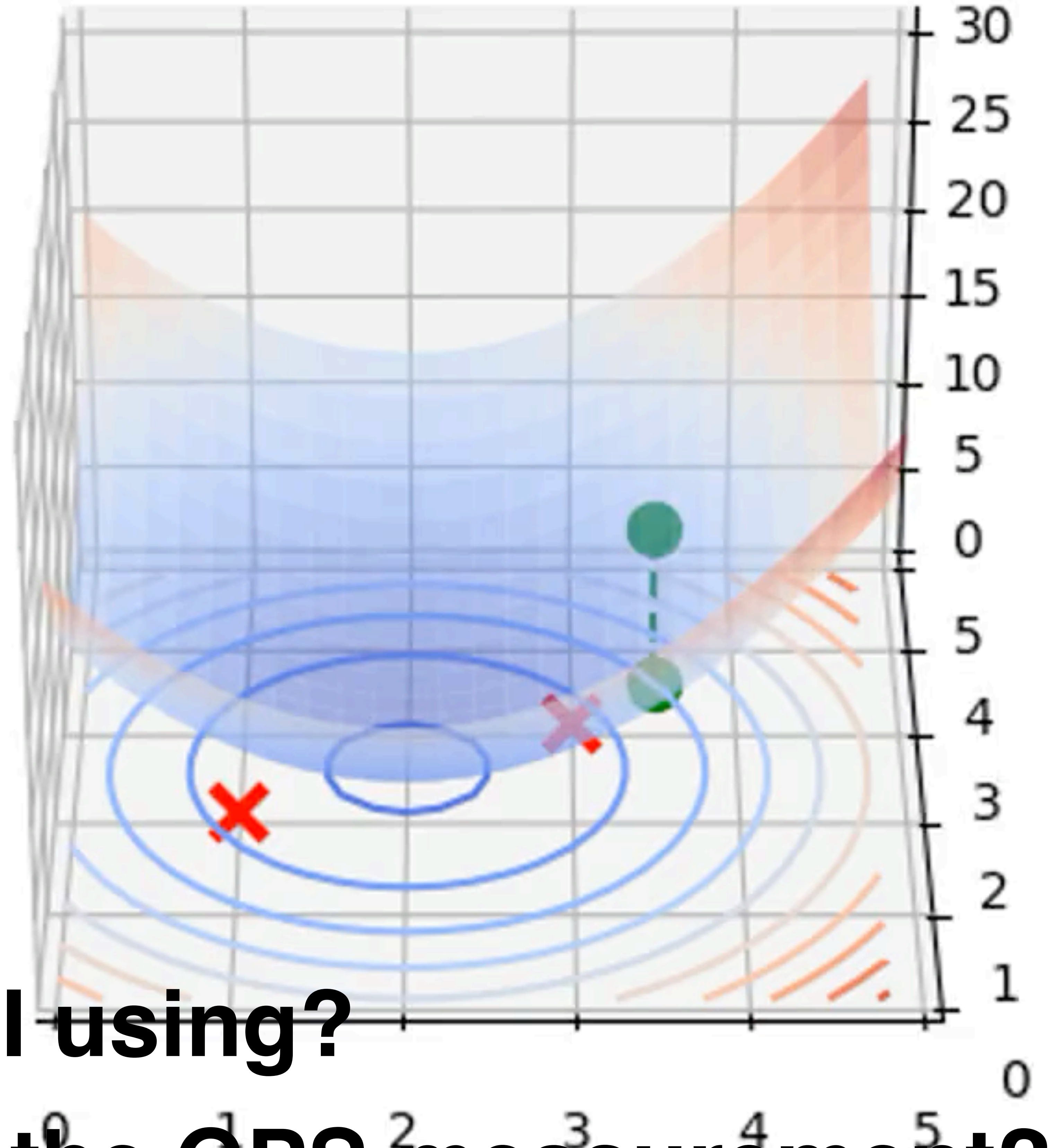
$$= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}} (\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1} (\mathbf{x}_t - \mathbf{x}_0)$$

$$= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2$$

$$\Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Sigma_t^{GPS} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \frac{1}{2} \cdot 1 \cdot \|\mathbf{x}_t - \mathbf{x}_0\|^2 + \frac{1}{2} \cdot 1 \cdot \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2$$





What kind of optimizer am I using?

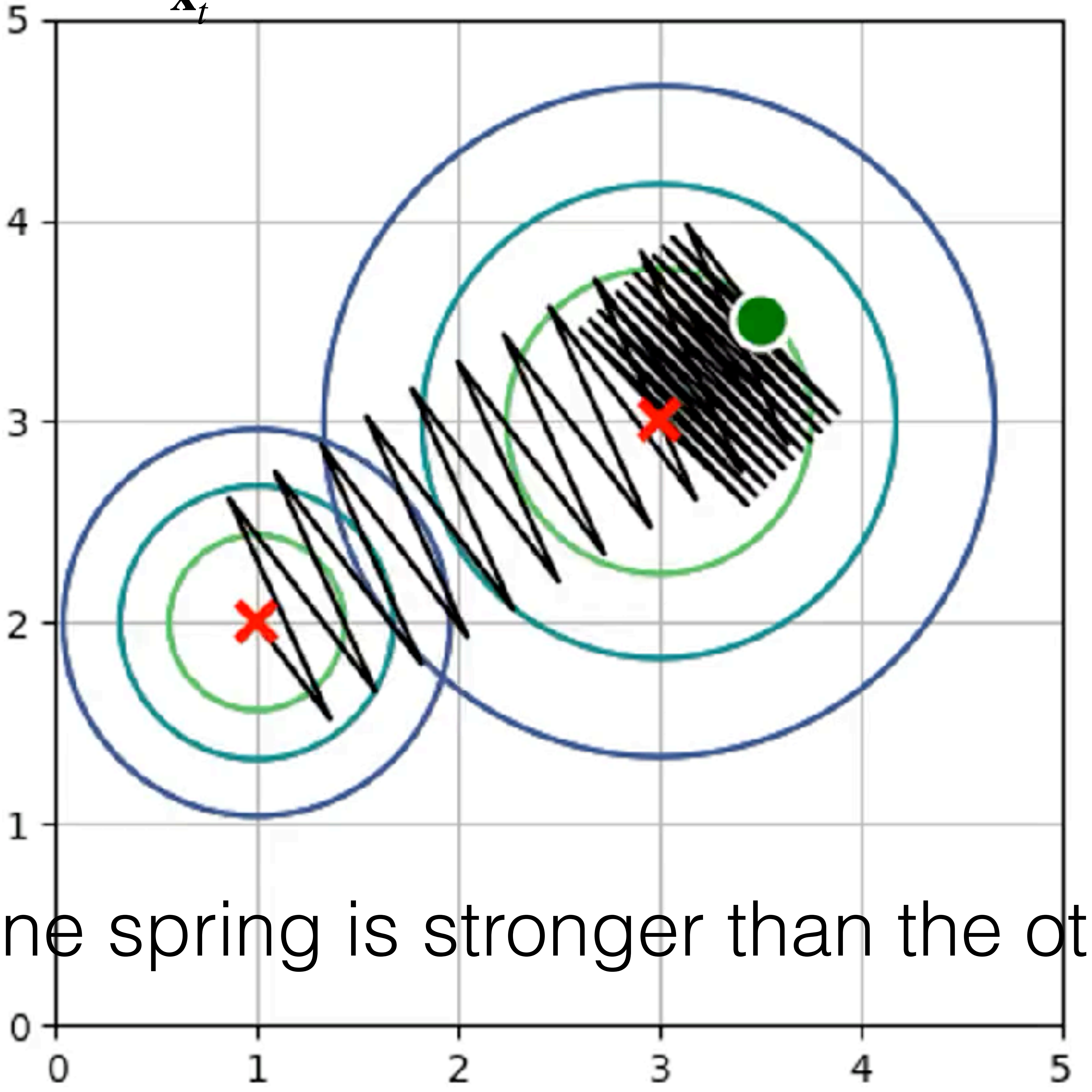
What happens if I decrease covariance of the GPS measurement?

Example: 2D Localisation from **GPS** measurements only in **single time instance**

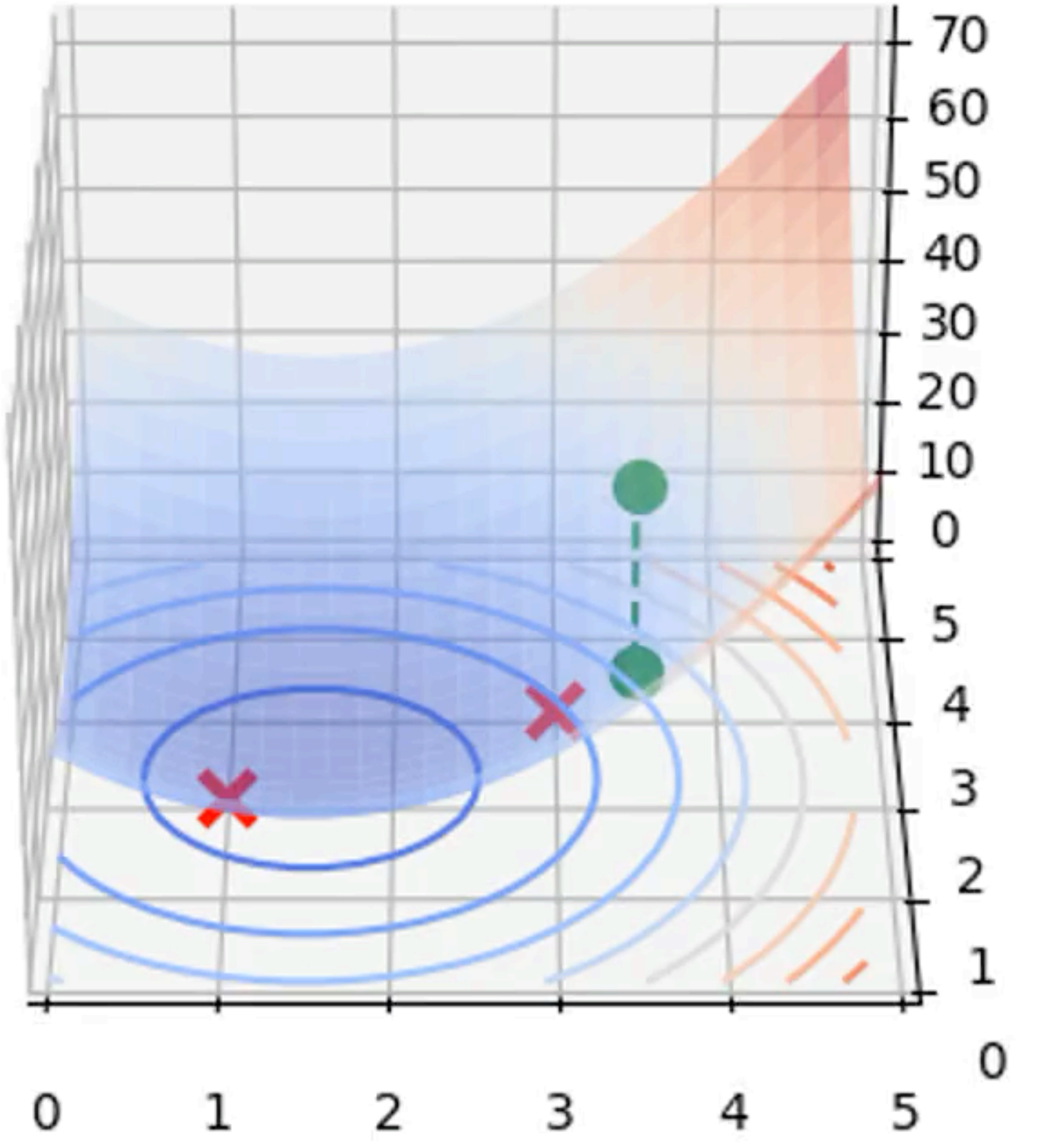
$$\begin{aligned}\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \\ &= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top \Sigma_t^{GPS^{-1}} (\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1} (\mathbf{x}_t - \mathbf{x}_0) \\ &= \arg\min_{\mathbf{x}_t} 3\|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2\end{aligned}$$

$$\Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma_t^{GPS} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$E = \frac{1}{2} \cdot 1 \cdot \|\mathbf{x}_t - \mathbf{x}_0\|^2 + \frac{1}{2} \cdot 3 \cdot \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2$$



One spring is stronger than the other

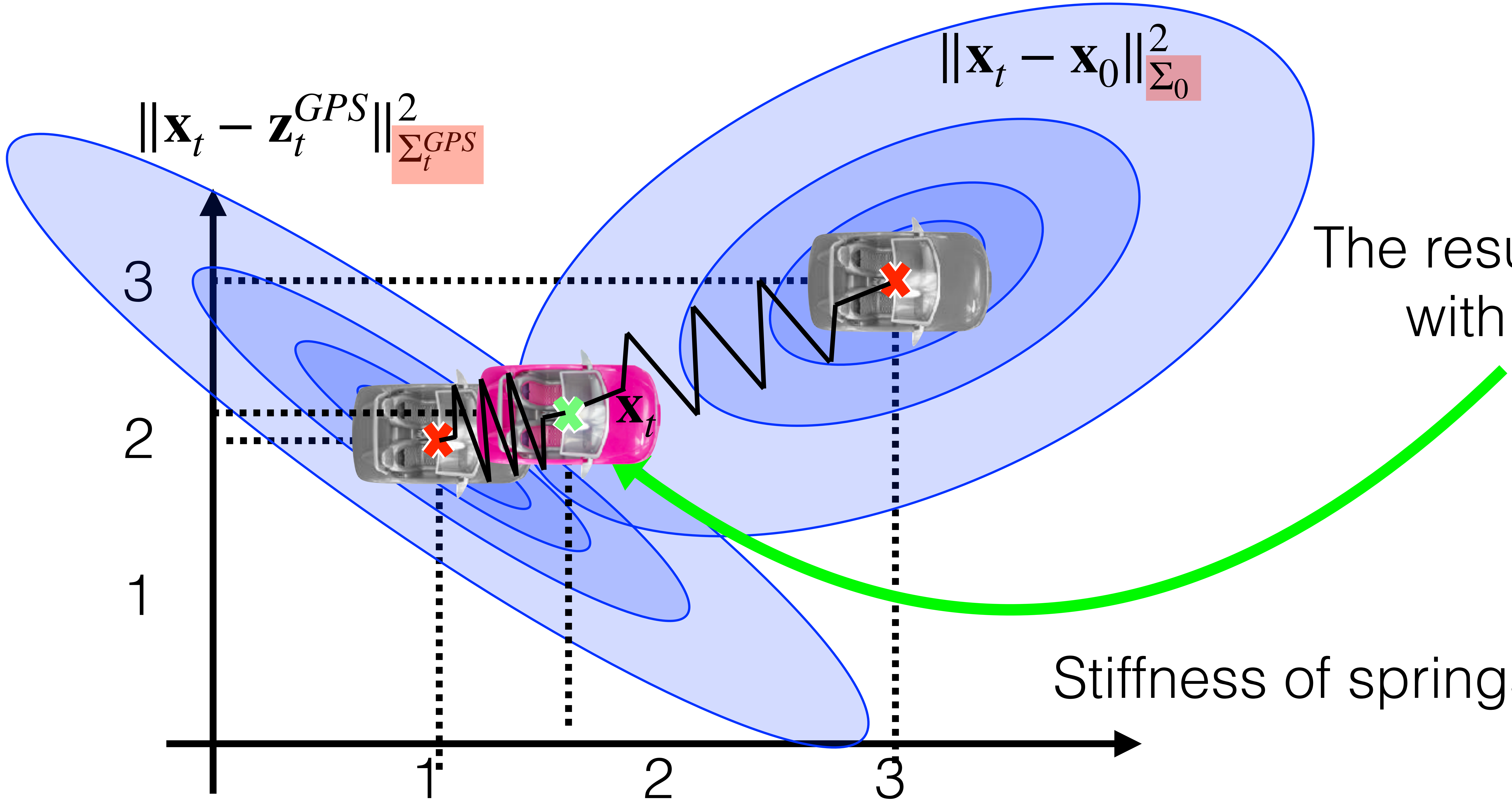


Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\begin{aligned}\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0)\\ &= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top {\Sigma_t^{GPS}}^{-1}(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top \Sigma_0^{-1}(\mathbf{x}_t - \mathbf{x}_0)\\ &= \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}} + \|\mathbf{x}_t - \mathbf{x}_0\|^2_{\Sigma_0}\end{aligned}$$

