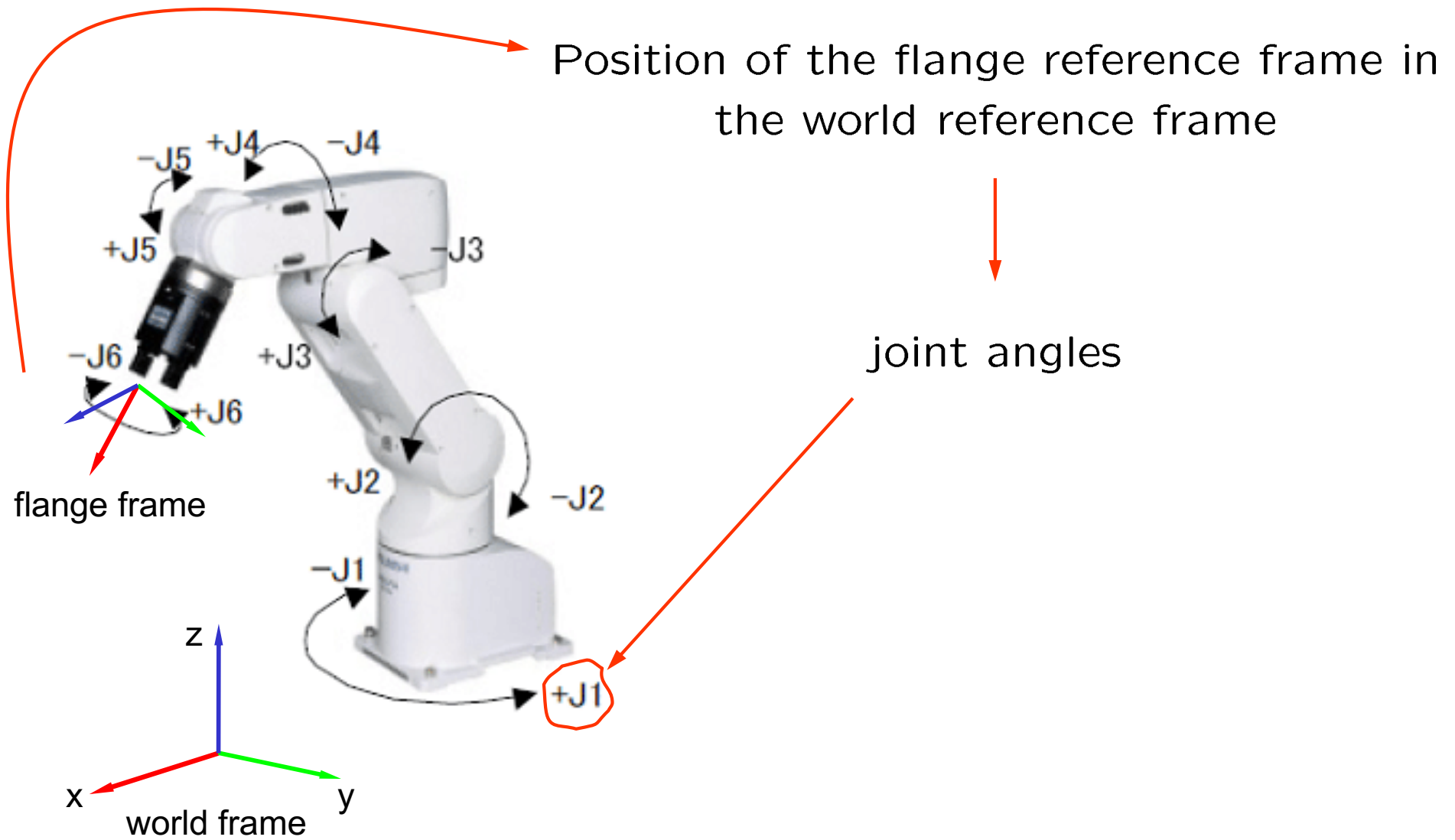


Advanced Robotics

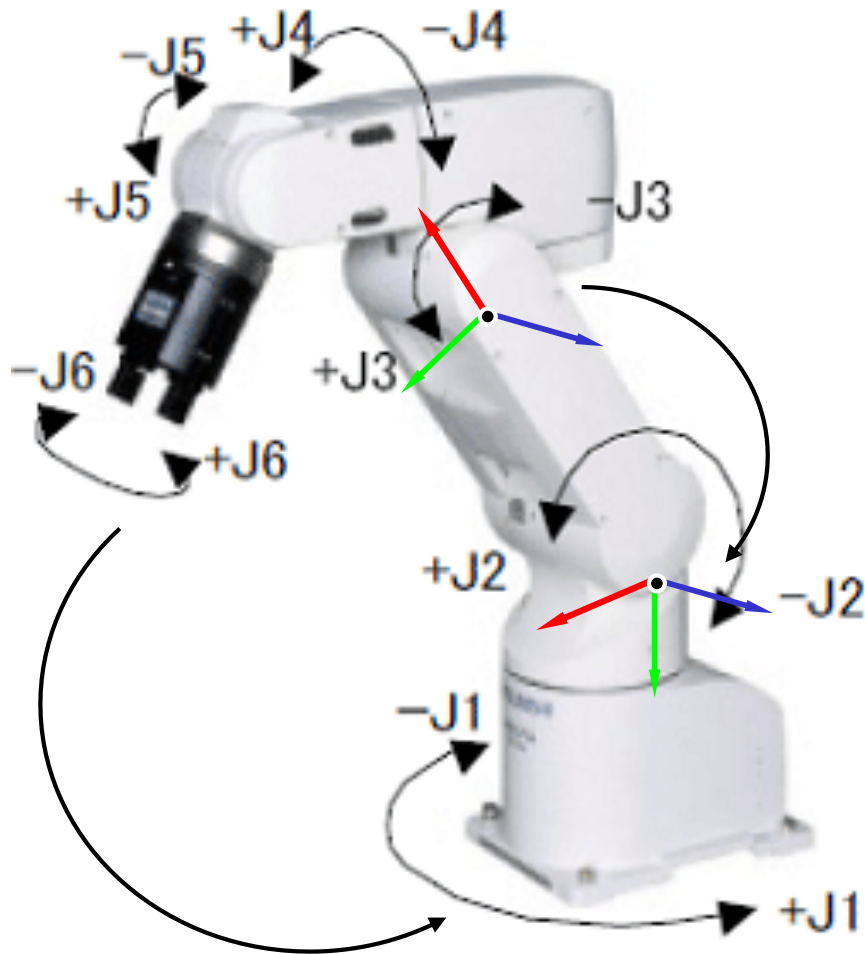
Lecture 12

Inverse kinematic task

Inverse kinematic task



