UIR: Game Theory in Robotics - Lab 1

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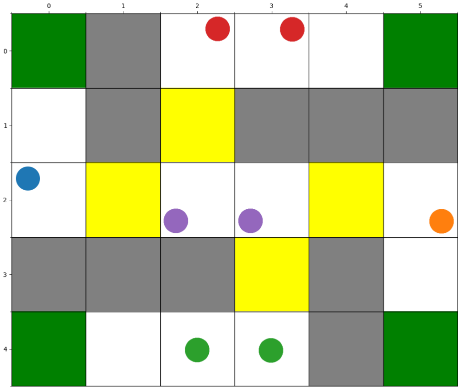
Overview

Game Theory

- We know how to control robots to perform some plan. However, in real world the robots must often cooperate or compete with some other agents. (For example competition with humans in security scenarios.) The outcome of our actions can then critically depend on the actions of other agents and game theory is a framework studying those cases.
- In the assignments we will focus on two-player zero-sum games. (no cooperation ☹)
- Optimal strategy in such games is described by Nash equilibrium.

Adversarial Planning Homework

- green - entrances
- yellow - goals
- gray - walls
- white - free spaces
- dots - sensors
- One player is finding path from entrance to goal and the other chooses sensors. Cost for path-finder is crossing chosen sensor and small cost for each move. The game is zero-sum.



Adversarial Planning Homework

- We saw general setting of this problem in the lecture. In our setting the defender can only choose one sensor.
- We will use double oracle algorithm to solve the problem.
- Components - the oracles
- Planning oracle: We need a planner which given the current sensor coverage can plan the best path.
- Sensor oracle: We need to be able to get the best sensor given a probability distribution over paths (the strategy)
- We will also need to evaluate value of the pair of path and sensor to fill the matrix.

Two-player Zero-sum Game

- Strategy sets M and N
- Utility matrix $\mathbf{U} = [c_{ij}]_{i \in M, j \in N}$
- Represented as matrix and also called Matrix game or Normal-form Game (NFG)
- Example: $|M| = 2, \quad |N| = 3, \quad \mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$
- Row player with 2 actions is maximizing and column player with 3 actions is minimizing

Double Oracle

- Used to find Nash Equilibrium of zero-sum two-player game (in our example normal form game).
- Pick randomly one action for each players and form restricted subgame using those actions.
- In each iteration solve the subgame and find the best responses to the current strategy profile in the subgame for both players and add them to the restricted subgame, then solve the subgame again.
- We stop when we encounter iteration where both best responses are already in the subgame.

Double Oracle

Try Double Oracle algorithm on the following matrix game

	A	B	C	D	E
V	-8	9	0	7	-6
W	6	9	6	5	6
X	1	-8	3	8	7
Y	5	2	6	-5	2
Z	4	3	3	0	8

Double Oracle

BRs $= \{W, A\}$ - A already in the subgame so we add W
Solution of the subgame gives A and W with probability 1

	A	B	C	D	E
V	-8	9	0	7	-6
W	6	9	6	5	6
X	1	-8	3	8	7
Y	5	2	6	-5	2
Z	4	3	3	0	8

BRs $= \{W, D\}$ so we add D to the subgame and solve it

	A	B	C	D	E
V	-8	9	0	7	-6
W	6	9	6	5	6
X	1	-8	3	8	7
Y	5	2	6	-5	2
Z	4	3	3	0	8

Double Oracle

Strategy for Player 1: $X(V) = \frac{1}{16}, X(W) = \frac{15}{16}$
Strategy for Player 2: $Y(A) = \frac{1}{8}, Y(D) = \frac{7}{8}$

	A	B	C	D	E	
V	-8	9	0	7	-6	5.125
W	6	9	6	5	6	5.125
X	1	-8	3	8	7	7.125
Y	5	2	6	-5	2	-3.75
Z	4	3	3	0	8	0.5
	$5\frac{1}{8}$	9	$5\frac{5}{8}$	$5\frac{1}{8}$	$5\frac{2}{8}$	

BRs $= \{X, (A, D)\}$ so we add X to the subgame and we solve it

	A	B	C	D	E	
V	-8	9	0	7	-6	5.125
W	6	9	6	5	6	5.125
X	1	-8	3	8	7	7.125
Y	5	2	6	-5	2	-3.75
Z	4	3	3	0	8	0.5
	$5\frac{1}{8}$	9	$5\frac{5}{8}$	$5\frac{1}{8}$	$5\frac{2}{8}$	

Strategy for Player 2: $S_1(A) = \frac{3}{8}, S_1(D) = \frac{5}{8}$
Strategy for Player 1: $S_2(V) = 0, S_2(W) = \frac{7}{8}, S_2(X) = \frac{1}{8}$

	A	B	C	D	E	
V	-8	9	0	7	-6	1.375
W	6	9	6	5	6	5.375
X	1	-8	3	8	7	5.375
Y	5	2	6	-5	2	-1.25
Z	4	3	3	0	8	1.5
	$5\frac{3}{8}$	$6\frac{7}{8}$	$5\frac{5}{8}$	$5\frac{3}{8}$	$6\frac{1}{8}$	

All best responses are already in the subgame
so we stop and we have a Nash equilibrium

	A	B	C	D	E	
V	-8	9	0	7	-6	1.375
W	6	9	6	5	6	5.375
X	1	-8	3	8	7	5.375
Y	5	2	6	-5	2	-1.25
Z	4	3	3	0	8	1.5
	$5\frac{3}{8}$	$6\frac{7}{8}$	$5\frac{5}{8}$	$5\frac{3}{8}$	$6\frac{1}{8}$	

Linear Program to Find Nash Equilibrium

Create Linear programs for both players in this game

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & -2 \end{bmatrix}$$

Primal

$$\begin{aligned} \text{Maximize} \quad & v \\ \text{subject to} \quad & \mathbf{x}^\mathsf{T}\mathbf{U}\mathbf{e}_j \geq v, \forall j \in N \\ & x_i \geq 0, \forall i \in M \\ & \sum_{i \in M} x_i = 1 \\ & v \in \mathbb{R} \end{aligned}$$

Dual

$$\begin{aligned} \text{Minimize} \quad & v \\ \text{subject to} \quad & \mathbf{y}^\mathsf{T}\mathbf{U}^\mathsf{T}\mathbf{e}_j \leq v, \forall j \in M \\ & y_i \geq 0, \forall i \in N \\ & \sum_{i \in N} y_i = 1 \\ & v \in \mathbb{R} \end{aligned}$$

Linear Program to Find Nash Equilibrium

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & -2 \end{bmatrix}, \quad \bar{x} = \left(\frac{4}{5}, \frac{1}{5}\right), \bar{y} = \left(\frac{1}{2}, 0, \frac{1}{2}\right), \bar{v} = 2$$

Primal

$$\begin{aligned} \text{maximize} \quad & v \\ \text{s.t.} \quad x_1 + 6x_2 &\geq v \\ 2x_1 + 5x_2 &\geq v \\ 3x_1 - 2x_2 &\geq v \\ x_1 + \ \ x_2 &= 1 \\ x_1, x_2 &\geq 0 \\ v &\in \mathbb{R} \end{aligned}$$

Dual

$$\begin{aligned} \text{minimize} \quad & v \\ \text{s.t.} \quad y_1 + 2y_2 + 3y_3 &\leq v \\ 6y_1 + 5y_2 - 2y_3 &\leq v \\ y_1 + \ \ y_2 + \ \ y_3 &= 1 \\ y_1, y_2, y_3 &\geq 0 \\ v &\in \mathbb{R} \end{aligned}$$

The End