$\mathbf{z}_t^{GPS} = [1,2]^\top$, $\mathbf{x}_0 = [3,3]^\top$

Σ_t^{GPS}, Σ_0



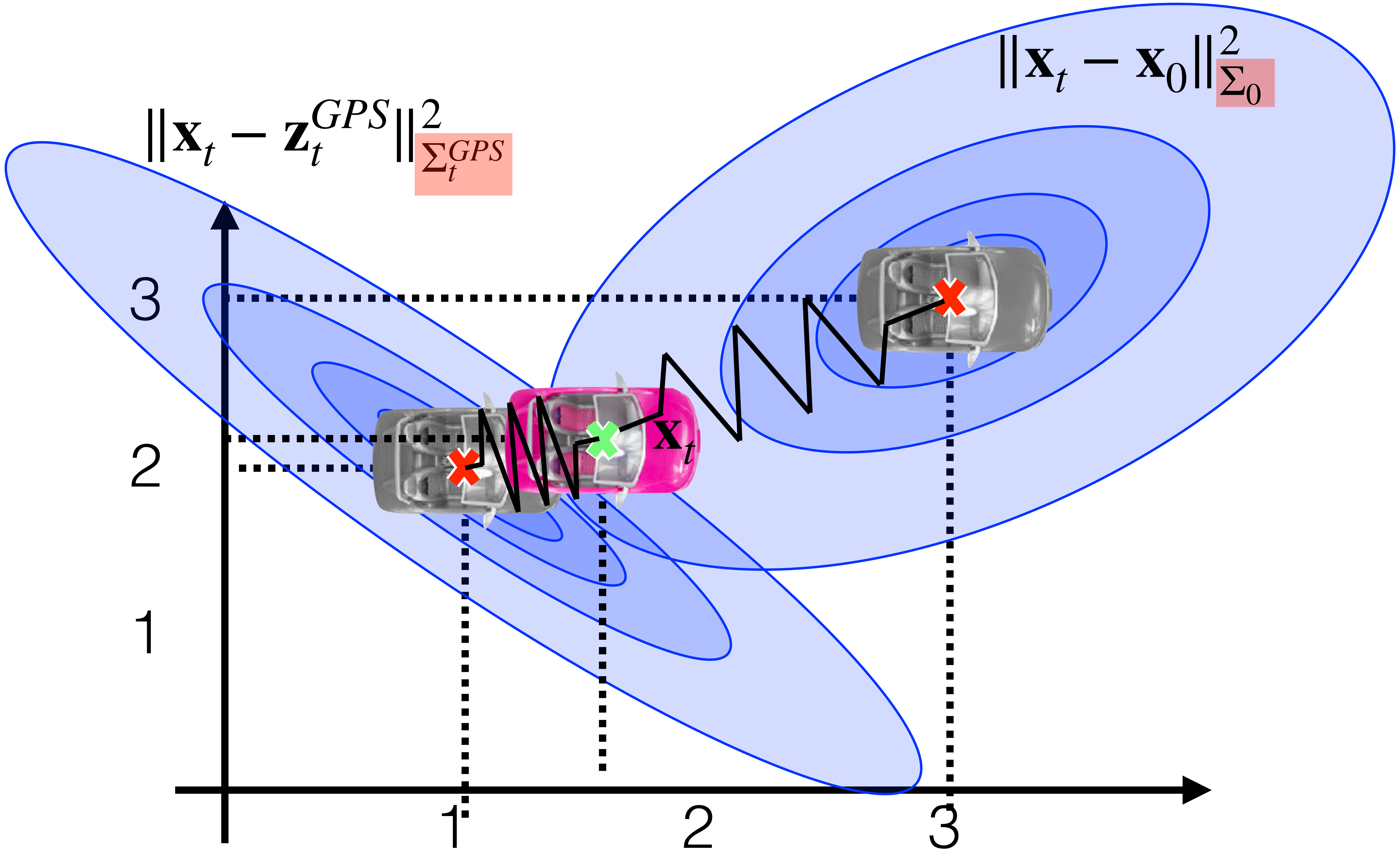
The result is linear least squares with closed-form solution

Stiffness of springs is orientation-specific

Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_t} \; \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}} + \|\mathbf{x}_t - \mathbf{x}_0\|^2_{\Sigma_0}$$

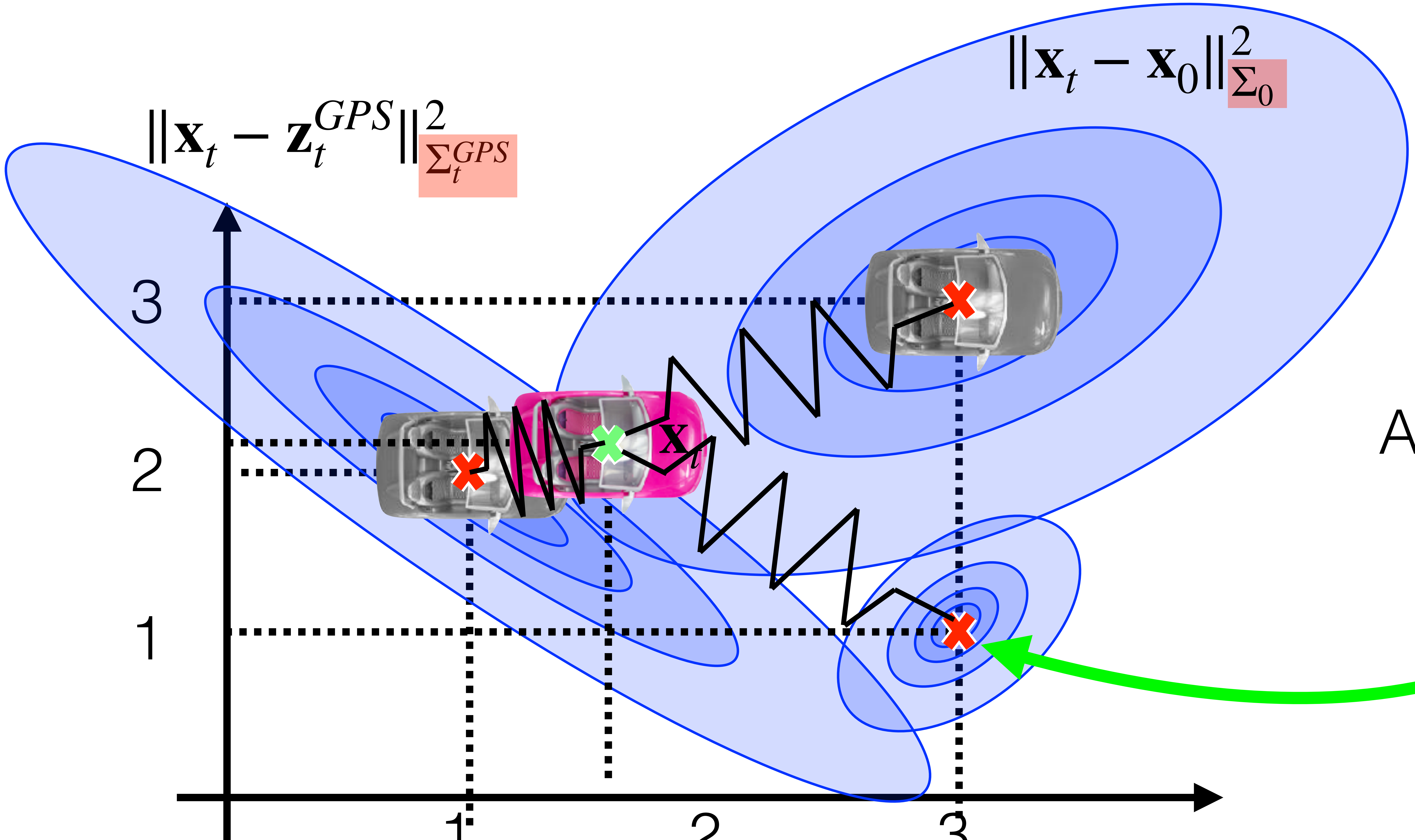
Two terms => two springs with orientation-dependent stiffness



Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_t} \ \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}} + \|\mathbf{x}_t - \mathbf{x}_0\|^2_{\Sigma_0}$$

Two terms => two springs with orientation-dependent stiffness

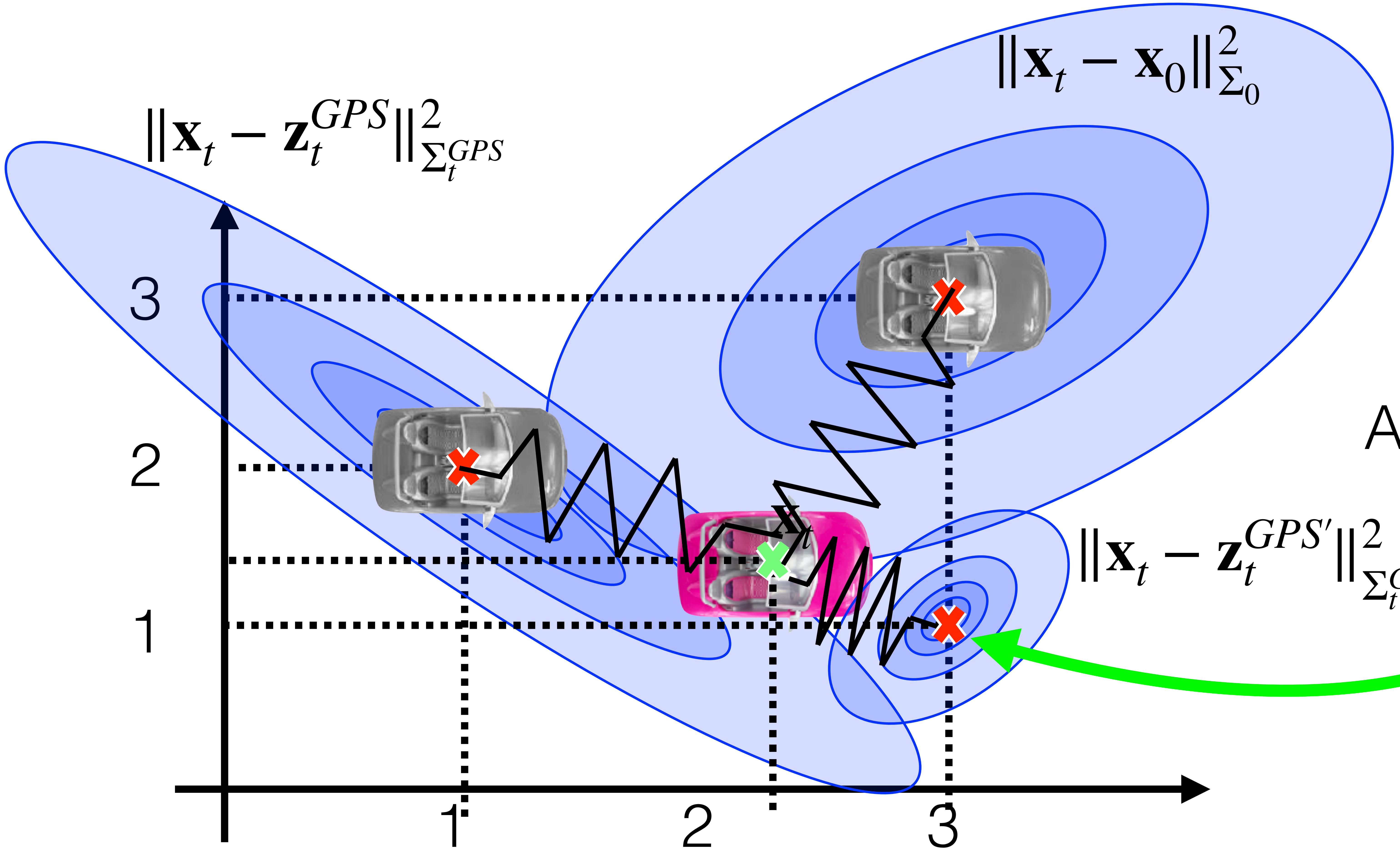


Adding more measurements …

Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\mathbf{x}^{\star} = \arg\min_{\mathbf{x}_t} \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}} + \|\mathbf{x}_t - \mathbf{x}_0\|^2_{\Sigma_0} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS'}\|^2_{\Sigma_t^{GPS'}}$$

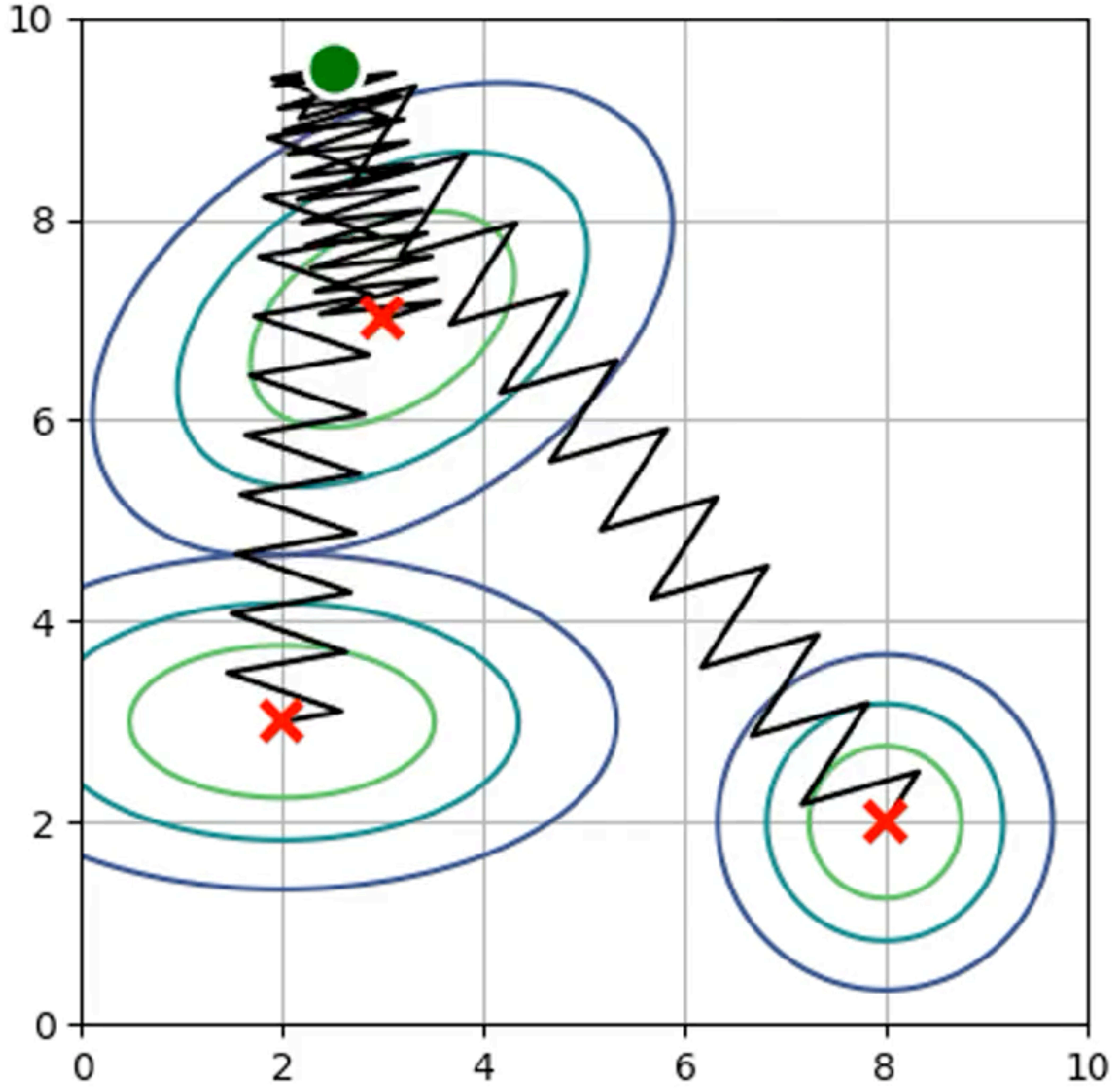
Three terms => three springs with orientation-dependent stiffness