Two consecutive bodies are related by a transform



4 parameters

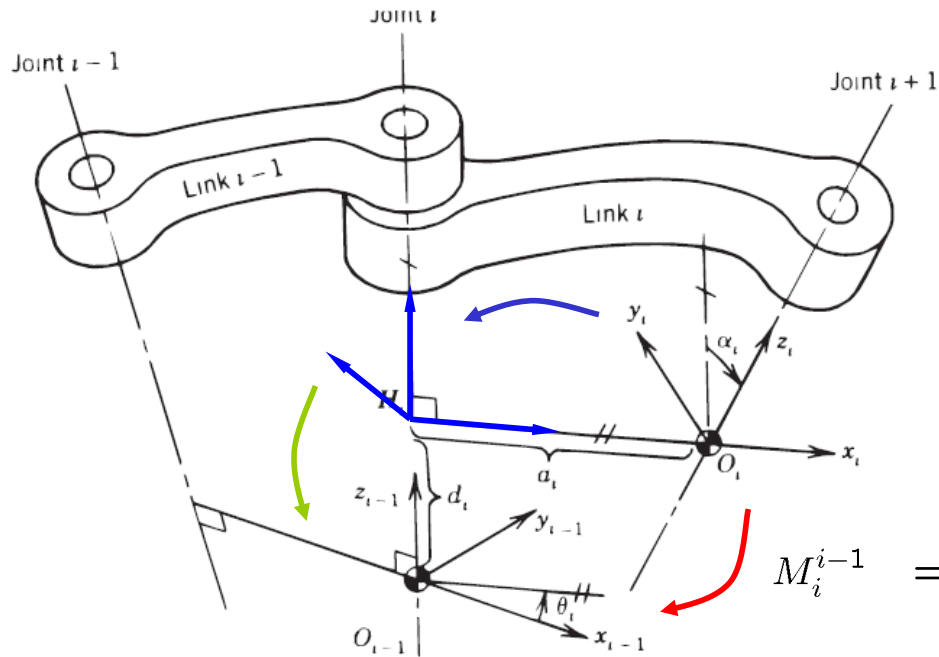
$$\alpha_i \mid a_i \mid \theta_i \mid d_i$$

