Random Walker Segmentation

Grady, IEEE PAMI 2006

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Interactive multilabel segmentation

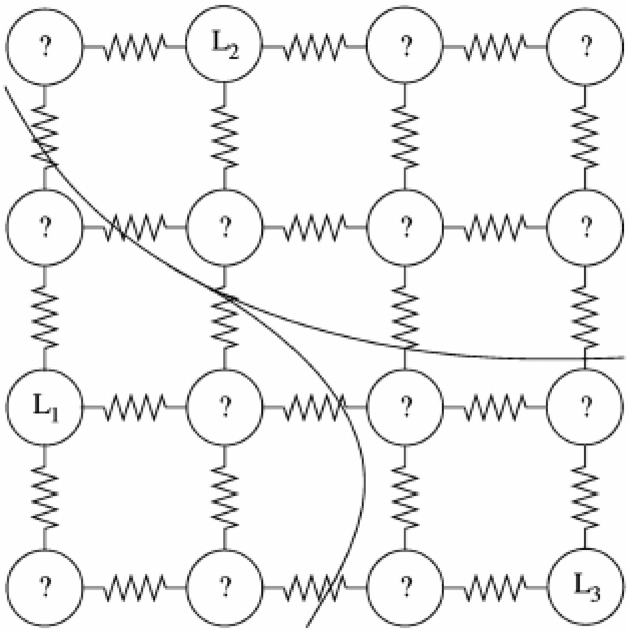
1. fast computation,
2. fast editing,
3. an ability to produce an arbitrary segmentation with enough interaction, and
4. intuitive segmentations.

seeds, linear system of equations with warm start

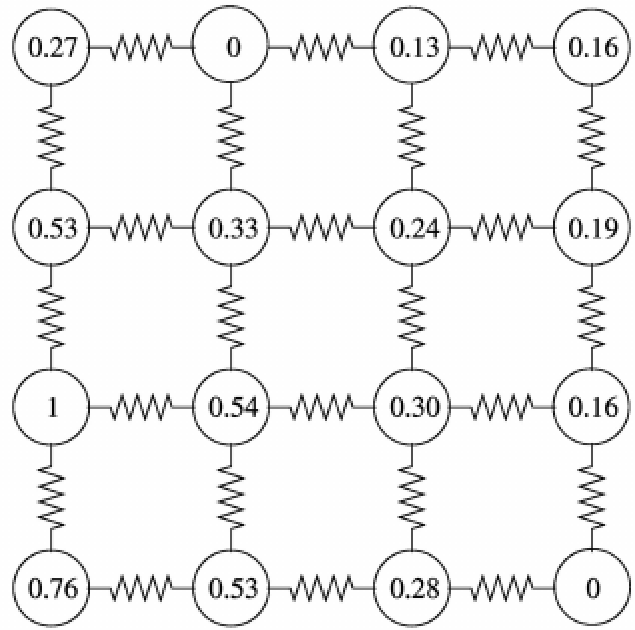
Random walker

- Image = graph, pixels = nodes, neighbors = edges
- Define a set of seedpoints for each class
- Start at any pixel
- Choose next pixel randomly — perceptually close with higher probability
- Class probability = probability of reaching seedpoints of a class
- No unary, only binary terms
- Simulation untractable
- Close solution exists

Electrical analogy



Electrical analogy (2)



Edge weights

$$w_{ij} = \exp\left(-\beta(g_i - g_j)^2\right)$$

and similar as seen in the normalized cut article

Continuous Dirichlet problem

$$\nabla^2 u = 0.$$

given boundary (seed) values.

Minimizes

$$D[u] = \frac{1}{2} \int_\Omega |\nabla u|^2 d\Omega,$$

Discrete formulation

Define the combinatorial Laplacian matrix [47] as

$$L_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -w_{ij} & \text{if } v_i \text{ and } v_j \text{ are adjacent nodes,} \\ 0 & \text{otherwise,} \end{cases}$$

where L_{ij} is indexed by vertices v_i and v_j.