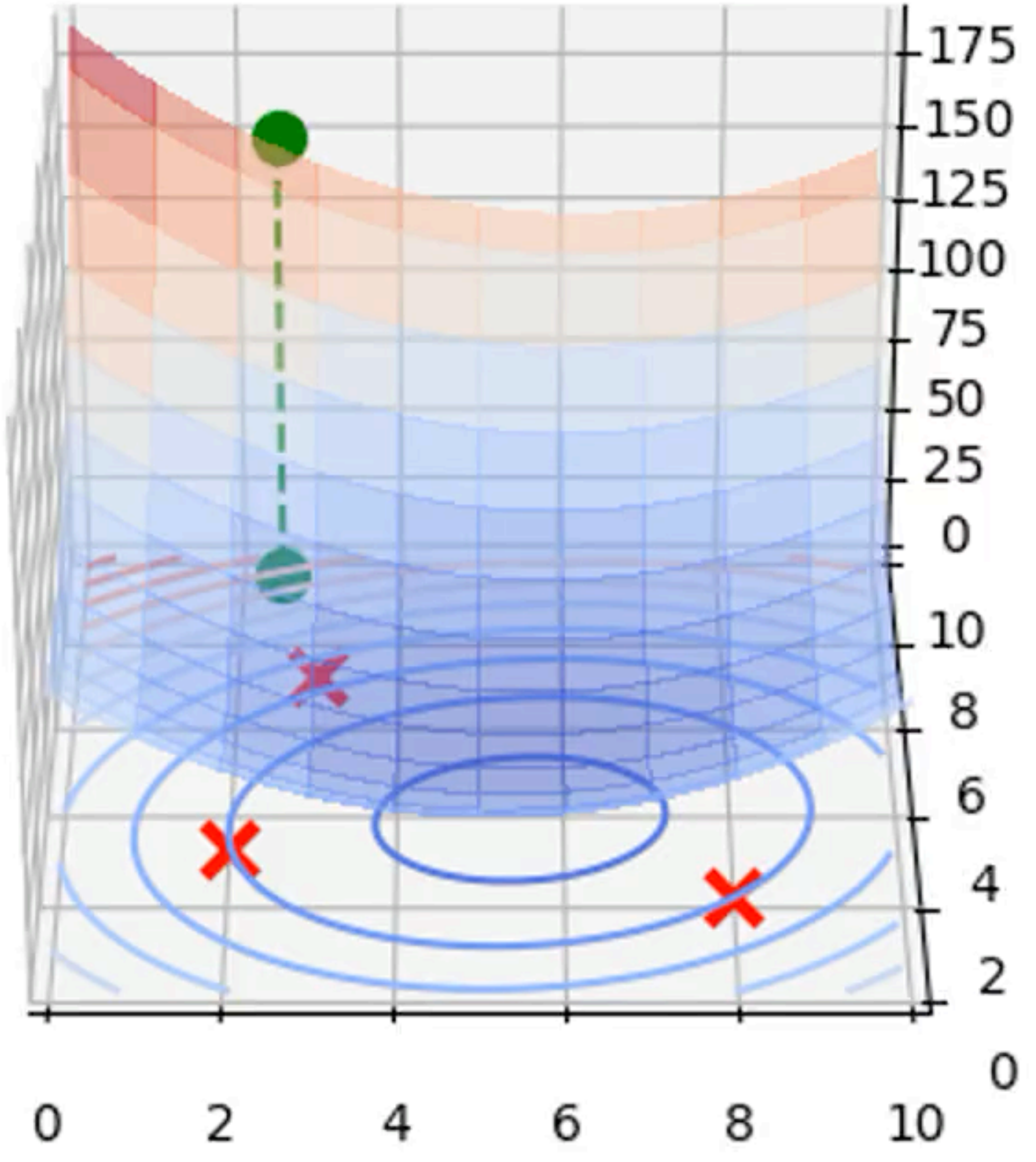


Adding more measurements …

Satelite navigation (GPS/GNSS)

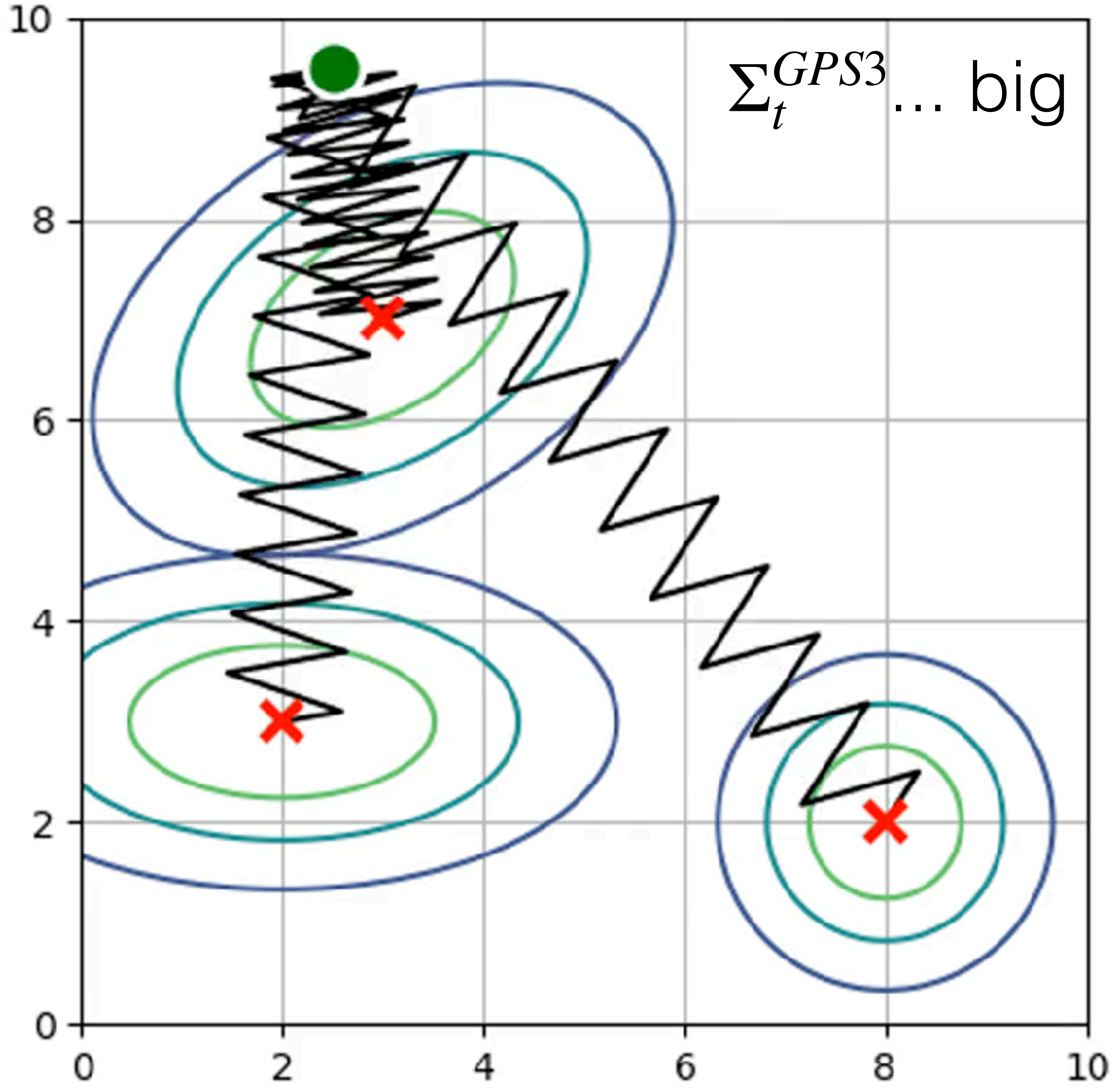
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_t} \; \|\mathbf{x}_t - \mathbf{z}_t^{GPS1}\|^2_{\Sigma_t^{GPS1}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS2}\|^2_{\Sigma_t^{GPS2}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS3}\|^2_{\Sigma_t^{GPS3}}$$



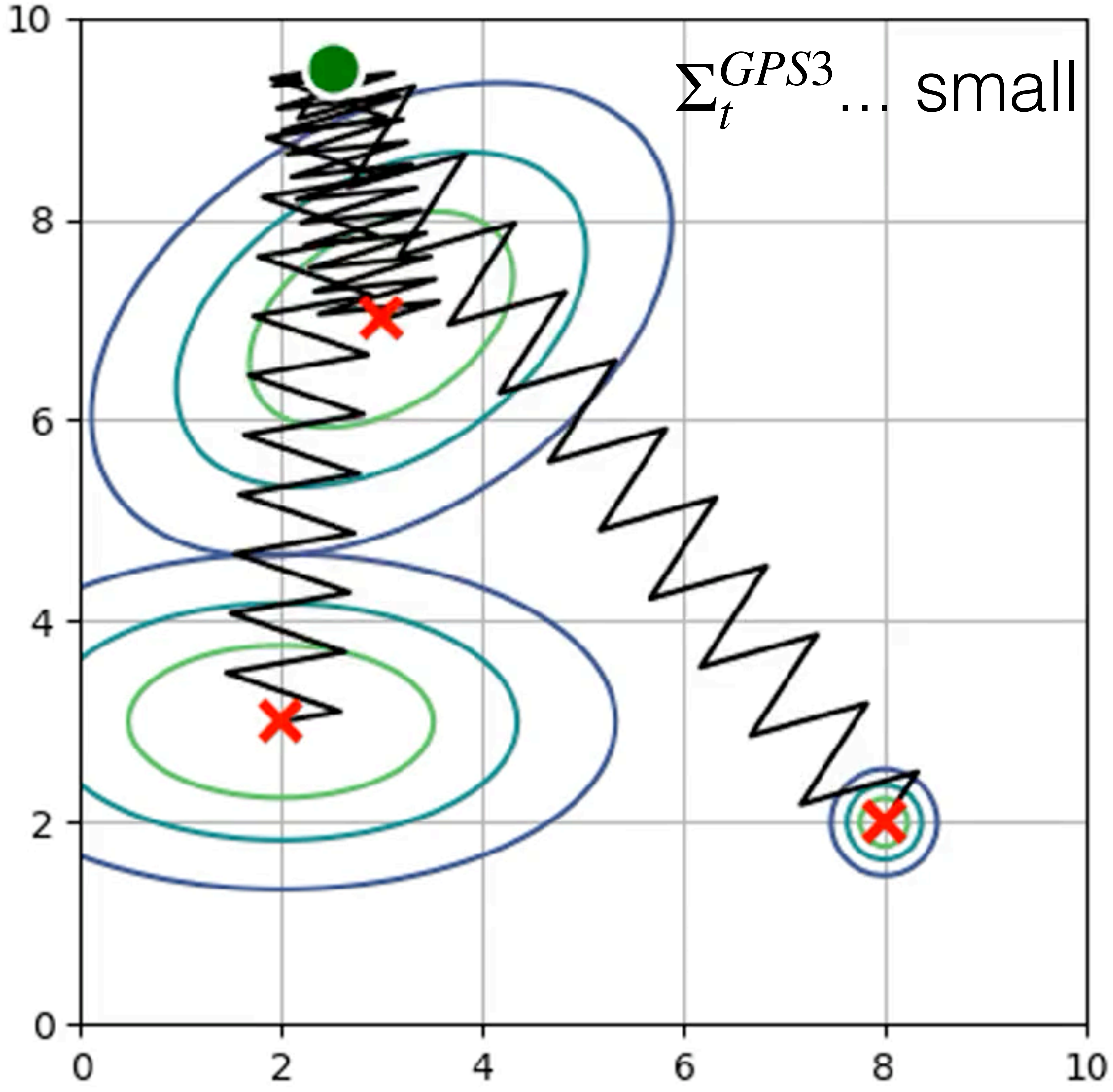


Satelite navigation (GPS/GNSS)

$$\mathbf{x}^{\star} = \arg\min_{\mathbf{x}_t} \ \|\mathbf{x}_t - \mathbf{z}_t^{GPS1}\|^2_{\Sigma_t^{GPS1}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS2}\|^2_{\Sigma_t^{GPS2}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS3}\|^2_{\Sigma_t^{GPS3}}$$



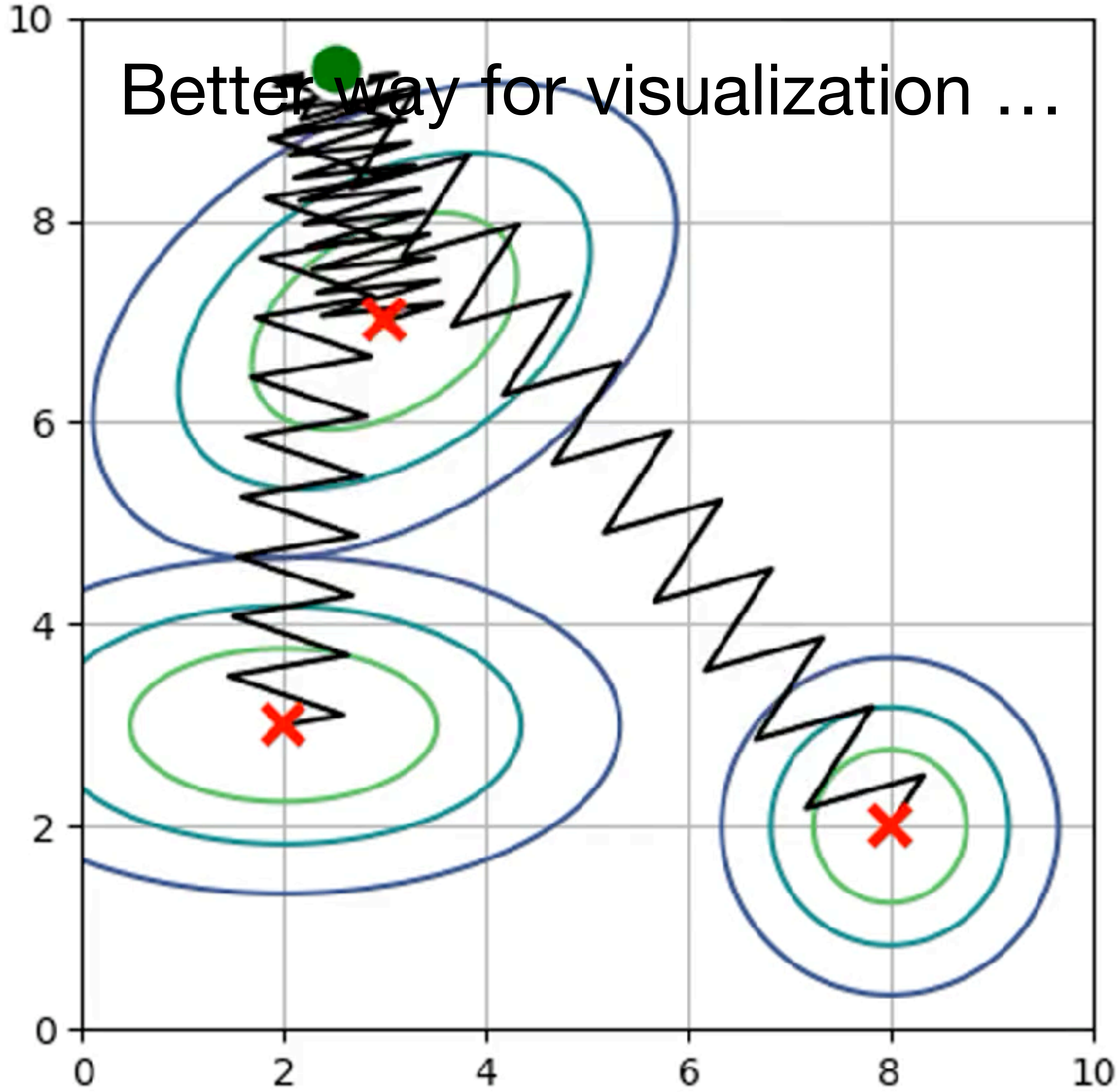
Σ_t^{GPS3}... big



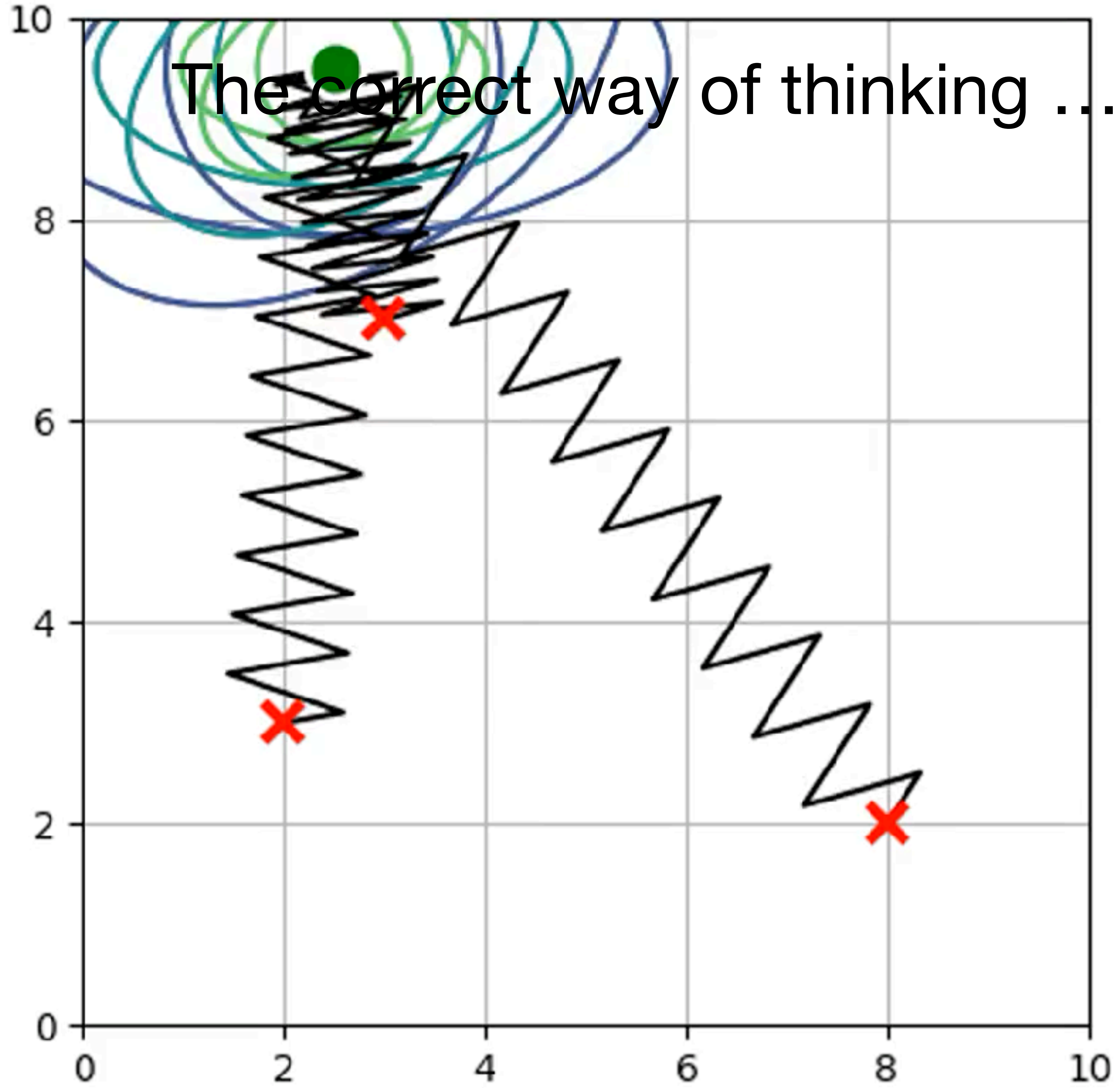
Σ_t^{GPS3}... small

Satelite navigation (GPS/GNSS)

$$\mathbf{x}^{\star} = \arg\min_{\mathbf{x}_t} \ \|\mathbf{x}_t - \mathbf{z}_t^{GPS1}\|^2_{\Sigma_t^{GPS1}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS2}\|^2_{\Sigma_t^{GPS2}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS3}\|^2_{\Sigma_t^{GPS3}}$$



