3D Computer Vision - Task 0-4 notes

Lab session materials for subjects B4M33TDV, BE4M33TDV, XP33VID

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Transformation between two projective planes

vector form	elementwise form
$\lambda \underline{\mathbf{x}}_2 = \mathbf{H} \underline{\mathbf{x}}_1$	$\lambda \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$

- bijection, $\mathbf{H}$ is regular, invertible
- $\lambda \neq 0$, any nonzero multiple of $\mathbf{H}$ represents the same transformation
- works for ideal points ($w_1 = 0$, $w_2 = 0$) as well
- in the affine plane (no ideal points) we can work with $w_1 = 1$, $w_2 = 1$ fixed

Notation of rows of $\mathbf{H}$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix}$$

i.e., $\mathbf{h}_i$ (without the $^\top$) is the i-th row of the matrix transposed to the column vector.

- Express particular elements of the vector $\underline{\mathbf{x}}_2$

vector form / elementwise form

$$\begin{aligned} \lambda u_2 &= \mathbf{h}_1^\top \underline{\mathbf{x}}_1 \quad (1) & \lambda u_2 &= h_{11} u_1 + h_{12} v_1 + h_{13} w_1 \\ \lambda v_2 &= \mathbf{h}_2^\top \underline{\mathbf{x}}_1 \quad (2) & \lambda v_2 &= h_{21} u_1 + h_{22} v_1 + h_{23} w_1 \\ \lambda w_2 &= \mathbf{h}_3^\top \underline{\mathbf{x}}_1 \quad (3) & \lambda w_2 &= h_{31} u_1 + h_{32} v_1 + h_{33} w_1 \end{aligned}$$

- Eliminate nonzero λ – multiply sides of eq. (1) and (2) by swapped sides of eq. (3) divided by λ, i.e., multiply the right side of (1) and (2) by w_2 and the left side of (1) and (2) by $\mathbf{h}_3^\top \underline{\mathbf{x}}_1 / \lambda$.
  Note: we do not divide by any of the coordinate entries (some can be zero).

$$\begin{aligned} u_2 \mathbf{h}_3^\top \underline{\mathbf{x}}_1 &= w_2 \mathbf{h}_1^\top \underline{\mathbf{x}}_1 & u_2(h_{31} u_1 + h_{32} v_1 + h_{33} w_1) &= w_2(h_{11} u_1 + h_{12} v_1 + h_{13} w_1) \\ v_2 \mathbf{h}_3^\top \underline{\mathbf{x}}_1 &= w_2 \mathbf{h}_2^\top \underline{\mathbf{x}}_1 & v_2(h_{31} u_1 + h_{32} v_1 + h_{33} w_1) &= w_2(h_{21} u_1 + h_{22} v_1 + h_{23} w_1) \end{aligned}$$

- Some manipulation – to the homogeneous form, 'transpose' dot products

$$\begin{aligned} w_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_1 \qquad\qquad\qquad & - u_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_3 = 0 \\ w_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_2 \quad & - v_2 \underline{\mathbf{x}}_1^\top \mathbf{h}_3 = 0 \end{aligned}$$

$$\begin{aligned} w_2 u_1 h_{11} + w_2 v_1 h_{12} + w_2 w_1 h_{13} \qquad\qquad\qquad\qquad & - u_2 u_1 h_{31} - u_2 v_1 h_{32} - u_2 w_1 h_{33} = 0 \\ w_2 u_1 h_{21} + w_2 v_1 h_{22} + w_2 w_1 h_{23} \quad & - v_2 u_1 h_{31} - v_2 v_1 h_{32} - v_2 w_1 h_{33} = 0 \end{aligned}$$