$$M_i^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

Serial manipulator with 6 motions

Denavit-Hartenberg motion decomposition will be useful



$$M_i^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_i^{i-1} = M_{int}^{i-1} M_i^{int}$$

$$M_{int}^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse kinematics – formulation 1

Given the position of the flange, i.e. the matrix M
and parameters of the mechanism, e.g. α_i, a_i, d_i
compute the control variables $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$

Inverse kinematics – solution

Matrix motion equation

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

known

function of

$$\alpha_i, a_i, d_i$$

and

$$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$$

Change of variables – from trigonometry to algebra

$$M_{int}^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos \theta_i \longrightarrow c_i$$

$$\sin \theta_i \longrightarrow s_i$$

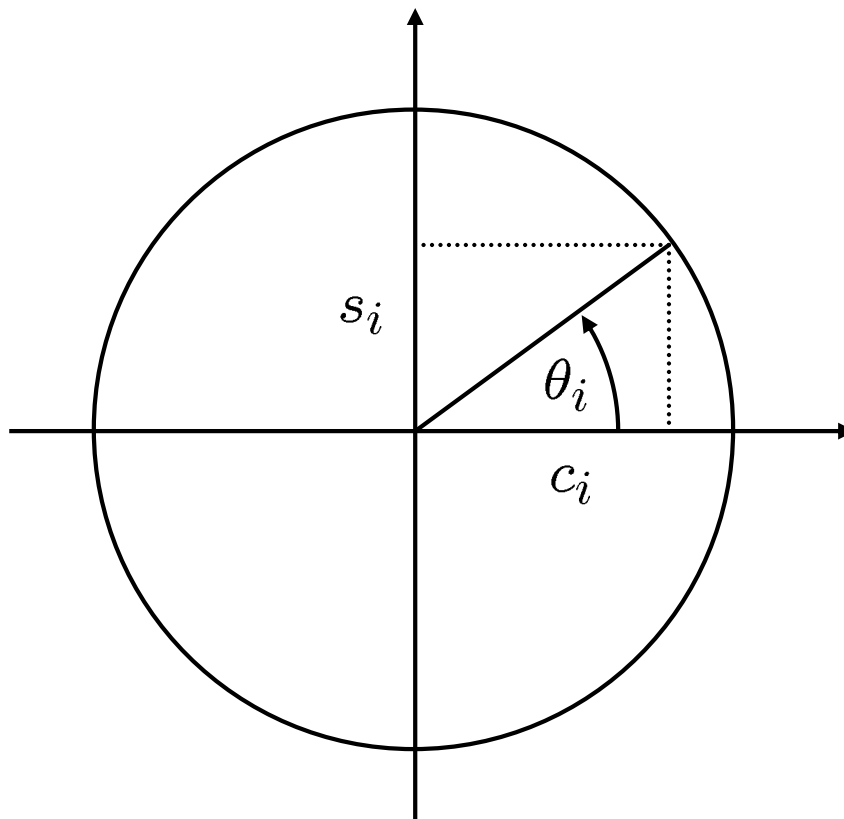
$$\cos \alpha_i \longrightarrow p_i$$

$$\sin \alpha_i \longrightarrow q_i$$

$$M_{int}^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & p_i & -q_i & 0 \\ 0 & q_i & p_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic identity