Better way for visualization …



The correct way of thinking …

Multiple time instances

+

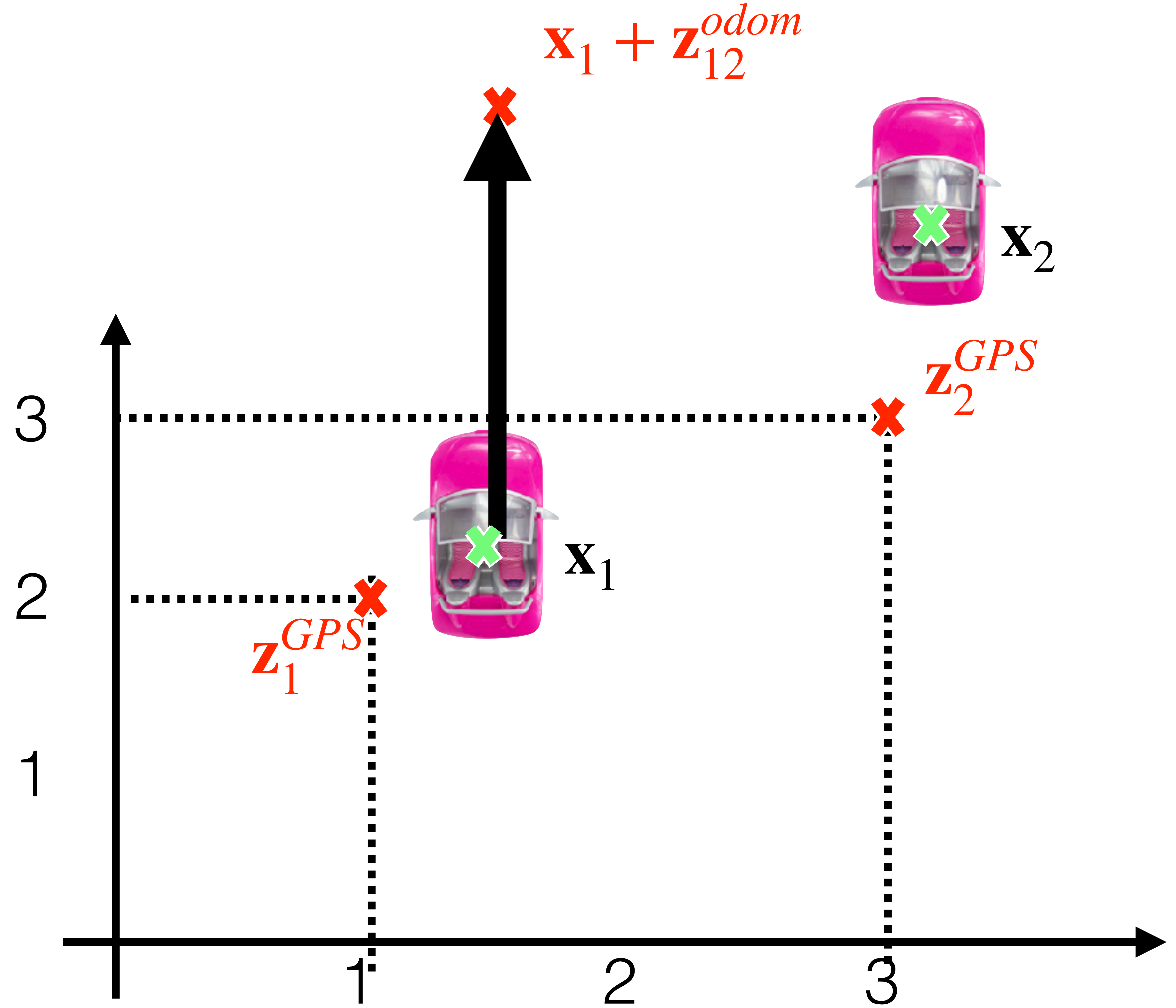
Absolute pose measurement (e.g. GPS)

+

Relative pose measurement (e.g.odometry from wheels/IMU/camera/lidar)

2D Localisation in **multiple time instances** from **GPS+odom**

$\mathbf{x}_1^\star, \mathbf{x}_2^\star \quad =???$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

Assume only two absolute gps measurements and one relative odom. measurement

$$\mathbf{x}_1^\star, \mathbf{x}_2^\star \;= \arg\max_{\mathbf{x}_1,\mathbf{x}_2} p(\mathbf{x}_1, \mathbf{x}_2 \,|\, \mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom})$$

Bayes theorem

↓

$$= \arg\max_{\mathbf{x}_1,\mathbf{x}_2} \frac{p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom} \,|\, \mathbf{x}_1, \mathbf{x}_2)\; p(\mathbf{x}_1, \mathbf{x}_2)}{p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom})}$$

Uniform prior

↓

$$= \arg\max_{\mathbf{x}_t} p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom} \,|\, \mathbf{x}_1, \mathbf{x}_2)$$

Conditional Independence

↓

$$= \arg\max_{\mathbf{x}_t} p(\mathbf{z}_1^{GPS} \,|\, \mathbf{x}_1) \cdot p(\mathbf{z}_2^{GPS} \,|\, \mathbf{x}_2) \cdot p(\mathbf{z}_{12}^{odom} \,|\, \mathbf{x}_1, \mathbf{x}_2)$$

unrealistic but useful

Normal likelihood

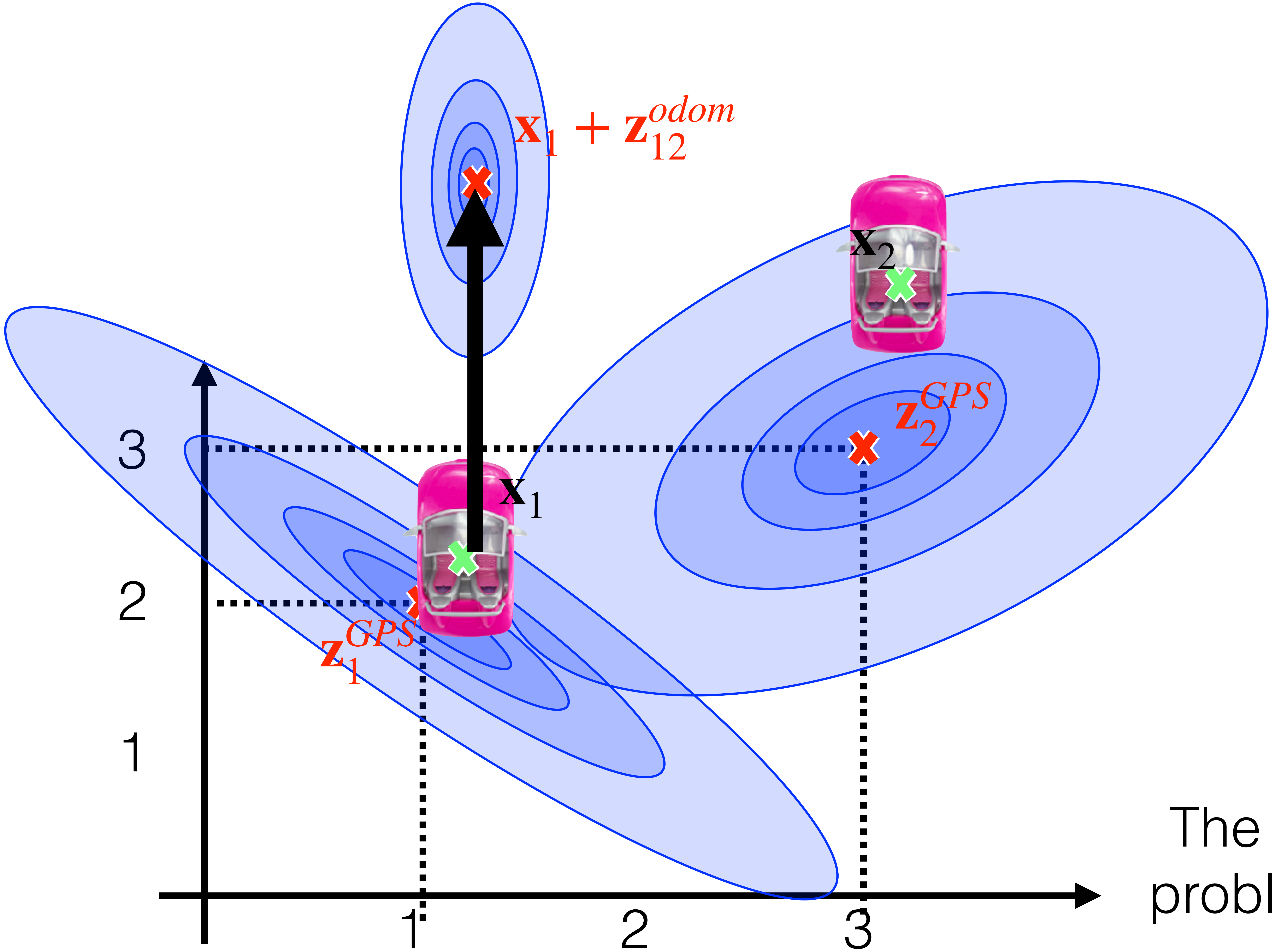
↓

$$= \arg\max_{\mathbf{x}_1,\mathbf{x}_2} \mathcal{N}(\mathbf{z}_1^{GPS}; \mathbf{x}_1, \Sigma_1^{GPS}) \mathcal{N}(\mathbf{z}_2^{GPS}; \mathbf{x}_2, \Sigma_2^{GPS}) \mathcal{N}(\mathbf{z}_{12}^{odom}; \mathbf{x}_2 - \mathbf{x}_1, \Sigma_{12}^{odom})$$

$$= \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \; \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$

2D Localisation in **multiple time instances** from **GPS+odom**

$$\mathbf{x}^{\star} = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$

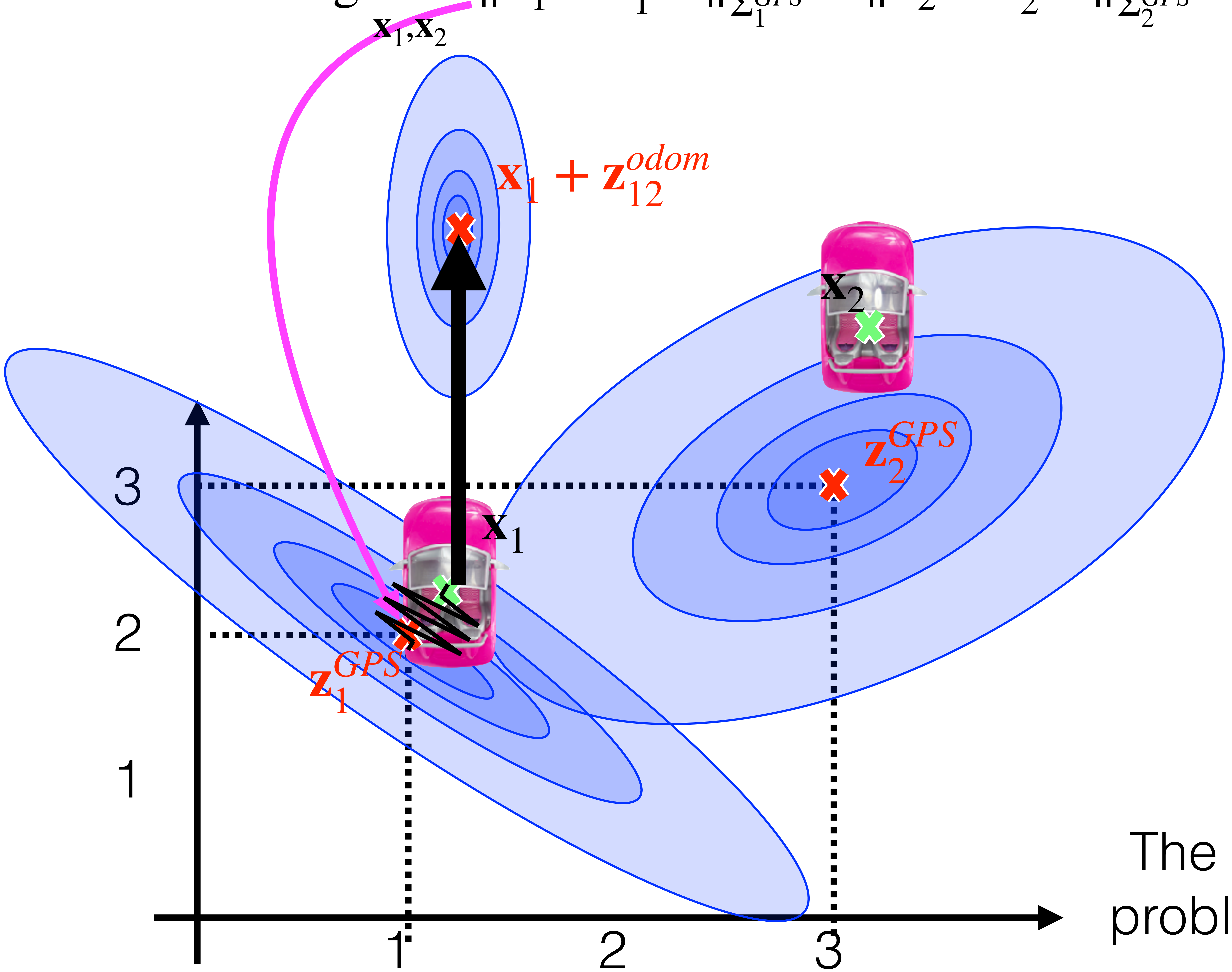


The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

unary

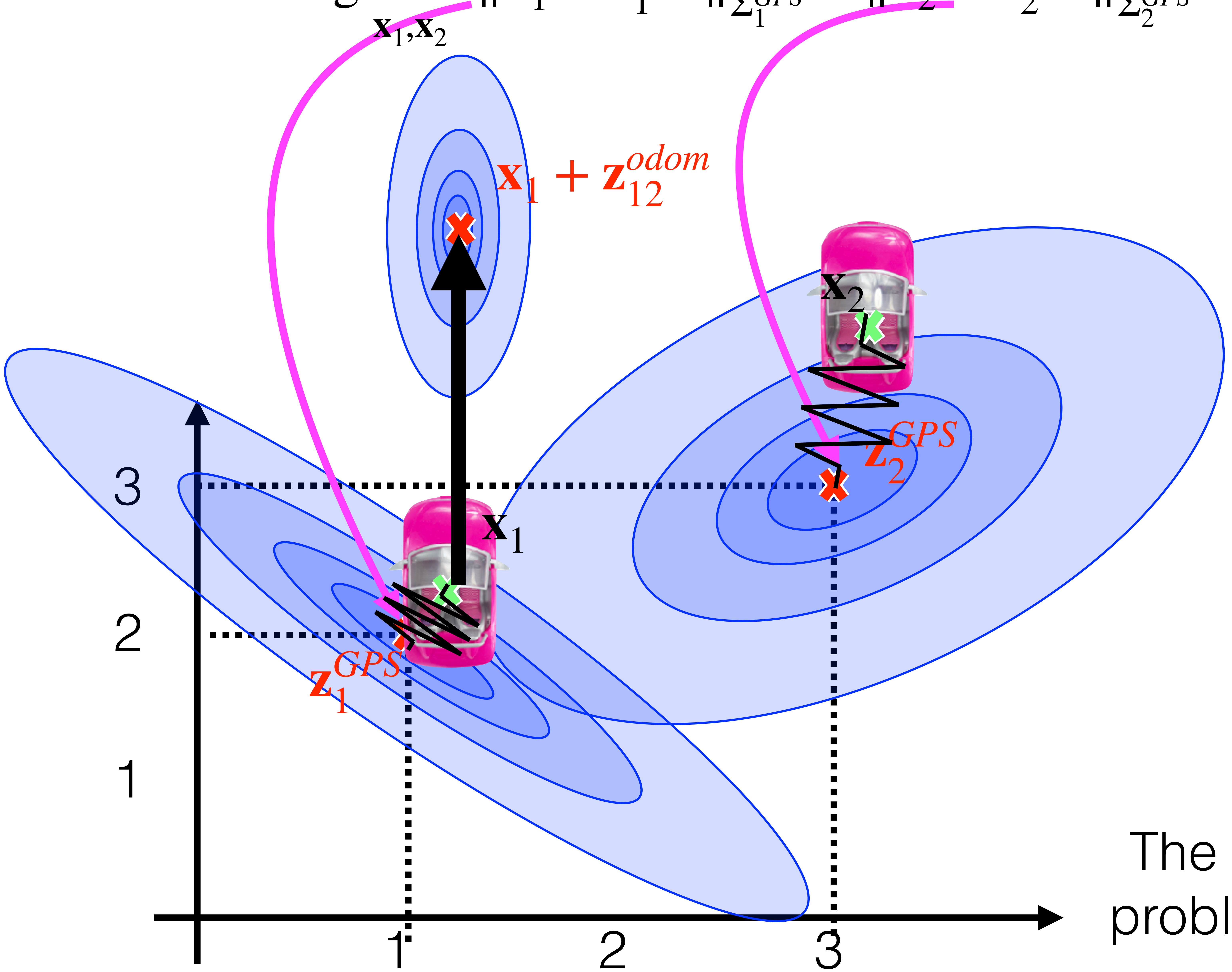
$$\mathbf{x}^{\star} = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

