GRAPHICAL MARKOV MODELS (WS2023)
5. SEMINAR

Assignment 1. Consider the task of finding the most probable sequence of (hidden) states for a (Hidden) Markov model on a chain.

a) Show that the Dynamic Programming approach applied for this task can be interpreted as an equivalent transformation (re-parametrisation) of the model.

b) Show that the transformed functions (potentials) encode an explicit description of *all* optimisers of the problem.

Assignment 2. Prove that a sum of submodular functions is submodular.

Assignment 3. Examine the following functions w.r.t. submodularity
a) $f(k, k') = |k - k'|$, where $k, k' \in \mathbb{Z}$.
b) $f(k, k') = (k - k')^2$, where $k, k' \in \mathbb{Z}$.
c) $f(k_1, \dots, k_n) = \max_i k_i - \min_i k_i$, where $k_i \in \mathbb{Z}$.

Assignment 4. Let K be a completely ordered finite set. We assume w.l.o.g. $K = \{1, 2, \dots, m\}$. For a function $u\colon K \to \mathbb{R}$ define its discrete "derivative" by $Du(k) = u(k+1) - u(k)$.

a) Let u be a function $u\colon K^2 \to \mathbb{R}$ and denote by D_1 and D_2 the discrete derivatives w.r.t. the first and second argument. Prove the following equality

$$D_1 D_2 u(k_1, k_2) = u(k_1 + 1, k_2 + 1) + u(k_1, k_2) - u(k_1 + 1, k_2) - u(k_1, k_2 + 1).$$

Conclude that all mixed derivatives $D_1 D_2 u(k_1, k_2)$ of a submodular functions are non-positive.

b) Prove that the condition established in a) is necessary and sufficient for a function to be submodular.
Hint: Start from the observation that the following equality holds for a function of one variable

$$u(k + l) - u(k) = \sum_{i=k}^{k+l-1} Du(i)$$

and generalise it for functions of two variables.

c) Prove that any function $u\colon K^2 \to \mathbb{R}$ can be represented as a sum of a submodular and a supermodular function.
Hint: Consider the mixed derivative $D_1 D_2 u(k_1, k_2)$, decompose it into its negative and positive part and "integrate" them back separately.

Assignment 5. Consider a GRF for binary valued labellings $x\colon V \to \{0, 1\}$ of a graph (V, E) given by

$$p(x) = \frac{1}{Z} \exp\Big[\sum_{ij \in E} u_{ij}(x_i, x_j) + \sum_{i \in V} u_i(x_i)\Big].$$

Show that is is always possible to find an equivalent transformation (re-parametrisation)

$$u_{ij} \to \tilde{u}_{ij}, \quad u_i \to \tilde{u}_i$$

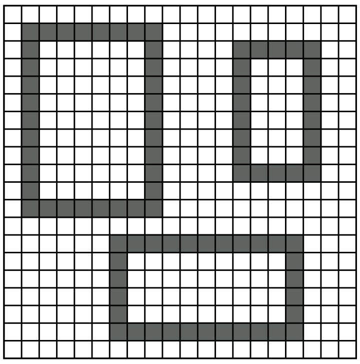
such that the new pairwise functions $\tilde{u}_{ij}$ have the form

$$\tilde{u}_{ij}(x_i, x_j) = \alpha_{ij}|x_i - x_j|$$

with some real numbers $\alpha_{ij} \in \mathbb{R}$.

Assignment 6. Transform the *Travelling Salesman Problem* into a $(\min, +)$-problem.

Assignment 7. Consider the language L of all b/w images $x\colon D \to \{b, w\}$ containing an arbitrary number of non-overlapping and non-touching one pixel wide rectangular frames (see figure).



a) Prove that L is not expressible by a locally conjunctive predicate

$$x \in L \quad \text{if and only if} \quad f(x) = \bigwedge_{c \in \mathcal{C}} f_c(x_c) = 1$$

with predicates f_c, defined on image fragments x_c, where the hyperedges $c \subset D$ have bounded size $|c| \leqslant m < |D|$.

b) Show that L can be expressed by introducing a field $s\colon D \to K$ of non-terminal symbols, a locally conjunctive predicate on them and pixel-wise predicates g relating the non-terminal and terminal symbol in each pixel

$$x \in L \quad \text{if and only if} \quad \bigvee_{s \in K^D} \Big[\bigwedge_{c \in \mathcal{C}} f_c(s_c) \wedge \bigwedge_{i \in D} g(x_i, s_i) \Big] = 1$$

Find a suitable structure $\mathcal{C}$, an alphabet of non-terminal symbols K and predicates f_c, g.