- Matrix representation – collect all known terms (point coordinates from the i-th corresponding pair of points) to a matrix $\mathbf{A}_i$ and all unknowns to a vector $\mathbf{h}$

$$\underbrace{\begin{bmatrix} w_2\underline{\mathbf{x}}_1^\top & \mathbf{0}^\top & -u_2\underline{\mathbf{x}}_1^\top \\ \mathbf{0}^\top & w_2\underline{\mathbf{x}}_1^\top & -v_2\underline{\mathbf{x}}_1^\top \end{bmatrix}}_{\mathbf{A}_i} \underbrace{\begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix}}_{\mathbf{h}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} w_2u_1 & w_2v_1 & w_2w_1 & 0 & 0 & 0 & -u_2u_1 & -u_2v_1 & -u_2w_1 \\ 0 & 0 & 0 & w_2u_1 & w_2v_1 & w_2w_1 & -v_2u_1 & -v_2v_1 & -v_2w_1 \end{bmatrix}}_{\mathbf{A}_i} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\mathbf{h}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\mathbf{A}_i$ – 2 equations, $\mathbf{h}$ – 8 unknowns (up to scalar multiple) $\longrightarrow$ 4 correspondences needed, stacked to 8×9 matrix $\mathbf{A}$. Then $\mathbf{h}$ is a solution of linear homogeneous system, i.e. right null-space of $\mathbf{A}$, and $\mathbf{H}$ is composed from $\mathbf{h}$.

$$\mathbf{A} = \begin{bmatrix} \vdots \\ \mathbf{A_i} \\ \vdots \end{bmatrix} \qquad \mathbf{A}\mathbf{h} = \mathbf{0}$$

Note: When working with points from the affine plane only, we can assume point coordinates normalized to $w_1 = 1$, $w_2 = 1$.

$$\mathbf{A} = \begin{bmatrix} & \vdots & \\ \underline{\mathbf{x}}_1^\top & \mathbf{0}^\top & -u_2\underline{\mathbf{x}}_1^\top \\ \mathbf{0}^\top & \underline{\mathbf{x}}_1^\top & -v_2\underline{\mathbf{x}}_1^\top \\ & \vdots & \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} & & & & \vdots & & & & \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u_2u_1 & -u_2v_1 & -u_2 \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v_2u_1 & -v_2v_1 & -v_2 \\ & & & & \vdots & & & & \end{bmatrix}$$

- Homography between two images does not depends on world coordinate system choice (chosen at the first camera matrix $\mathbf{P_1}$)
- Two cameras $\mathbf{P}_1$ and $\mathbf{P}_2$ observing a plane $\mathbf{\Pi}$

$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \quad \underline{\mathbf{\Pi}}^\top = \begin{bmatrix} \mathbf{n}^\top & d \end{bmatrix}$$

1. Reconstruct $\underline{\mathbf{X}}$ constrained by the plane $\underline{\mathbf{\Pi}}$ from $\underline{\mathbf{u}}_1$
 (projection equation augmented by plane constraint row $\underline{\mathbf{\Pi}}^\top \underline{\mathbf{X}} = 0$ to obtain 4×4 invertible matrix

$$\lambda_1 \begin{bmatrix} \underline{\mathbf{u}}_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} \\ \mathbf{n}^\top & d \end{bmatrix}}_{\mathbf{M}} \underline{\mathbf{X}}$$

$$\underline{\mathbf{X}} = \lambda_1 \underbrace{\begin{bmatrix} \mathbf{K}_1^{-1} & \mathbf{0} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} & \frac{1}{d} \end{bmatrix}}_{\mathbf{M}_{-1}} \begin{bmatrix} \underline{\mathbf{u}}_1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} \mathbf{K}_1^{-1} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} \end{bmatrix} \underline{\mathbf{u}}_1$$

2. Project $\underline{\mathbf{X}}$ to the second camera to obtain $\mathbf{H}$

$$\lambda_2 \underline{\mathbf{u}}_2 = \mathbf{P_2} \underline{\mathbf{X}} = \lambda_1 \mathbf{K_2} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{K_1}^{-1} \\ \frac{-\mathbf{n}^\top \mathbf{K}_1^{-1}}{d} \end{bmatrix} \underline{\mathbf{u}}_1 = \lambda_1 \underbrace{\mathbf{K}_2 \left(\mathbf{R} - \mathbf{t}\mathbf{n}^\top / d \right) \mathbf{K}_1^{-1}}_{\mathbf{H}_{12}} \underline{\mathbf{u}}_1$$