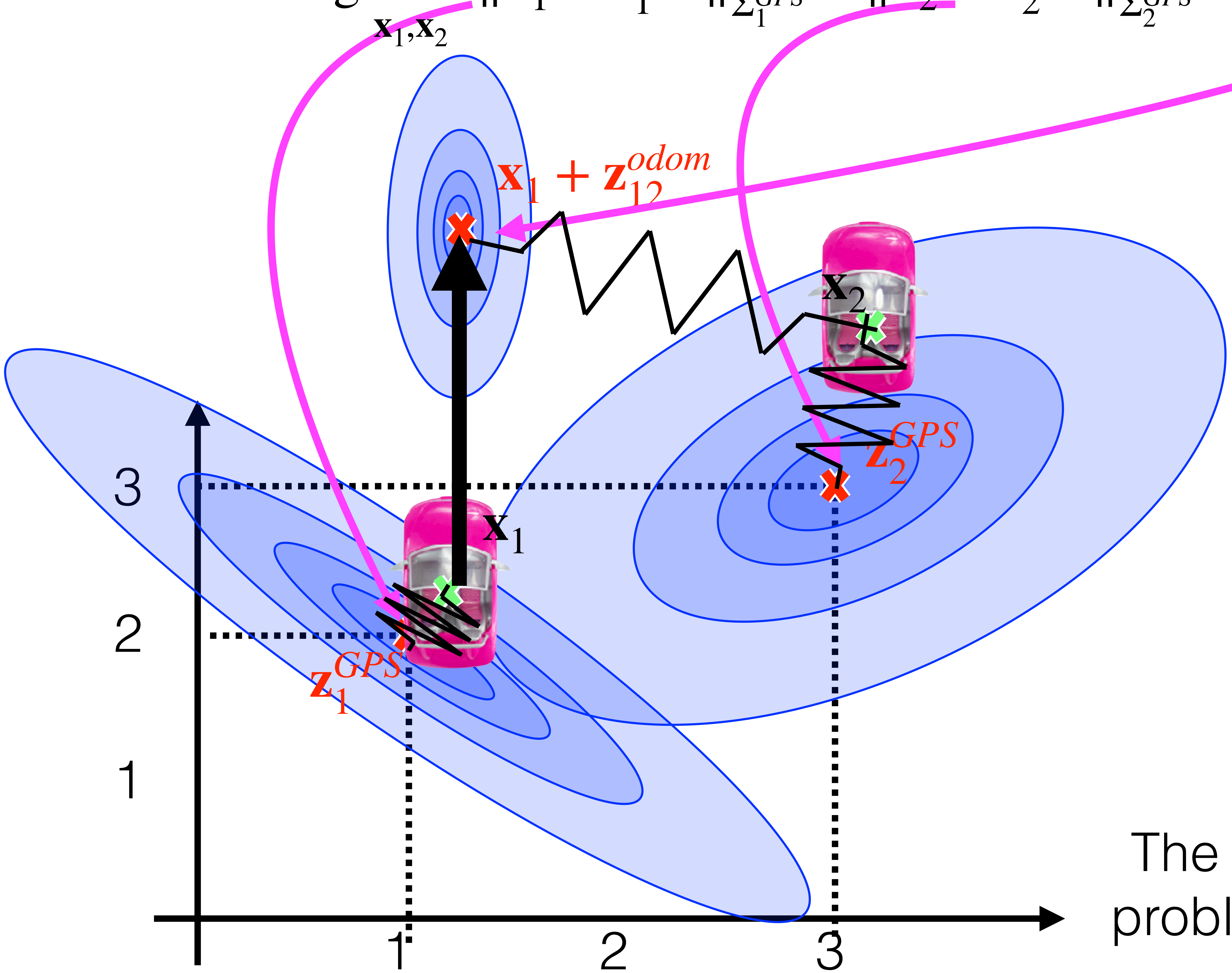
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \overset{\text{unary}}{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}} + \overset{\text{unary}}{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

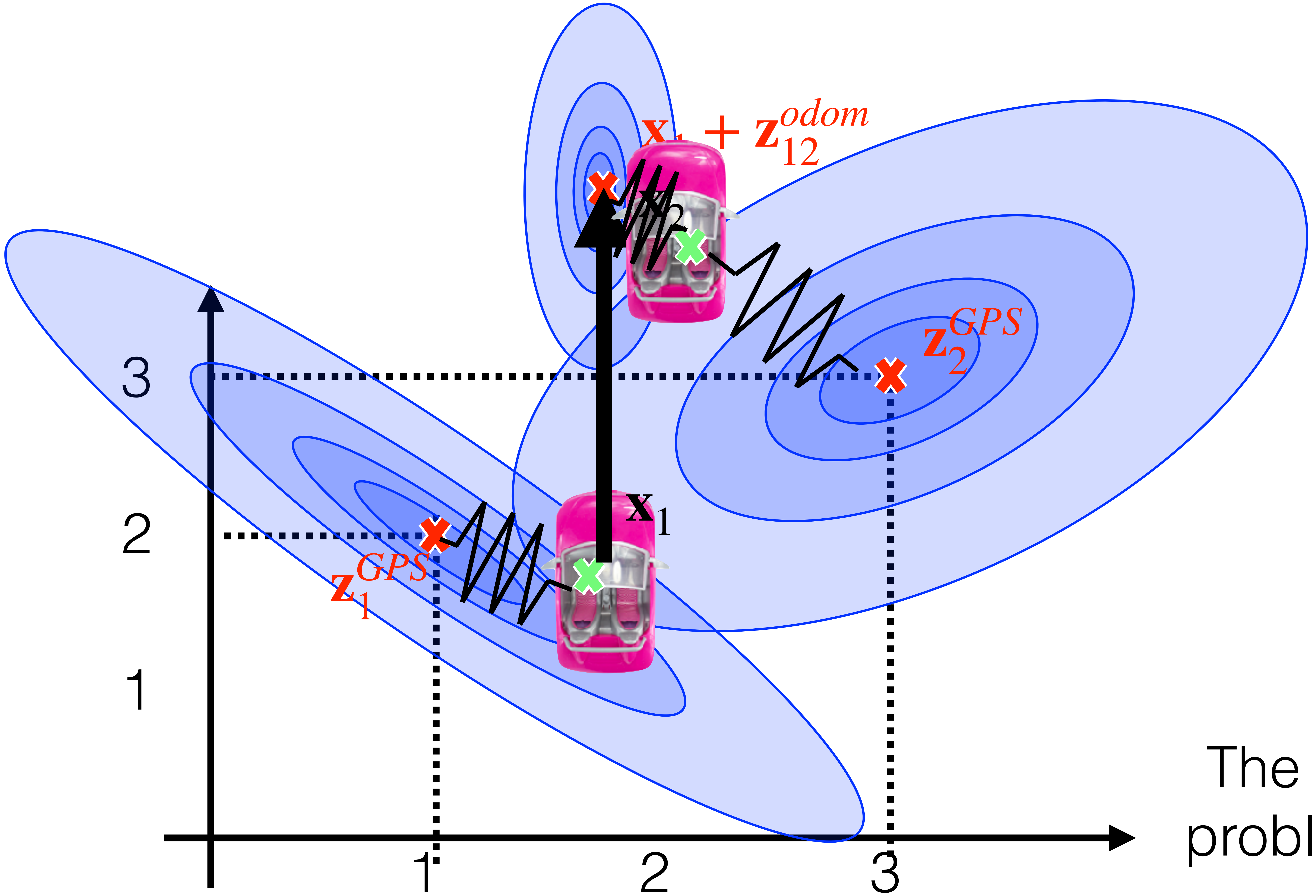
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \overset{\text{unary}}{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}} + \overset{\text{unary}}{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}} + \overset{\text{pair-wise}}{\|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}}$$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

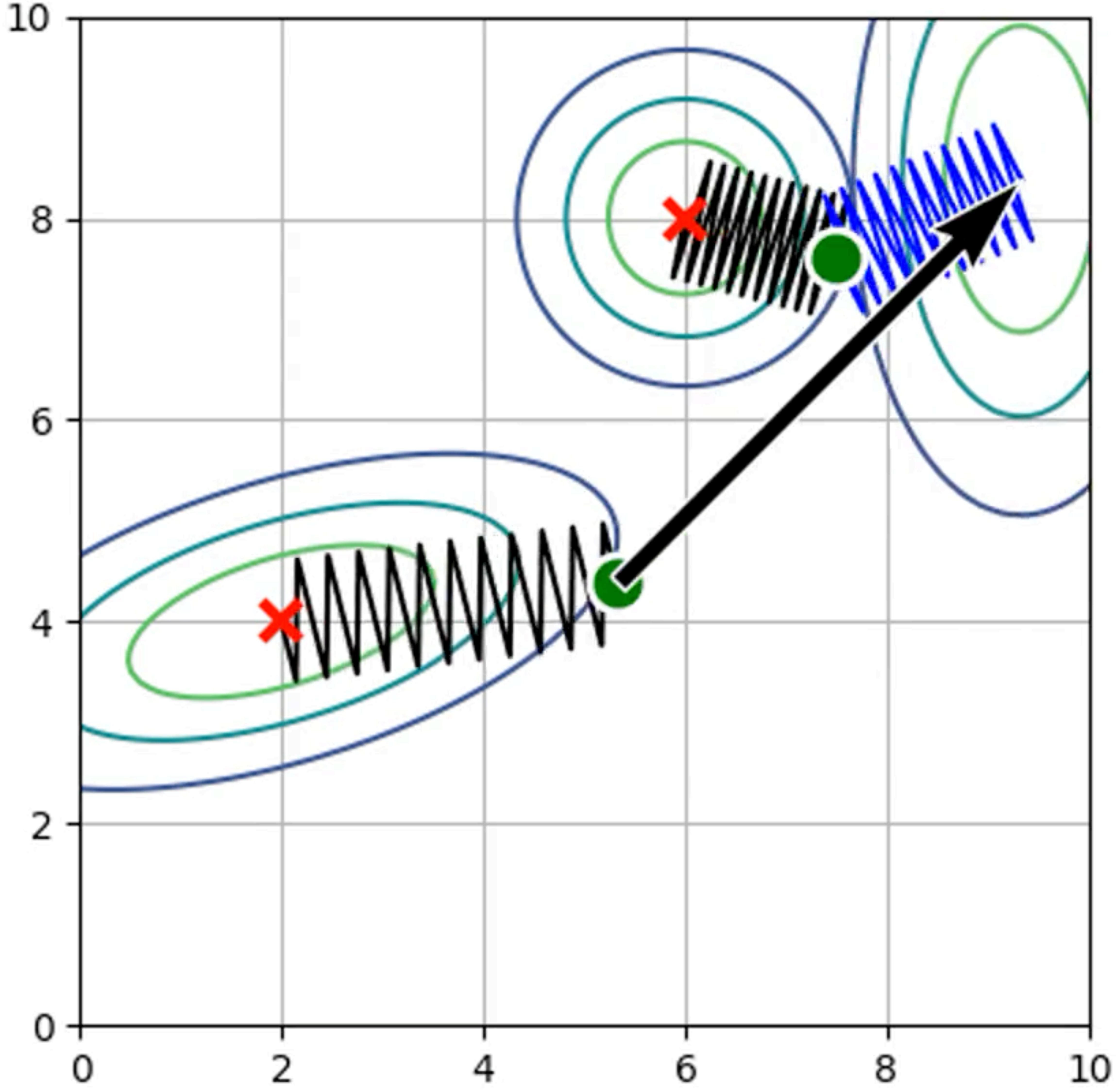
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \underbrace{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}}_{\text{unary}} + \underbrace{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}}_{\text{unary}} + \underbrace{\|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}}_{\text{pair-wise}}$$



The result is linear least squares problem with closed-form solution

Mechanical machine example

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \overset{\text{unary}}{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}} + \overset{\text{unary}}{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}} + \overset{\text{pair-wise}}{\|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}}$$



● $\mathbf{x}_t$ …robot poses

✕ $\mathbf{z}_t^{GPS}$ …GPS measurement

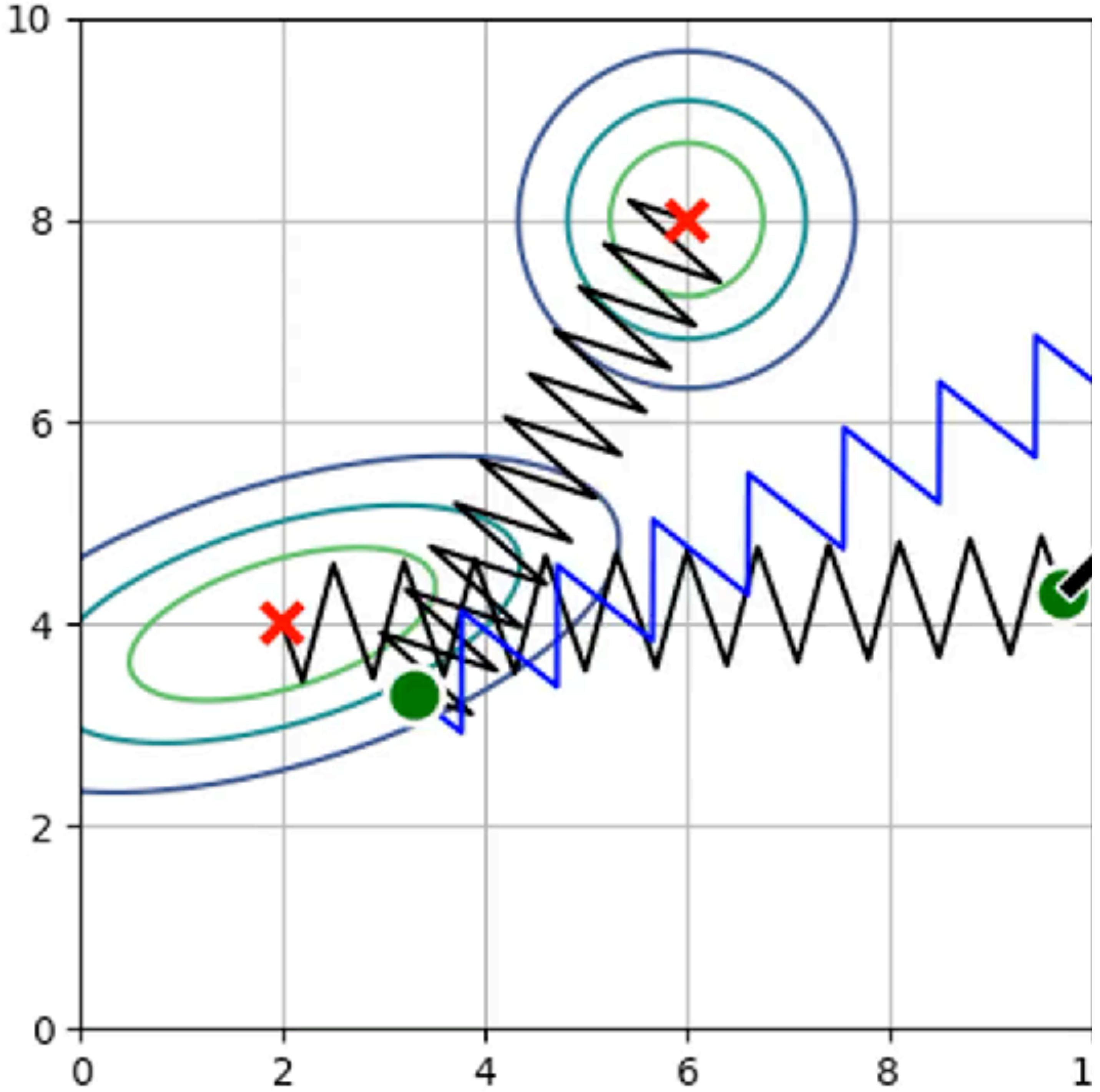
→ $\mathbf{z}_t^{odom}$…odometry measurements

〰 $\sum_t \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}}$ … GPS loss

〰 $\|\mathbf{x}_2 + \mathbf{z}_{12}^{odom} - \mathbf{x}_2\|^2_{\Sigma_t^{odom}}$ … odom loss

What happens to resulting loss if GPS and odom are inconsistent?