1 unknown θ_i \longrightarrow 2 unknowns c_i, s_i + 1 algebraic identity



$$c_i^2 + s_i^2 = 1$$

Change of variables – from trigonometry to algebra

$$M_{int}^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos \theta_i \longrightarrow c_i$$

$$\sin \theta_i \longrightarrow s_i$$

$$\cos \alpha_i \longrightarrow p_i$$

$$\sin \alpha_i \longrightarrow q_i$$

$$M_{int}^{i-1} = \begin{bmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & p_i & -q_i & 0 \\ 0 & q_i & p_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_i^2 + s_i^2 = 1$$

$$p_i^2 + q_i^2 = 1$$

Inverse kinematics – formulation 2

Given the position of the arm, i.e. the matrix M

and parameters of the mechanism, e.g. α_i, a_i, d_i

compute the control variables

$$s_1, c_1 ; s_2, c_2 ; s_3, c_3 ; s_4, c_4 ; s_5, c_5 ; s_6, c_6$$

subject to the constraint

$$M = M_1^0(c_1, s_1) M_2^1(c_2, s_2) M_3^2(c_3, s_3) M_4^3(c_4, s_4) M_5^4(c_5, s_5) M_6^5(c_6, s_6)$$

and


$$\begin{array}{ll} c_1^2 + s_1^2 = 1 & c_4^2 + s_4^2 = 1 \\ c_2^2 + s_2^2 = 1 & c_5^2 + s_5^2 = 1 \\ c_3^2 + s_3^2 = 1 & c_6^2 + s_6^2 = 1 \end{array}$$

Counting unknowns and equations

12 unknowns

$s_1, c_1 ; s_2, c_2 ; s_3, c_3 ; s_4, c_4 ; s_5, c_5 ; s_6, c_6$

12 equations (3 x 4 matrix) but only 6 independent (M contains rotation)


$$M = M_1^0(c_1, s_1) M_2^1(c_2, s_2) M_3^2(c_3, s_3) M_4^3(c_4, s_4) M_5^4(c_5, s_5) M_6^5(c_6, s_6)$$

6 equations

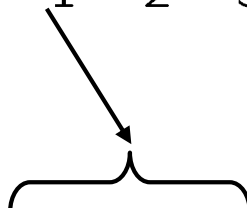
$$\begin{array}{ll} c_1^2 + s_1^2 = 1 & c_4^2 + s_4^2 = 1 \\ c_2^2 + s_2^2 = 1 & c_5^2 + s_5^2 = 1 \\ c_3^2 + s_3^2 = 1 & c_6^2 + s_6^2 = 1 \end{array}$$

There is 12 unknowns and 12 equations \longrightarrow can be solved

Decomposition to elementary motions

Decomposition to elementary motions

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$


$$M = M_{int}^0 M_1^{int} M_{int}^1 M_2^{int} M_{int}^2 M_3^{int} M_{int}^3 M_4^{int} M_{int}^4 M_5^{int} M_{int}^5 M_6^{int}$$

Decomposition to elementary motions

and rename matrices to make it shorter

$$M_i^{i-1} \longrightarrow M_i$$

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5 \longrightarrow M = M_1 M_2 M_3 M_4 M_5 M_6$$

$$M_{int}^{i-1} M_i^{int} \longrightarrow M_{i1} M_{i2}$$

$$M = M_{int}^0 M_1^{int} M_{int}^1 M_2^{int} M_{int}^2 M_3^{int} M_{int}^3 M_4^{int} M_{int}^4 M_5^{int} M_{int}^5 M_6^{int}$$