- General cameras – derivation is the same, but a bit ugly

$$\lambda_1 \begin{bmatrix} \underline{\mathbf{u}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{K_1 R_1} & -\mathbf{K_1 R_1 C_1} \\ \mathbf{n}^\top & d \end{bmatrix} \underline{\mathbf{X}} \rightarrow \underline{\mathbf{X}} = \lambda_1 \begin{bmatrix} \mathbf{R}_1^\top \mathbf{K}_1^{-1} - \frac{\mathbf{C}_1 \mathbf{n}^\top \mathbf{R}_1^\top \mathbf{K}_1^{-1}}{d + \mathbf{n}^\top \mathbf{C}_1} \\ \frac{-\mathbf{n}^\top \mathbf{R}_1^\top \mathbf{K}_1^{-1}}{d + \mathbf{n}^\top \mathbf{C}_1} \end{bmatrix} \underline{\mathbf{u}}$$

$$\lambda_2 \underline{\mathbf{u}}_2 = \mathbf{P}_2 \underline{\mathbf{X}} = \lambda_1 \mathbf{K}_2 \begin{bmatrix} \mathbf{R}_2 & -\mathbf{R}_2 \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1^\top - \frac{\mathbf{C}_1 \mathbf{n}^\top \mathbf{R}_1^\top}{d + \mathbf{n}^\top \mathbf{C}_1} \\ \frac{-\mathbf{n}^\top \mathbf{R}_1^\top}{d + \mathbf{n}^\top \mathbf{C}_1} \end{bmatrix} \mathbf{K}_1^{-1} \underline{\mathbf{u}}$$

$$\mathbf{H} = \mathbf{K}_2 \left(\mathbf{R}_2 \mathbf{R}_1^\top - \frac{\mathbf{R}_2 (\mathbf{C}_1 - \mathbf{C}_2) \mathbf{n}^\top \mathbf{R}_1^\top}{d + \mathbf{n}^\top \mathbf{C}_1} \right) \mathbf{K_1}^{-1}$$

Inverse homography

1. Change of coordinate frame - apply $\mathbf{T}$ (4×4): $\underline{\mathbf{X}}' = \mathbf{T}\underline{\mathbf{X}}$, $\mathbf{P}' = \mathbf{P}\mathbf{T}^{-1}$, $\underline{\boldsymbol{\Pi}}'^\top = \underline{\boldsymbol{\Pi}}\mathbf{T}^{-1}$

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{P}'_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top\mathbf{t} \end{bmatrix} \quad \mathbf{P}'_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \underline{\boldsymbol{\Pi}}'^\top = \begin{bmatrix} \underbrace{\mathbf{n}^\top\mathbf{R}^\top}_{\mathbf{n}'^\top} & \underbrace{-\mathbf{n}^\top\mathbf{R}^\top\mathbf{t} + d}_{d'} \end{bmatrix}$$

$\mathbf{n}'$, d' are parameters of the plane w.r.t. coordinate system of the second camera

2. Use eq. from Page 6, substitute $\mathbf{P}_1 \leftarrow \mathbf{P}'_2$, $\mathbf{P}_2 \leftarrow \mathbf{P}'_1$, $\underline{\boldsymbol{\Pi}} \leftarrow \underline{\boldsymbol{\Pi}}'$

$$\mathbf{H}_{21} = \mathbf{K}_1 \left(\mathbf{R}^\top + \frac{\mathbf{R}^\top\mathbf{t}\mathbf{n}^\top\mathbf{R}^\top}{d - \mathbf{n}^\top\mathbf{R}^\top\mathbf{t}} \right) \mathbf{K}_2^{-1} = \mathbf{K}_1 \left(\mathbf{R}^\top + \frac{\mathbf{R}^\top\mathbf{t}\mathbf{n}'^\top}{d'} \right) \mathbf{K}_2^{-1}$$

(can be easily verified that $\mathbf{H}_{21} = \mathbf{H}_{12}^{-1}$ from Page 6)

Alternative: use the general equation on Page 7 for $\mathbf{P}_1$, $\mathbf{P}_2$ swapped and the same $\underline{\boldsymbol{\Pi}}$.