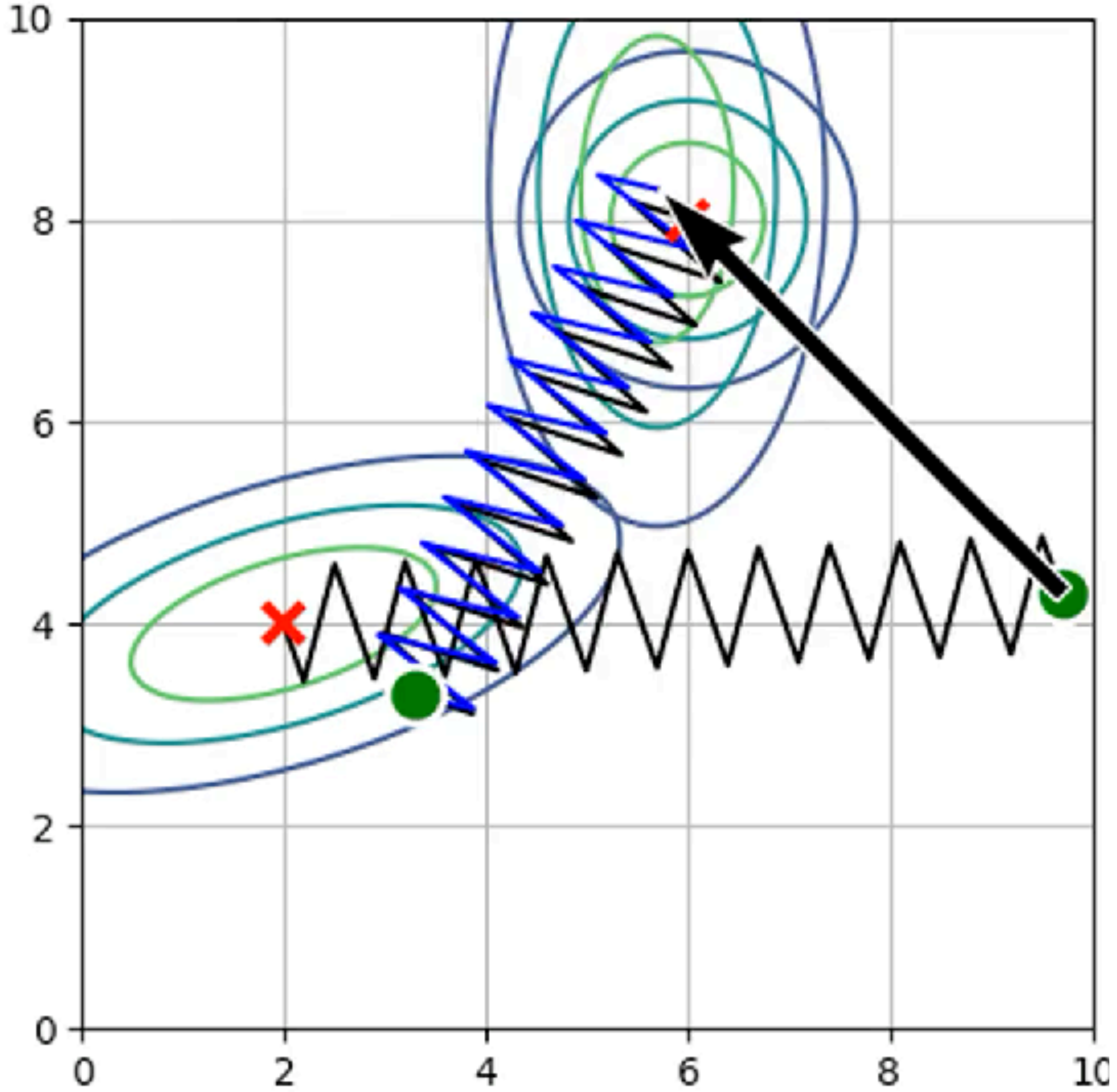
consistent



loss_opt= 0.002646

inconsistent

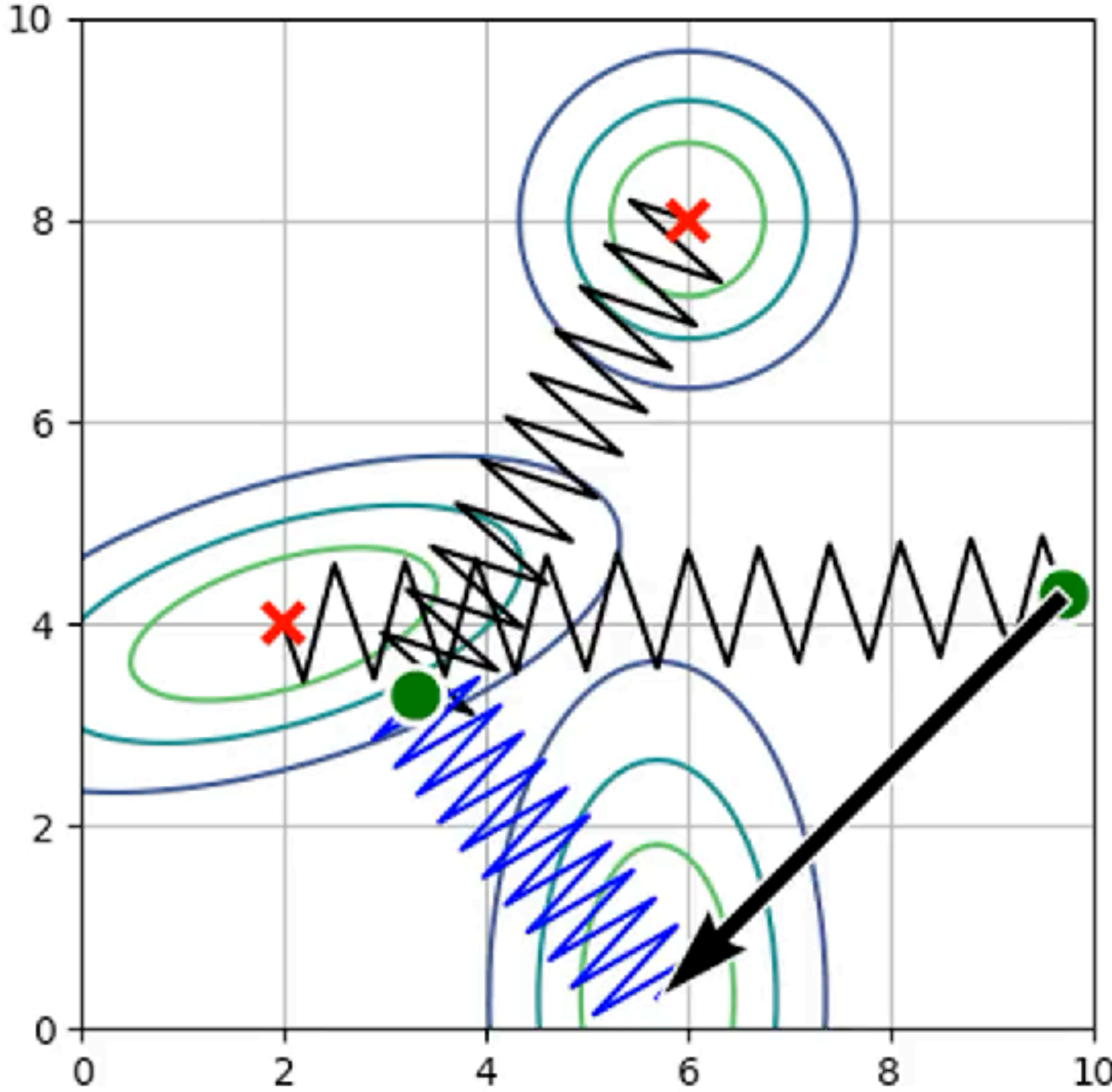
odom rotated by 90degs



loss_opt= 10.972076

more inconsistent

odom rotated by180degs



loss_opt= 18.28627

Does it happen to humans?

Motion sickness

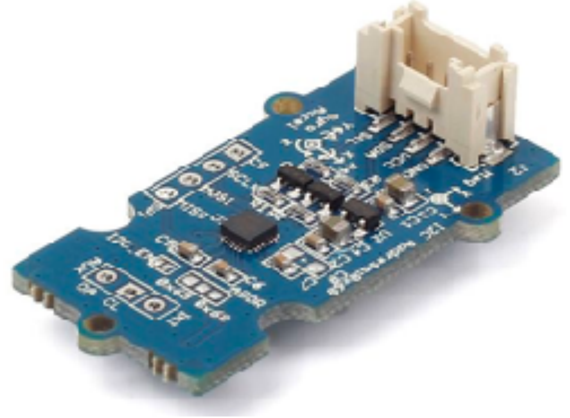
https://friendsofwords.com/2023/01/24/why-i-dont-agree-with-enjoy-the-journey-not-the-destination/

Why does the body react so weirdly?

https://www.reddit.com/r/comics/comments/gsn4u1/fun_guys/

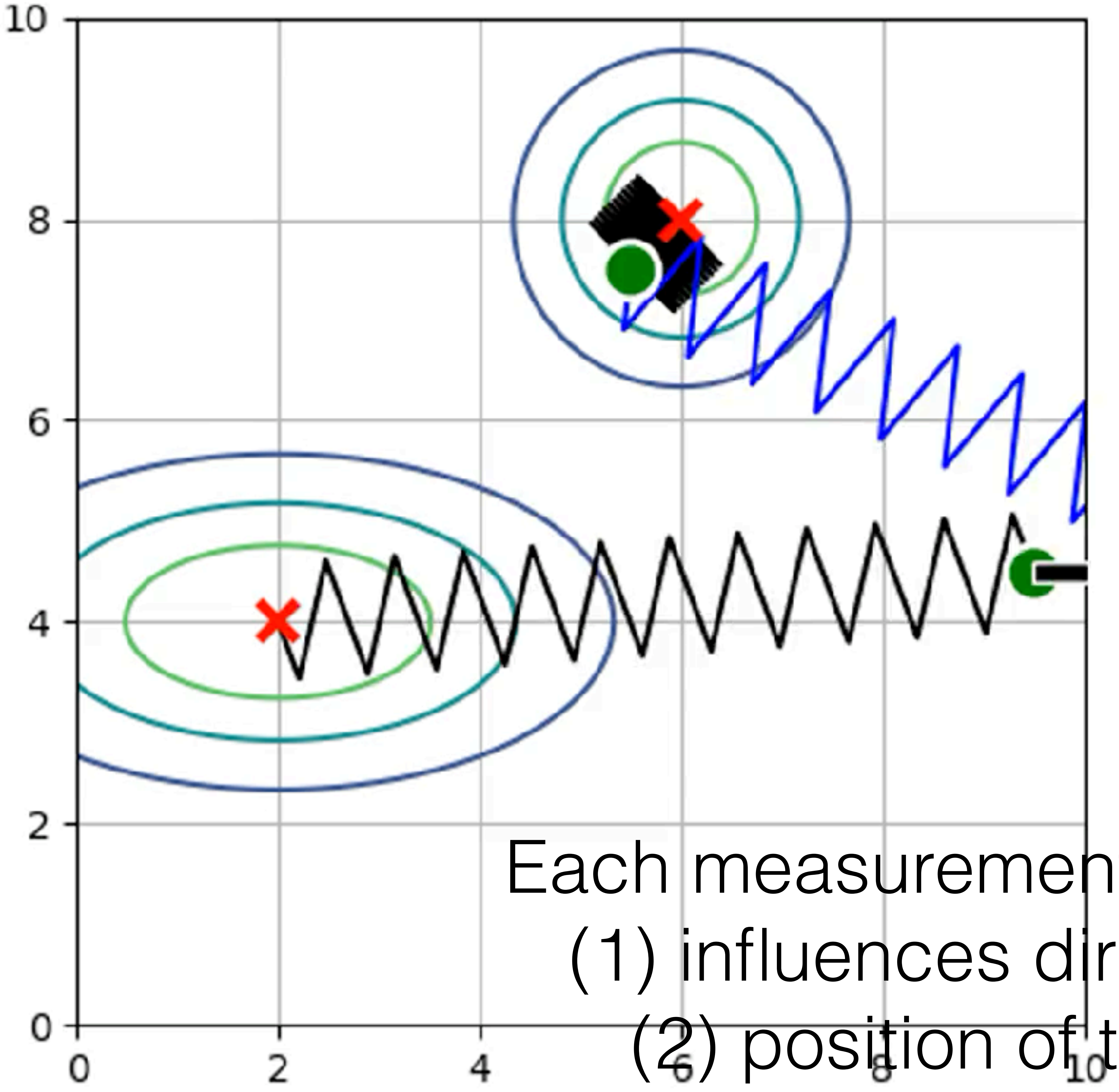
real world physical inconsistencies

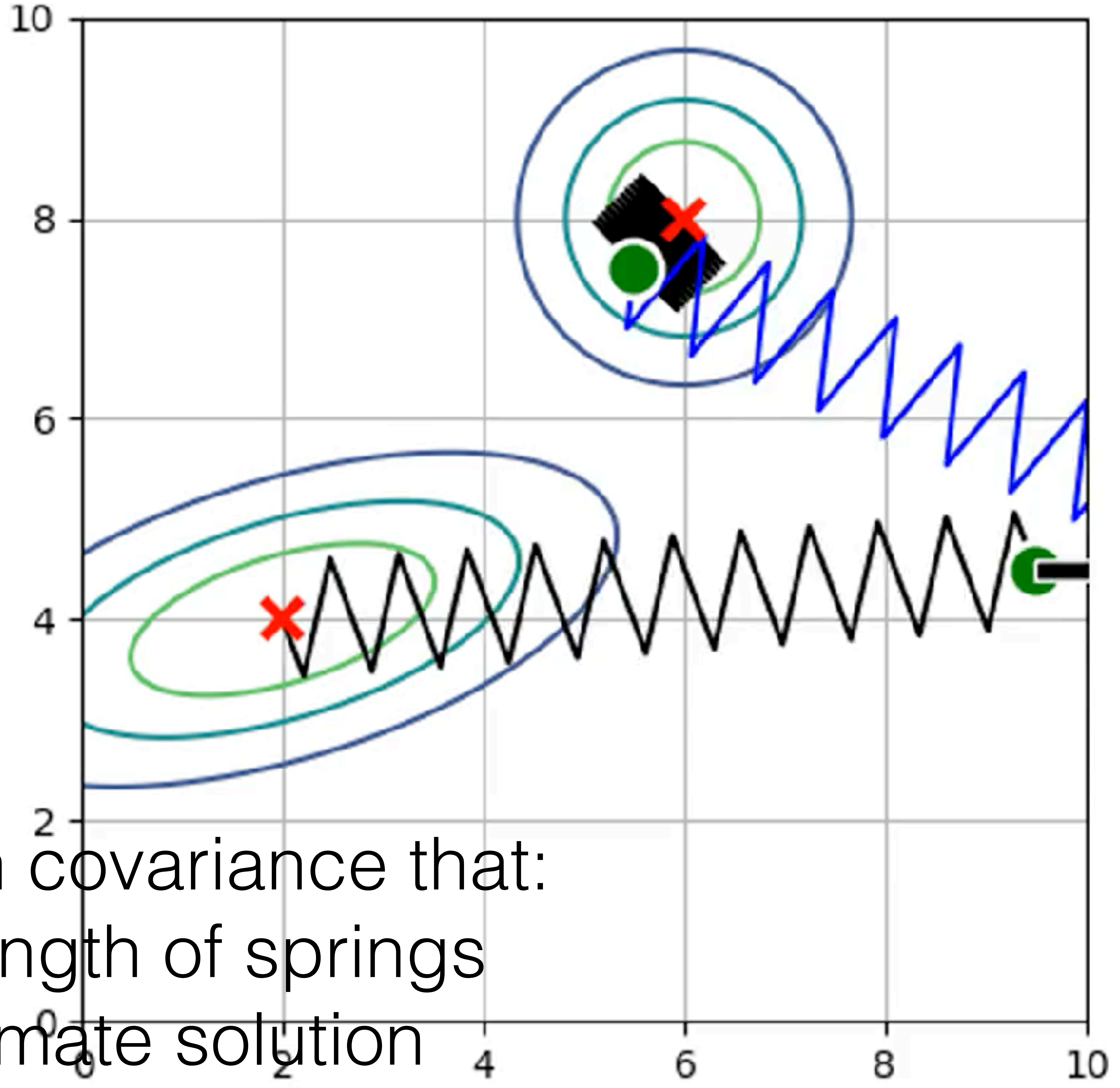
Satelite navigation (GPS/GNSS) +



Odometry (IMU)