↓

$$M = M_{11} M_{12} M_{21} M_{22} M_{31} M_{32} M_{41} M_{42} M_{51} M_{52} M_{61} M_{62}$$

Inversion of D-H motion matrix preserves “linearity”

$$M_i^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_i & -s_i p_i & s_i q_i & a_i c_i \\ s_i & c_i p_i & -c_i q_i & a_i s_i \\ 0 & q_i & p_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{linear in } c_i, s_i}$$

$$= \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

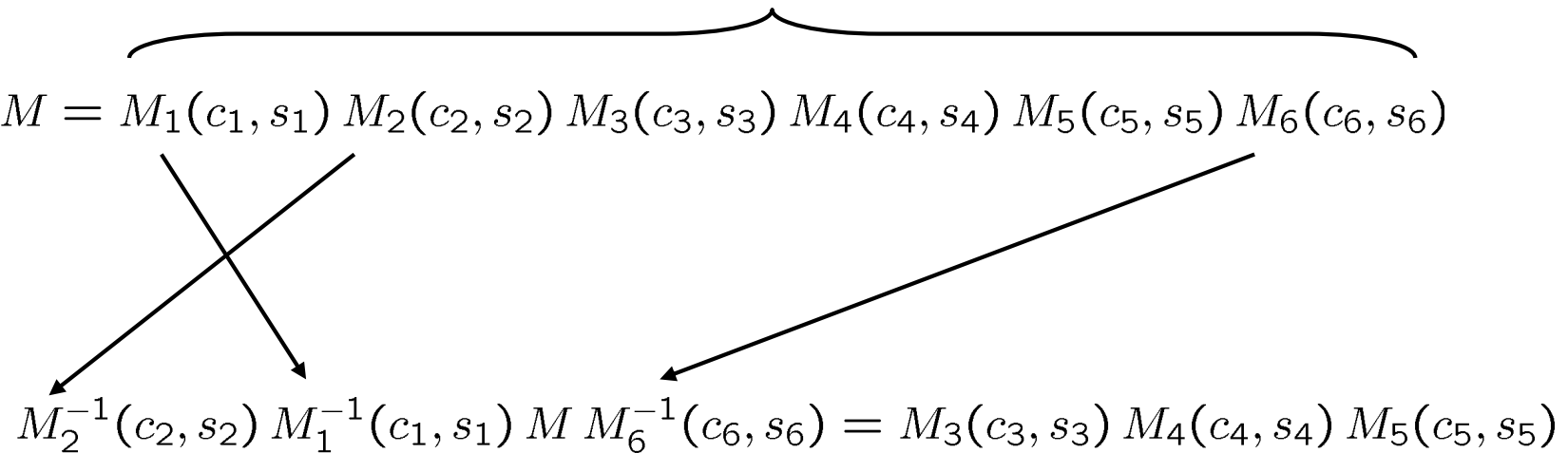
Inversion of D-H motion matrix preserves “linearity”

$$\begin{aligned}
 \text{inv}(M_i^{i-1}) &= \text{inv} \left(\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{inv} \left(\begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 1 & 0 & 0 & -a_i \\ 0 & \cos \alpha_i & \sin \alpha_i & 0 \\ 0 & -\sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 & 0 \\ -\sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & -d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 & -a_i \\ -\sin \theta_i \cos \alpha_i & \cos \theta_i \cos \alpha_i & \sin \alpha_i & -d_i \sin \alpha_i \\ \sin \theta_i \sin \alpha_i & -\cos \theta_i \sin \alpha_i & \cos \alpha_i & -d_i \cos \alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \underbrace{\begin{bmatrix} c_i & s_i & 0 & -a_i \\ -s_i p_i & c_i p_i & q_i & -d_i q_i \\ s_i q_i & -c_i q_i & p_i & -d_i p_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{linear in } c_i, s_i}
 \end{aligned}$$

Separate unknowns as much as possible

products of 6 unknowns

$$M = M_1(c_1, s_1) M_2(c_2, s_2) M_3(c_3, s_3) M_4(c_4, s_4) M_5(c_5, s_5) M_6(c_6, s_6)$$


$$M_2^{-1}(c_2, s_2) M_1^{-1}(c_1, s_1) M M_6^{-1}(c_6, s_6) = M_3(c_3, s_3) M_4(c_4, s_4) M_5(c_5, s_5)$$

product of 3 unknowns

product of 3 unknowns

algebraic equations of degree 3