GRAPHICAL MARKOV MODELS (WS2023/24)
1. SEMINAR

Assignment 1 (Segmentation)**.** Consider the finite language in $\{a, b\}^n$ containing all words of the form

$$\underbrace{\overbrace{a \dots a}^{n_a} b \dots b}_{n}$$

where $0 \leqslant n_a \leqslant n$. Informally, this model describes all segmentations of a sequence into two consecutive intervals.
a) Construct a Markov chain assigning a strictly positive probability to all admissible words and zero probability to all inadmissible words.
b) Construct a Markov chain such that all admissible words are equiprobable, whereas all other words have probability zero.
Hint: Determine the p.d. for the first state in the chain and all transition probabilities (the latter may depend on the position).

Assignment 2. A tetrahedron with differently coloured facets lies with the blue side down on a table. The tetrahedron is tilted n times over a randomly chosen edge (At each time instance there are three edges incident with the table to choose from). What is the probability to have the blue side down at the end?

Assignment 3 (Gambler's ruin)**.** Consider a random walk on the set $L = \{0, 1, 2, \dots, a\}$ starting in some point $x \in L$. The position jumps by either ± 1 in each time period (with equal probabilities). The walk ends if either of the boundary states $0, a$ is hit. Compute the probability $u(x)$ to finish in state a if the process starts in state x.
Hints:

(1) What are the values of $u(0)$ and of $u(a)$?
(2) Find a difference equation for $u(x)$, $0 < x < a$ by relating it with $u(x - 1)$ and $u(x + 1)$.
(3) Translate the difference equation into a relation between the successive differences $u(x + 1) - u(x)$ and $u(x) - u(x - 1)$.
(4) Deduce that the solution is a linear function of x and find its coefficients from the boundary conditions $u(0)$ and $u(a)$.

Assignment 4. Consider a homogeneous Markov chain model with a doubly stochastic matrix of transition probabilities P. (A matrix is doubly stochastic if all its elements are non-negative and each of its rows and columns sums up to one.)
a) Prove that the uniform distribution on its states is a stationary distribution for P.
a) Conclude that the model has the detailed balance property if P is symmetric.

Assignment 5. Consider the Ehrenfest model (Example 1., Section 1. of the lecture). Prove that the distribution

$$p(s_i = k) = \frac{1}{2^N} \binom{N}{k}$$

is a stationary distribution for the corresponding Markov chain model. Prove that the model does not fulfil the conditions of Theorem 1 (Section 1. of the lecture).

A homogeneous Markov chain is *irreducible* if each state $k \in K$ is accessible from each state $k' \in K$, i.e., for each pair of states k, k', there exists an $m > 0$ for which $P^m_{kk'} > 0$. (P denotes the matrix of transition probabilities of the model.)

Let $\mathcal{T}_k = \{t > 0 \mid P^t_{kk} > 0\}$ be the set of all time steps for which a Markov chain can start and end in a state $k \in K$. The *period* τ_k of a state $k \in K$ is defined as the greatest common divisor of the numbers in $\mathcal{T}_k$. If $\tau_k = 1$, the state is called a-periodic.

If the Markov chain is irreducible, then the period of all states is equal (w/o proof). If an irreducible Markov chain has a-periodic states, then it fulfils the conditions of Theorem 1. (Section 1. of the lecture)

Assignment 6. Consider a random walk on an undirected graph (see Example 2., Section 1. of the lecture). What conditions on the graph and on the probabilities W_{ij} guarantee irreducibility of this Markov model and a-periodicity of its states?