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_t} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 + \mathbf{z}_{12}^{odom} - \mathbf{x}_2\|^2_{\Sigma_t^{odom}}$$



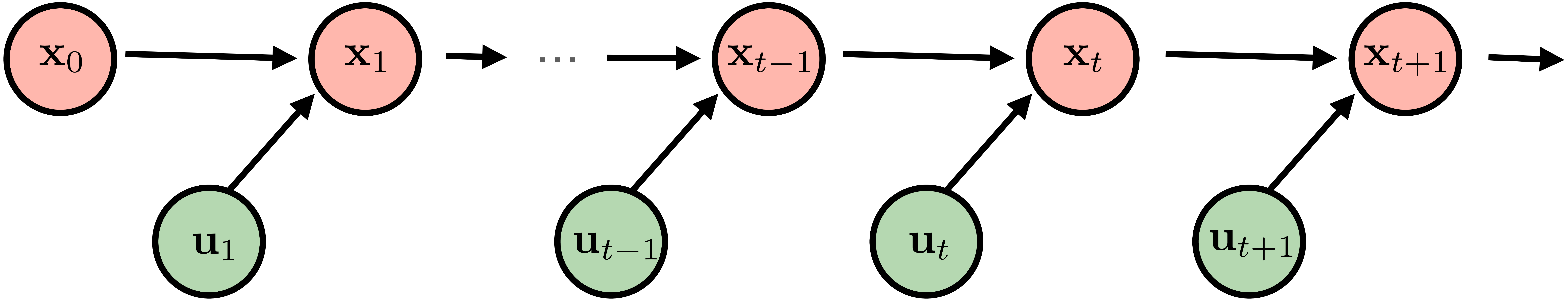


Each measurement has its own covariance that:
(1) influences directional strength of springs
(2) position of the MAP estimate solution

Motion model

Localisation in **multiple time instances** from **actions and motion model**

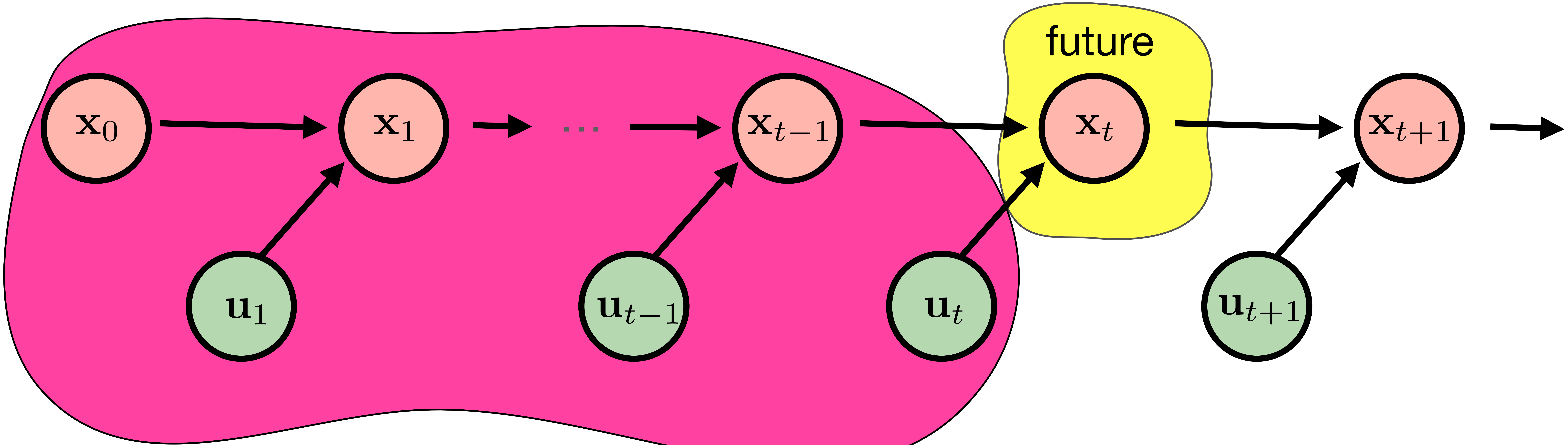
Actions: $\mathbf{u}_1 \ldots \mathbf{u_t}$ (generated by external source)



Localisation in **multiple time instances** from **actions and motion model**

Actions: $\mathbf{u}_1 \ldots \mathbf{u_t}$ (generated by external source)

State-transition prob.: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \ldots, \mathbf{x}_0, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_1, \mathbf{z}_{t-1}, \ldots, \mathbf{z}_1)$



Localisation in **multiple time instances** from **actions and motion model**

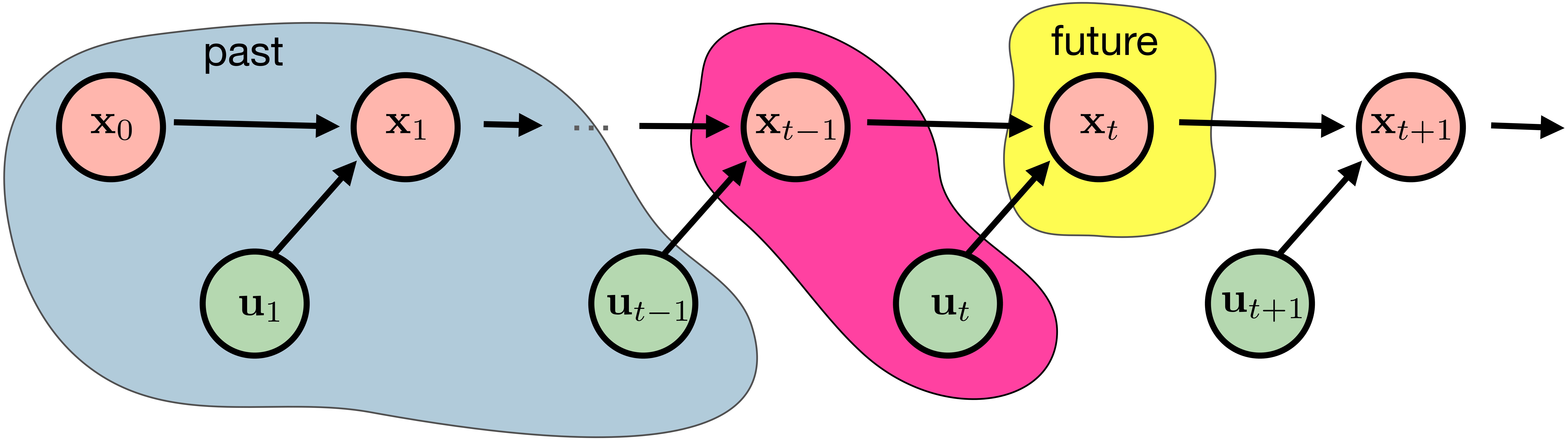
Actions: $\mathbf{u}_1 \ldots \mathbf{u}_\mathbf{t}$ (generated by external source)

State-transition prob.: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \ldots, \mathbf{x}_0, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_1, \mathbf{z}_{t-1}, \ldots, \mathbf{z}_1)$

Markov assumption: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \ldots, \mathbf{x}_0, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_1, \mathbf{z}_{t-1}, \ldots, \mathbf{z}_1)$

Motion model: $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \epsilon_{\text{noise}}$ (prior about robot's behaviour)

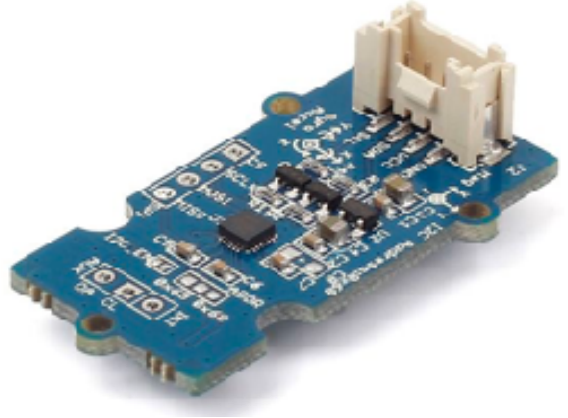
Example: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t; g(\mathbf{x}_{t-1}, \mathbf{u}_t), \Sigma_t^g)$ e.g. linear $\mathcal{N}(\mathbf{x}_t; \mathbf{x}_{t-1} + \mathbf{u}_t, \Sigma_t^g)$



Localisation from **GPS + IMU + actions**

GPS/GNSS

+



IMU

+

motion model

$$\mathbf{x}^* = \arg\max_{\mathbf{x}_0 \ldots \mathbf{x}_t} p(\mathbf{x}_0 \ldots \mathbf{x}_t | \mathbf{z}_1 \ldots \mathbf{z}_t, \mathbf{u}_1 \ldots \mathbf{u}_t) \quad = \; ???$$

Localisation from **actions + GPS + IMU**

Bayes theorem

↓

$$\mathbf{x}^* = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{x}_0\ldots\mathbf{x}_t \,|\, \mathbf{z}_1\ldots\mathbf{z}_t, \mathbf{u}_1\ldots\mathbf{u}_t) = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} \frac{p(\mathbf{z}_1\ldots\mathbf{z}_t \,|\, \mathbf{x}_0\ldots\mathbf{x}_t, \cancel{\mathbf{u}_1\ldots\mathbf{u}_t})\,p(\mathbf{x}_0\ldots\mathbf{x}_t \,|\, \mathbf{u}_1\ldots\mathbf{u}_t)}{\cancel{p(\mathbf{z}_0\ldots\mathbf{z}_t \,|\, \mathbf{u}_1\ldots\mathbf{u}_t)}}$$

Conditional independence of z on u given x

↓

$$= \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{z}_1\ldots\mathbf{z}_t \,|\, \mathbf{x}_0\ldots\mathbf{x}_t)\; p(\mathbf{x}_0\ldots\mathbf{x}_t \,|\, \mathbf{u}_1\ldots\mathbf{u}_t)$$

Normal likelihoods + conditional independences

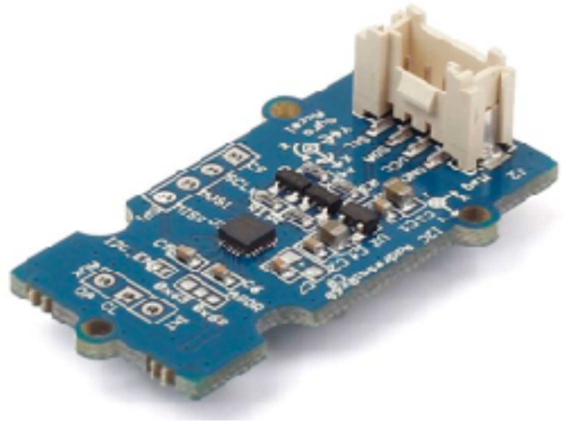
↓

$$= \arg\max_{\mathbf{x}_0,\ldots\mathbf{x}_t} \prod_i \mathcal{N}(\mathbf{z}_i^{GPS}; \mathbf{x}_i, \Sigma_i^{GPS}) \prod_i \mathcal{N}(\mathbf{z}_i^{odom}; \mathbf{x}_i - \mathbf{x}_{i-1}, \Sigma_i^{odom}) \prod_i \mathcal{N}(\mathbf{x}_i; g(\mathbf{x}_{i-1}, \mathbf{u}_i), \Sigma_i^{g})$$

$$= \arg\min_{\mathbf{x}_0,\ldots\mathbf{x}_t} \sum_{i=1}^{t} \|\mathbf{x}_i - \mathbf{z}_i^{GPS}\|^2_{\Sigma_i^{GPS}} + \sum_{i=1}^{t} \|\mathbf{x}_i - \mathbf{x}_{i-1} - \mathbf{z}_i^{odom}\|^2_{\Sigma_i^{odom}} + \sum_{i=1}^{t} \|\mathbf{x}_i - g(\mathbf{x}_{i-1}, \mathbf{u}_i)\|^2_{\Sigma_i^{g}}$$

$$= \arg\min_{\mathbf{x}_0,\ldots\mathbf{x}_t} \sum_j f_j(\mathbf{x}, \mathbf{z})^2$$

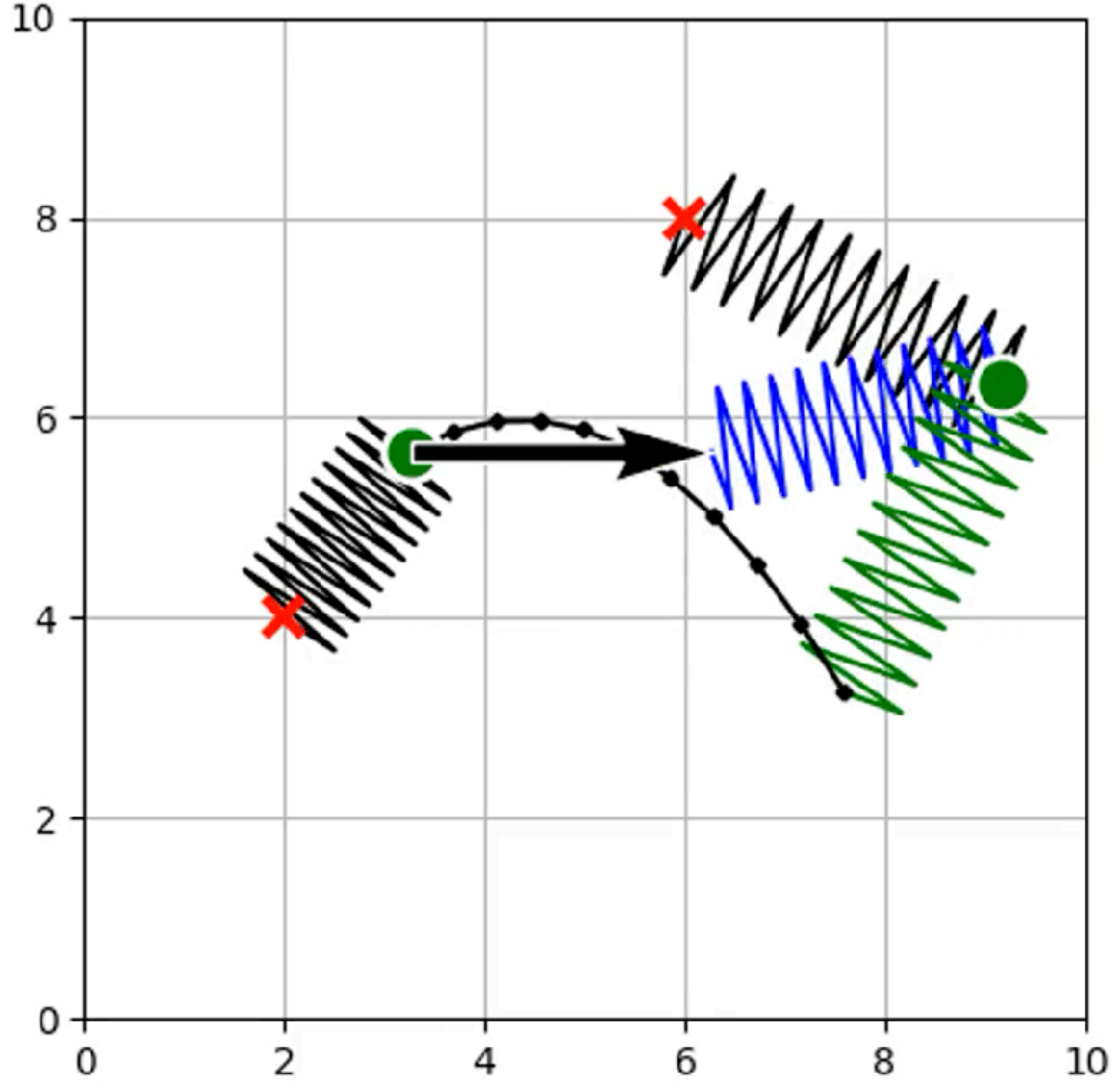
GPS/GNSS +



IMU +

motion model

$$\mathbf{x}_1^\star, \mathbf{x}_2^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 + \mathbf{z}_{12}^{odom} - \mathbf{x}_2\|^2_{\Sigma_t^{odom}} + \|g(\mathbf{x}_1, \mathbf{u}_2) - \mathbf{x}_2\|^2_{\Sigma_t^g}$$