Define the $m \times n$ edge-node **incidence matrix** as

$$A_{e_{ij}v_k} = \begin{cases} +1 & \text{if } i = k, \\ -1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

It holds that $A^T CA = L$

Constitutive matrix C - edge weights on the diagonal

Discrete formulation (2)

$$D[x] = \frac{1}{2}(Ax)^T C(Ax) = \frac{1}{2}x^T Lx = \frac{1}{2}\sum_{e_{ij}\in E} w_{ij}(x_i - x_j)^2,$$

$$\begin{aligned} D[x_U] &= \frac{1}{2}\begin{bmatrix} x_M^T x_U^T \end{bmatrix}\begin{bmatrix} L_M & B \\ B^T & L_U \end{bmatrix}\begin{bmatrix} x_M \\ x_U \end{bmatrix} \\ &= \frac{1}{2}\left(x_M^T L_M x_M + 2x_U^T B^T x_M + x_U^T L_U x_U\right) \end{aligned}$$

Deriving wrt x_u

$$L_U x_U = -B^T x_M,$$

Multiple labels

Therefore, for label s, the solution to the combinatorial Dirichlet problem may be found by solving

$$L_U x^s = -B^T m^s, \tag{10}$$

for one label or

$$L_U X = -B^T M, \tag{11}$$

Circuit analogy again

$$
\begin{aligned}
A^T z &= f && \text{(Kirchhoff's Current Law)}, && (13)\\
Cp &= z && \text{(Ohm's Law)}, && (14)\\
p &= Ax + b && \text{(Kirchhoff's Voltage Law)}, && (15)
\end{aligned}
$$

for a vector of branch currents, z, current sources, f, voltage sources, b, and potential drops (voltages), p. These three equations may be combined into the linear system

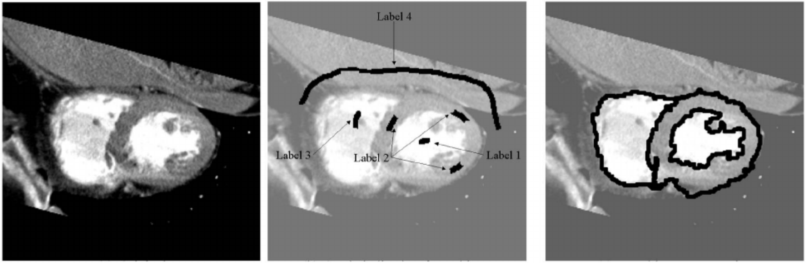
$$
\begin{aligned}
A^T CAx + A^T Cb &= f, && (16)\\
Lx &= f - A^T Cb, && (17)
\end{aligned}
$$

Numerical calculation

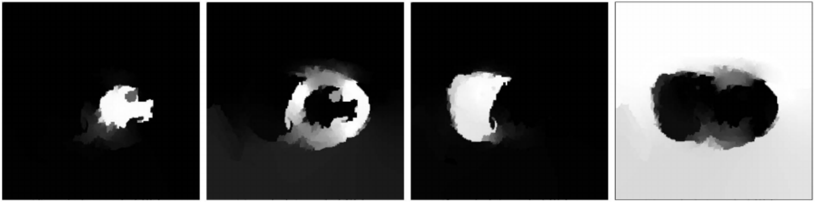
$$L_U x_U = -B^T x_M,$$

- Matrices big ($N \times N$) but sparse, $N \sim 10^4 - 10^8$
- LU decomposition and similar too slow for big images.
- Iterative solution - conjugated gradients, L not stored.

Example

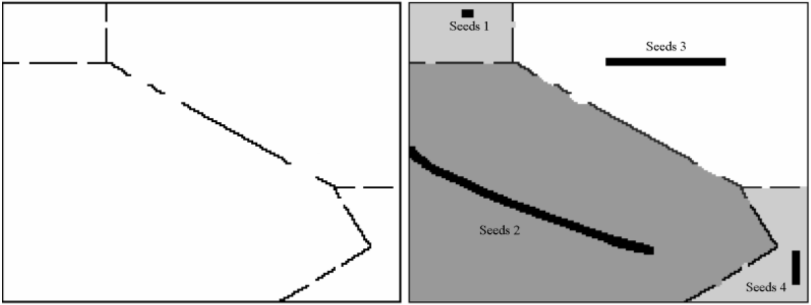


Example (cont)



class probabilities

Weak boundaries



Weak boundaries (2)