- A pair of cameras $\mathbf{K_1}[\mathbf{I}|\mathbf{0}]$, $\mathbf{K_2}[\mathbf{R}|\mathbf{t}]$ observes a pair of planes $\underline{\mathbf{\Pi}}_a$, $\underline{\mathbf{\Pi}}_b \to \mathbf{H}^a_{12}$, $\mathbf{H}^b_{12}$
- Consider a composed homography 'there-and-back': $1 \to 2$ via $\underline{\mathbf{\Pi}}_a$ and $2 \to 1$ via $\underline{\mathbf{\Pi}}_b$

$$\mathbf{H} = \mathbf{H}^b_{21}\mathbf{H}^a_{12} = \mathbf{K}_1(\mathbf{R}^\top + \mathbf{R}^\top\mathbf{t}\mathbf{n}'^\top_b/d'_b)\mathbf{K_2}^{-1}\,\mathbf{K_2}(\mathbf{R} - \mathbf{t}\mathbf{n}^\top_a/d_a)\mathbf{K}_1^{-1} =$$

$$= \mathbf{I} + \underbrace{(-\mathbf{K_1}\mathbf{R}^\top\mathbf{t})}_{\mathbf{v}}\underbrace{\left(\frac{\mathbf{n}^\top_a}{d_a} - \frac{\mathbf{n}'^\top_b\mathbf{R}}{d'_b} + \frac{\mathbf{n}'^\top_b\mathbf{t}\mathbf{n}^\top_a}{d_a\,d'_b}\right)\mathbf{K_1}^{-1}}_{\mathbf{a}^\top} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$$

Planar homology $\mathbf{H}$ with vertex $\mathbf{v}$, axis $\mathbf{a}$, eigen value $\mu = 1 + \mathbf{a}^\top\mathbf{v}$: $\mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$

$\mathbf{H}\mathbf{v} = \mathbf{v} + \mathbf{v}\mathbf{a}^\top\mathbf{v} = (1 + \mathbf{a}^\top\mathbf{v})\mathbf{v} = \mu\mathbf{v}$ — $\mathbf{v}$ is eigenvector of $\mathbf{H}$ corresponding to eigenvalue μ, as a planar point it is fixed w.r.t. the transformation

$\mathbf{a}^\top\underline{\mathbf{x}} = 0 \Longrightarrow \mathbf{H}\underline{\mathbf{x}} = \underline{\mathbf{x}} + \mathbf{v}\mathbf{a}^\top\underline{\mathbf{x}} = \underline{\mathbf{x}}$ — $\mathbf{a}$ is a line of fixed points - 2D eigenspace of $\mathbf{H}$ with double eigenvalue 1

- $\mathbf{a}$ and $\mathbf{v}$ represents homogeneous image entities but their scale matters in $\mathbf{I} + \mathbf{v}\mathbf{a}^\top$
- the point $\mathbf{v}$ and all points on a line $\mathbf{a}$ in one image are mapped to the second image to a points same for both $\mathbf{H}^a_{12}$ and $\mathbf{H}^b_{12}$: $\quad \mathbf{H}^a_{12}\mathbf{v} \simeq \mathbf{H}^b_{12}\mathbf{v} \quad (\mathbf{H}^a_{12})^{-\top}\mathbf{a} \simeq (\mathbf{H}^b_{12})^{-\top}\mathbf{a}$
- $\mathbf{a}$ is the image of the common line of the planes
- $\mathbf{v} = -\mathbf{K_1}\mathbf{R}^\top\mathbf{t}$ is the epipole in the first image
- eigenvectors of $\mathbf{H}$ are $(\mathbf{v}, \mathbf{x}_1, \mathbf{x}_2)$ corresponding to eigenvalues $(\mu, 1, 1)$, and $\mathbf{a} = \mathbf{x}_1 \times \mathbf{x}_2$ (when the matrix is multiplied by an unknown scale λ, the eigenvalues become $(\lambda\mu, \lambda, \lambda)$)