● $\mathbf{x}_t$ .... robot poses

✕ $\mathbf{z}_t^{GPS}$......... GPS measurement

→ $\mathbf{z}_t^{odom}$........ odometry measurements

$g(\mathbf{x}_1, \mathbf{u}_2)$ …motion model

$\sum_t \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}}$ … GPS loss

$\|\mathbf{x}_2 + \mathbf{z}_{12}^{odom} - \mathbf{x}_2\|^2_{\Sigma_t^{odom}}$ … odom loss

$\|g(\mathbf{x}_1, \mathbf{u}_2) - \mathbf{x}_2\|^2_{\Sigma_t^g}$ … motion loss

Factor graph

Factor graph

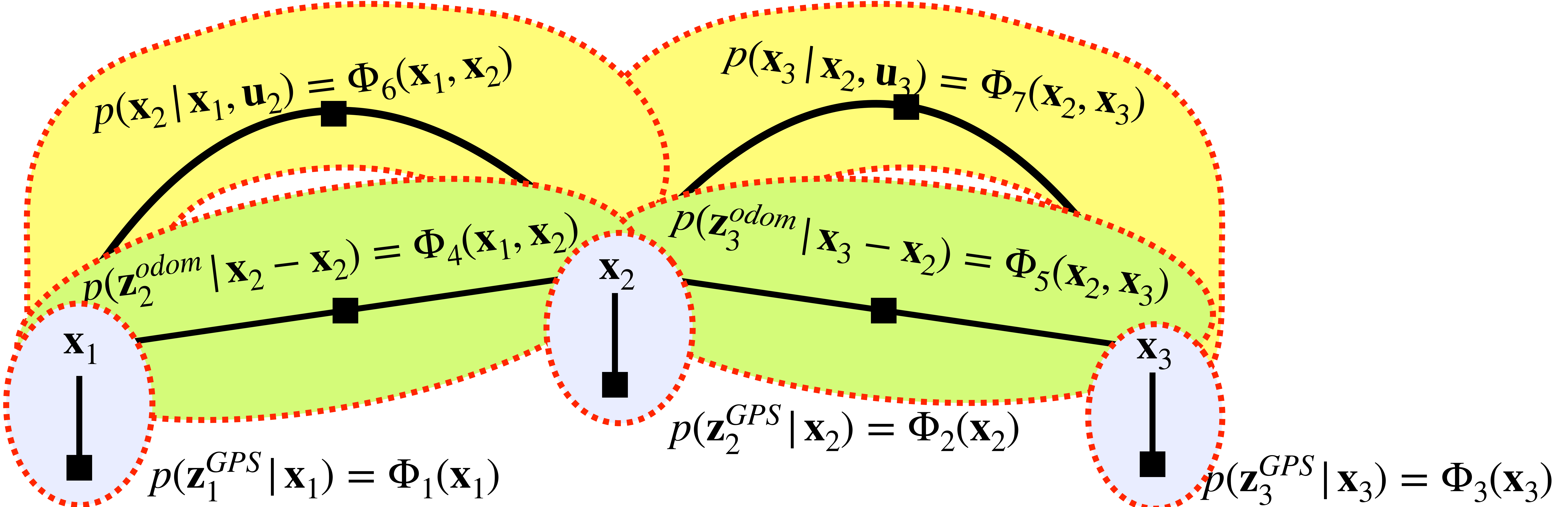
- Design choices (Markov assumption, cond. independence) yielded sparse model

$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_0,\dots\mathbf{x}_t} \prod_i p(\mathbf{z}_i^{GPS} \,|\, \mathbf{x}_i) \cdot \prod_i p(\mathbf{z}_i^{odom} \,|\, \mathbf{x}_i, \mathbf{x}_{i-1}) \cdot \prod_i p(\mathbf{x}_i \,|\, \mathbf{x}_{i-1}, \mathbf{u}_i)$$

unary pair-wise pair-wise

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_0,\dots\mathbf{x}_t} \sum_i \|\mathbf{x}_i - \mathbf{z}_i^{GPS}\|^2_{\Sigma_i^{GPS}} + \sum_i \|\mathbf{x}_i - \mathbf{x}_{i-1} - \mathbf{z}_i^{odom}\|^2_{\Sigma_i^{odom}} + \sum_i \|\mathbf{x}_i - g(\mathbf{x}_{i-1}, \mathbf{u}_i)\|^2_{\Sigma_i^g}$$

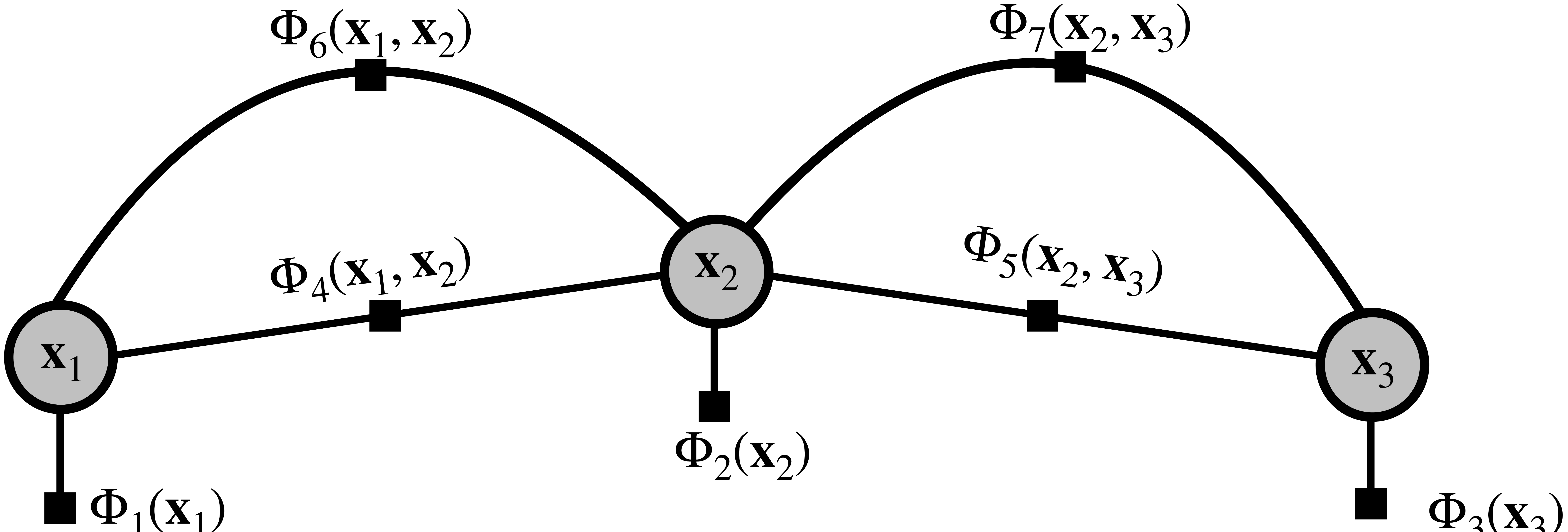
- The structure can be more complicated (e.g. ternary terms, loop closers)



Factor graph

Def: Factor graph is bipartite graph $\mathscr{G} = \{\mathscr{U}, \mathscr{V}, \mathscr{E}\}$ with

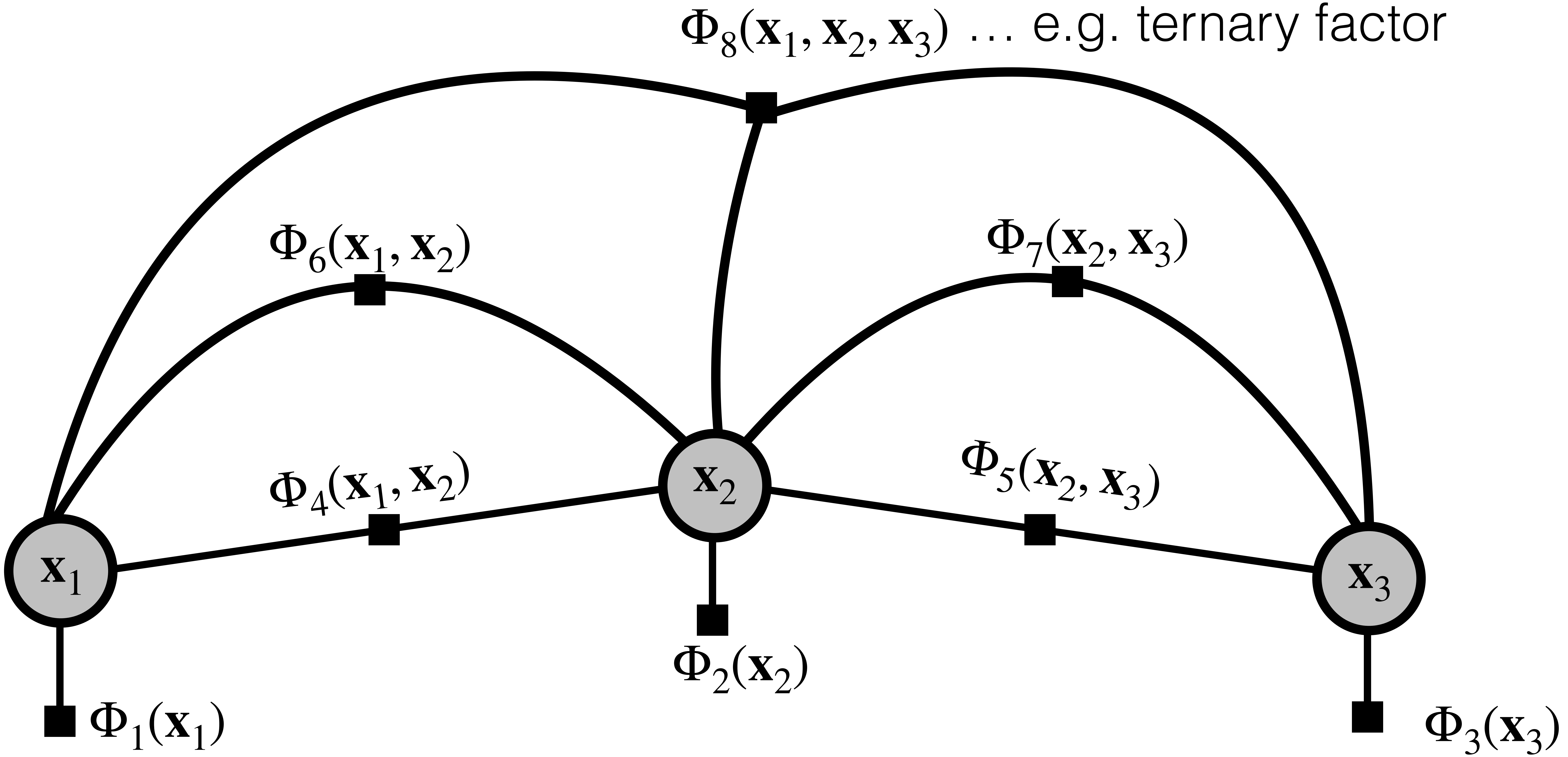
- Two types of nodes: factors ■ $\Phi_i \in \mathscr{U}$ and ● variables $\mathbf{x}_j \in \mathscr{V}$.



Factor graph

Def: Factor graph is bipartite graph $\mathscr{G} = \{\mathscr{U}, \mathscr{V}, \mathscr{E}\}$ with

- Two types of nodes: factors ■ $\mathbf{\Phi}_i \in \mathscr{U}$ and ● variables $\mathbf{x}_j \in \mathscr{V}$.
- Edges $\mathbf{e}_{ij} \in \mathscr{E}$ are always between factor nodes and variable nodes.

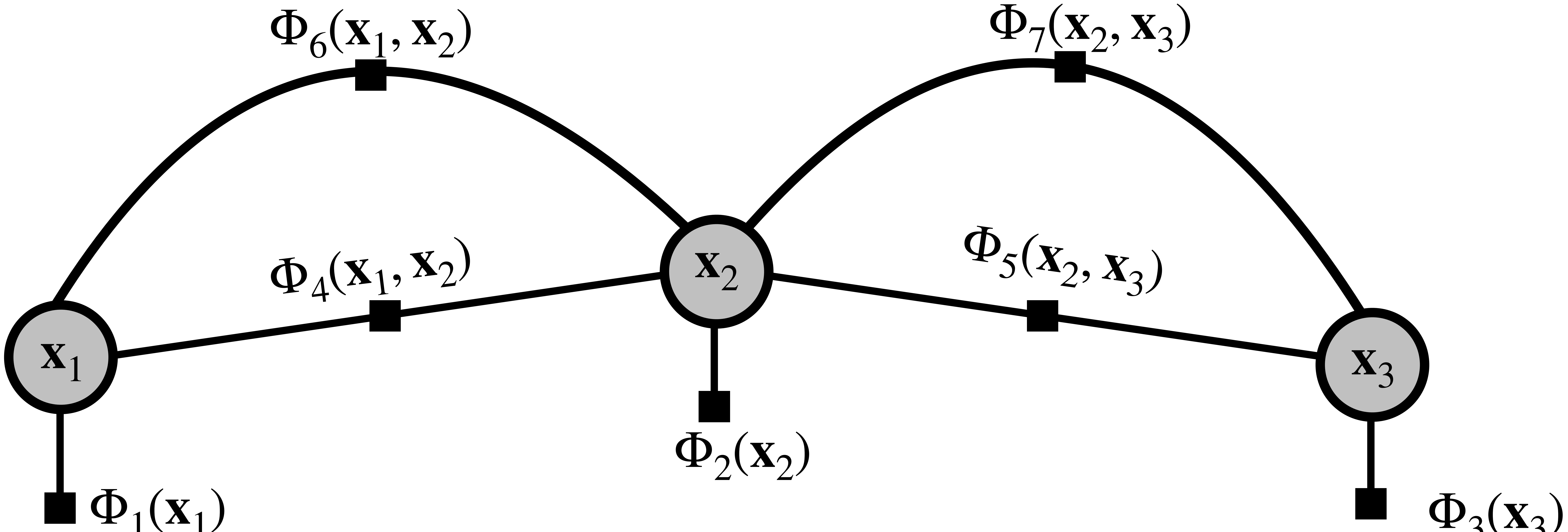


Factor graph

- Convenient visualisation of the (sparse) problem structure
- Simple formulation of MAP estimation problem in negative log-space

$$\mathbf{x}_0^\star \ldots \mathbf{x}_t^\star = \arg\max_{\mathbf{x}_0 \ldots \mathbf{x}_t} \prod_i \Phi_i(X_i) = \arg\min_{\mathbf{x}_0 \ldots \mathbf{x}_t} \sum_i -\log\Big(\Phi_i(X_i)\Big)$$

- Optimisation (continuous var. => local gradient opt., discr. var. => graph search)
- Sampling of $p(\mathbf{x}_0 \ldots \mathbf{x}_t)$ (MCMC Gibbs sampling, ancestral sampling for dir. acyclic
- If factors are linear => closed-form solution available (e.g. LS, KF)



Factor graph

- Convenient visualisation of the (sparse) problem structure
- Simple formulation of MAP estimation problem in negative log-space

$$\mathbf{x}_0^\star \ldots \mathbf{x}_t^\star = \arg\max_{\mathbf{x}_0 \ldots \mathbf{x}_t} \prod_i \Phi_i(X_i) = \arg\min_{\mathbf{x}_0 \ldots \mathbf{x}_t} \sum_i -\log\Big(\Phi_i(X_i)\Big)$$

- Optimisation (continuous var. => local gradient opt., discr. var. => graph search)
- Sampling of $p(\mathbf{x}_0 \ldots \mathbf{x}_t)$ (MCMC Gibbs sampling, ancestral sampling for dir. acyclic
- If factors are linear => closed-form solution available (e.g. LS, KF)
- Graphical model useful for MAP estimation:
  - SLAM
  - optimal control
  - tracking
  - self-supervised learning
  - …

Summary

- **Understand** localisation problem in robotics as MAP estimate of unknown variables
- **Model measurement probability** of simplified relative and absolute measurements
- **Model state-transition probability** for linear and nonlinear motion models
- **Write down optimisation** criterion in negative log-space for gaussian prob. distr.
- **Solve** underlying opt. problem using least squares / gradient descend algorithm in your favourite optimisation tool (MATLAB, Scipy, Pytorch, Julia, Mosek)
- **Next lecture:** Adds rotation and solve the optimization in SE(2) manifold