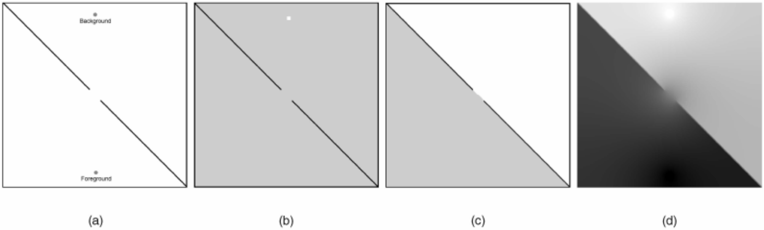
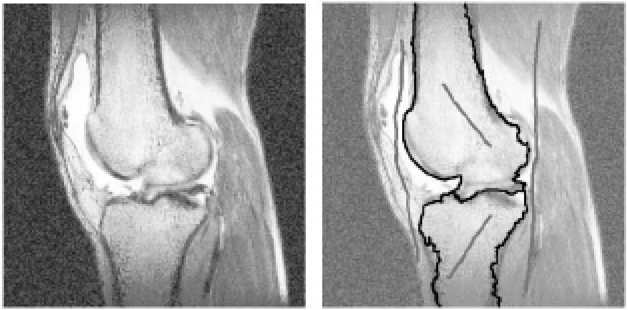
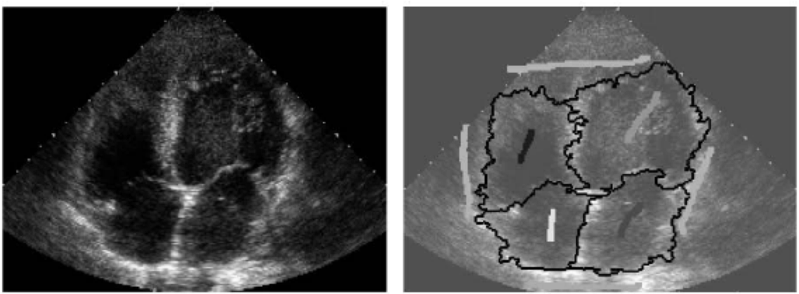


Fig. 5. Comparison of random walker algorithm to graph cuts for a weak boundary with small seeds. Note that a 4-connected graph was used in these experiments. (a) Original (synthetic) image created with a diagonal black line with a section completely erased. (b) Graph cuts solution—Since surface area of seeds is smaller than the weak boundary, the smallest cut minimally surrounds the seeds. (c) Random walker solution. (d) Probabilities associated with the random walker algorithm offer a notion of segmentation confidence at each pixel.

Medical examples

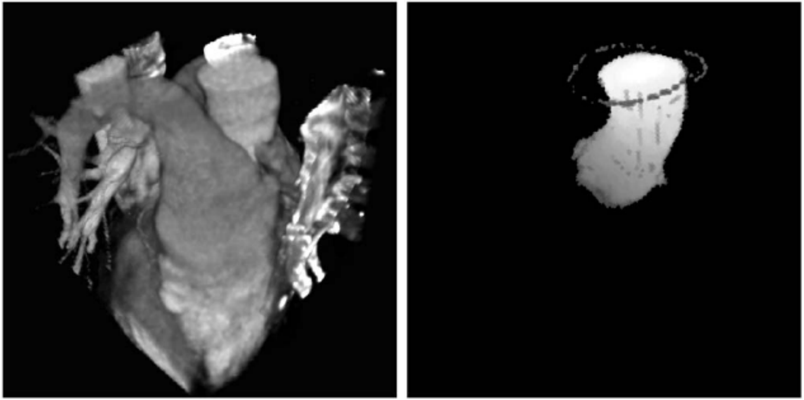


Medical examples (2)



Other examples

3D example



Conclusions

- supervised segmentation algorithm based on differences
- hints in the form of seeds/scribbles
- globally optimal (in the mathematical sense)
- only one parameter β
- robust
- Alternatives: GraphCuts, (fast) Levelsets, deep learning (Unet)