1. A point $\underline{\mathbf{u}}$ and its image $\lambda\underline{\mathbf{u}}' = \mathbf{H}\underline{\mathbf{u}}$ define a common line $\underline{\mathbf{k}}$ (if not lying on $\mathbf{a}$ or $\mathbf{v}$). Since $\mathbf{a}^\top\underline{\mathbf{u}} \neq 0$, the vertex must lie on this line as well:

$$\underline{\mathbf{k}}^\top\underline{\mathbf{u}} = 0\,, \quad \underline{\mathbf{k}}^\top\mathbf{H}\mathbf{u} = 0$$

$$\underline{\mathbf{k}}^\top\left(\underline{\mathbf{u}} + \mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}}\right) = 0 \quad\Longrightarrow\quad \underline{\mathbf{k}}^\top\mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}} = 0 \quad\Longrightarrow\quad \underline{\mathbf{k}}^\top\mathbf{v} = 0$$

The vertex is estimated from two correspondences: $\mathbf{v} = (\underline{\mathbf{u}}_1 \times \underline{\mathbf{u}}'_1) \times (\underline{\mathbf{u}}_2 \times \underline{\mathbf{u}}'_2)$

2. When the vertex $\mathbf{v}$ is known, a linear non-homogeneous system can be used for computing $\mathbf{a}$. Let $\mathbf{H} = \mathbf{I} + \mathbf{v}\mathbf{a}^\top$, $\underline{\mathbf{u}}_i^\top = [x_i, y_i, w_i]$, $\underline{\mathbf{u}}_i'^\top = [x'_i, y'_i, w'_i]$, $\mathbf{v}^\top = [x_v, y_v, w_v]$.

$$\lambda \begin{bmatrix} x'_i \\ y'_i \\ w'_i \end{bmatrix} = \underline{\mathbf{u}}_i + \mathbf{v}\mathbf{a}^\top\underline{\mathbf{u}}_i = \underline{\mathbf{u}}_i + \mathbf{v}\underline{\mathbf{u}}_i^\top\mathbf{a} = \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} + \begin{bmatrix} x_v\,\underline{\mathbf{u}}_i^\top \\ y_v\,\underline{\mathbf{u}}_i^\top \\ w_v\,\underline{\mathbf{u}}_i^\top \end{bmatrix}\mathbf{a}$$

$$\begin{aligned} x'_i(w_i + w_v\underline{\mathbf{u}}_i^\top\mathbf{a}) &= w'_i x_i + w'_i x_v \underline{\mathbf{u}}_i^\top\mathbf{a} \\ y'_i(w_i + w_v\underline{\mathbf{u}}_i^\top\mathbf{a}) &= w'_i y_i + w'_i y_v \underline{\mathbf{u}}_i^\top\mathbf{a} \end{aligned} \qquad \begin{aligned} (x'_i w_v - w'_i x_v)\underline{\mathbf{u}}_i^\top\mathbf{a} &= x_i w'_i - x'_i w_i \\ (y'_i w_v - w'_i y_v)\underline{\mathbf{u}}_i^\top\mathbf{a} &= y_i w'_i - y'_i w_i \end{aligned}$$

The left sides of these eqs. consists of the same vector $\underline{\mathbf{u}}_i^\top$ multiplied by a scalar, i.e., they are linearly dependent, and only one eq. can be used $\rightarrow$ three points needed.

$$\mathbf{A} = \begin{bmatrix} (x'_1 w_v - w'_1 x_v)\underline{\mathbf{u}}_1^\top \\ (x'_2 w_v - w'_2 x_v)\underline{\mathbf{u}}_2^\top \\ (x'_3 w_v - w'_3 x_v)\underline{\mathbf{u}}_3^\top \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} x_1 w'_1 - w_1 x'_1 \\ x_2 w'_2 - w_2 x'_2 \\ x_3 w'_3 - w_3 x'_3 \end{bmatrix} \quad \mathbf{a} = \mathbf{A}^{-1}\mathbf{b}$$