Exam from ARO **Name:** ______________________

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Number of submitted sheets: ______________________

June 1, 2023

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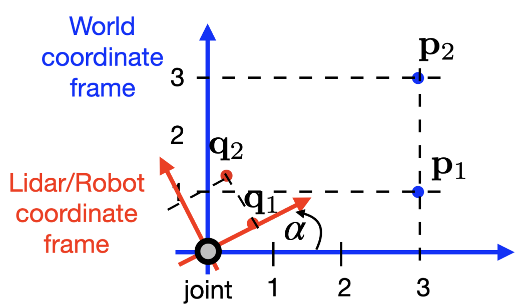
1. **Pose from known correspondences:** Consider the problem of estimating the rotation of a robot in a 2D world from two known correspondences; see Figure below. Two blue markers are known to be located at positions $\mathbf{p}_1 = [3,1]^\top, \mathbf{p}_2 = [3,3]^\top$ in the "World coordinate frame" (blue arrows). The robot (which is connected by joint to the origin of the w.c.f.) detects these markers and measures its positions $\mathbf{q}_1 = [1,0]^\top, \mathbf{q}_2 = [1,1]^\top$ in its own "Lidar/Robot coordinate frame" (red arrows). These measurements are obviously wrong - in particular, each measurement is assumed to have zero-mean Gaussian noise with the diagonal covariance $\Sigma = \mathbf{I}$, and measurements are assumed to be independent of each other. The unknown robot's rotation is parameterized by scalar value α such that the corresponding rotation matrix that transforms points from r.c.f. to w.c.f. is as follows: $\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. Only *rotation* α of the robot is considered to be unknown (any translation is technically impossible; therefore, it stays in the origin of w.c.f.).



a) (2 points) Let us first assume that α is provided. Given this α, how can you transform lidar measurements $\mathbf{q}$ into w.c.f?

$$\begin{bmatrix} \mathbf{q}_x^{\text{wcf}} \\ \mathbf{q}_y^{\text{wcf}} \end{bmatrix} = \qquad$$ (the answer is a mathematical expression)

b) (2 points) Draw a corresponding spring-like mechanical machine, the equilibrium of which will be the solution (feel free to draw springs directly into the figure above).

c) (2 points) What is an optimization-friendly problem, we can solve to get the optimal α?

$$\alpha^* = \arg\min_{\alpha} \sum_{i}$$ (fill in criterion)

d) (2 points) Analytically derive the solution of the underlying absolute orientation problem and substitute the values. Don't use any black-box covariance-matrix-like formulas from slides!

Derivation:

$\alpha^* =$ (the answer is one real number)

e) (2 points) Assume that you also have the prior knowledge that the robot's rotation comes from Gaussian distribution with mean $\overline{\alpha} = \pi/2$ and the same covariance $\sigma = 1$. What will be the formulation of the maximum aposteriori estimate of the $\alpha^\star$ and the corresponding optimization problem?

$\alpha^* = \arg\max \prod_i$	(maximum aposteriori estimate)
$\alpha^* = \arg\min \sum_i$	(corresponding opt. problem)

2. **Factorgraph localization with priors:**

Consider factorgraph localization of a robotic train on a circular track of given radius R. The trajectory of the train in the world frame is described by polar coordinates $(R, \phi_t)^T$, where $t \in 0, \dots T$. The initial pose of the train $(R, \phi_0)^T$ is given as a parameter of the algorithm. The robotic train has only a single control variable – the angular velocity of the wheels ω_t around their axle. For simplicity, assume that the train has only one row of wheels and there is only 1 track under the middle part of the train, and all effects such as balance, friction and slippage should be neglected.

a) (1 point) To perform localization with only the odometer sensor, there is one missing parameter that needs to be given. What parameter is it?

b) (1 point) Let us define further by s_t the travelled distance by the train by the time t. Express the train motion model ($s_{t+1} = \dots$) in terms of wheel angular velocity ω_t, the missing parameter, and traveling time Δt.

$s_{t+1} =$

c) (1 point) Express Δs in the world polar coordinate system, using R and $\Delta\phi = \phi_{t+1} - \phi_t$.

d) (1 point) Formulate the factor given by the motion prior as a stochastic motion model with zero-mean Gaussian noise.

$p(s_{t+1} | s_t, \omega_{t+1}) =$

e) (2 points) Write down formulation of the localization problem as a factorgraph, assuming the measurements are independent and identically distributed Gaussians.

$$\arg\max_{s_1 \dots s_T} p(s_1 \dots s_T | s_0, \omega_1 \dots \omega_T) =$$

f) (1 point) Draw corresponding factor graph with denoted states, actions and measurements and factors.

g) (1 point) Write down the maximum likelihood reformulation of the problem that is suitable for gradient optimization.

h) (1 point) Write down residuals for the constraints. What is the dimensionality of each residual?

i) (2 points) Sketch a Jacobian for the residuals and fill at least one row with exact expressions. What is the dimensionality of the Jacobian?

j) (1 point) Is gradient optimization necessary to solve this problem? Why?

k) (1 points) Consider that you would like to estimate the missing parameter instead of getting it as input. In order to do it, you will need an absolute localization device. What kind of device would you choose?

l) (2 points) How would formulation of the factorgraph localization change for your choice in the previous question? What would be the states, actions and measurements? What would be the constraints between them?

3. **Probabilistic completeness:**

   a) (1 point) What is probabilistic completeness of motion planning algorithms?

4. **Visibility graph (VG):**

   a) (3 points) Describe (by text/sketch/pseudocode) how planning using Visibility Graph (VG) works.

b) (1 point) For which types of robots/maps/obstacles is VG approach applicable?

c) (2 point) Consider a 2D point robot: $q = (x, y) \in \mathcal{C}$, with forward motion model

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= u_2 \end{aligned}$$

where $u_1, u_2 \in \mathbf{R}$ are control inputs. Can be VG used to plan path for this robot? Explain your answer (yes/no answer is not enough).

d) (1 point) What is the time-complexity for constructing VG on map with n obstacles?

5. **Rapidly-exploring Random Tree (RRT):**

    a) (3 points) Describe (by text/sketch/pseudocode) how planning using RRT works.

    b) (2 points) What of the following is true for RRT?

    - Complete
    - Complete and optimal
    - Probabilistic complete and non-optimal
    - Complete and also probabilistic complete
    - Deterministic
    - Stochastic
    - A plan (path) after a steps is always better (shorter) than a plan after b steps if $a > b$

6. **KD-tree**

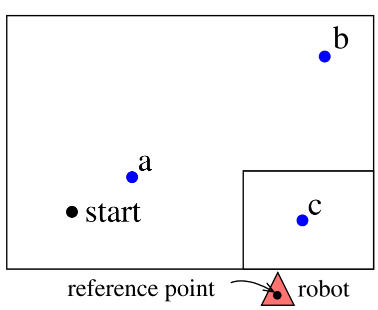
    a) (2 points) Describe the principle of KD-tree (assume that you have a set of 2D points).

b) (1 point) What is the purpose of KD-tree?

c) (1 point) What is the complexity of basic KD-tree operations?

7. **Triangle robot**

Consider the 2D triangle robot (robot can rotate) and a polygonal map (black=walls).



a) (3 points) Draw a cumulative distribution function of basic RRT planner for planning between pairs: start-a, start-b and start-c. Describe axes of the graph!