RECAP:

**Polynomials**: \( f = \sum a_{\alpha} x^\alpha \quad f \in \mathbb{Q}[x_1, \ldots, x_n] \)
\( \alpha \in \mathbb{Z}^n_{\geq 0} \quad a_{\alpha} \in \mathbb{Q} \)

**Nonomials**: \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \alpha \quad \text{multi-degree} \)

Total degree: \( d = \alpha_1 + \cdots + \alpha_n \)

Polynomials cannot be in general divided

**Monomial ordering** - Lexicographic, Graded reverse Lex ordering

**Leading term**: \( \text{LT}(f) = \text{LC}(f) \cdot \text{LT}(f) \)

Leading coefficient \( \sim \) Leading monomial

\( \text{LC}(f) = a_{\text{multideg}(f)} \quad \text{LT}(f) = x_{\text{multideg}(f)} \)

\( \text{multideg}(f) = \max \{ \alpha \in \mathbb{Z}^n_{\geq 0} | \alpha \neq 0 \} \)

**Division theorem + algorithm**

\( f_i, F \in \mathbb{Q}(x_1, \ldots, x_n) \)

\( f = a_1 f_1 + \cdots + a_s f_s + r \)

\( a_i, r \in \mathbb{Q}(x_1, \ldots, x_n) \), either \( r = 0 \) or none of the monomials is divisible by any of \( \text{LT}(f_1), \ldots, \text{LT}(f_s) \)

Furthermore, \( a_i f_i \neq 0 \Rightarrow \text{multideg}(f) \geq \text{multideg}(a_i f_i) \)
One non-linear polynomial eq. in one unknown

- is well understood
- the problem can be formulated as a computation of eigenvalues of a matrix
- simple example

\[ f = x^3 - 6x^2 + 11x - 6 = 0 \]

We can construct a companion matrix

\[
M_x = \begin{bmatrix}
0 & 0 & 6 \\
1 & 0 & -11 \\
0 & 1 & 6
\end{bmatrix}
\]

the characteristic polynomial of \( M_x \) is

\[
\det(M_x - xI) = \det\left( \begin{bmatrix}
-\frac{6}{x} & 0 & \frac{6}{x} \\
-1 & -\frac{11}{x} & 0 \\
0 & 1 & -\frac{6}{x}
\end{bmatrix} \right) = x^3 - 6x^2 + 11x - 6 = f
\]

Therefore eigenvalues of \( M_x \) (1, 2, 3) are the solutions to \( f(x) = 0 \)
Linear mapping represented by a matrix $M \in \mathbb{R}^{n\times n}$

Eigenvalues: 

$Mx = \lambda x$

$Mx - \lambda x = 0$

$Mx - \lambda Ix = 0$

$(M - \lambda I)x = 0$

$x \neq 0 \Rightarrow \uparrow$

$\text{rank } (M - \lambda I) < n$

$\Rightarrow \det (M - \lambda I) = 0$
- This procedure applies in general when the coefficient at the monomial of $f$ with the highest degree is equal to 1 (when we normalize the equation).

- Obviously, such normalization using division by a non-zero coefficient at the monomial of the highest degree produces an equivalent equation with the same solutions.

The general rule for constructing the companion matrix $M_x$ for polynomial
\[ f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \]

\[
M_x = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & -a_{n-1} \\
0 & \cdots & 0 & 1 & -a_n
\end{bmatrix}
\]

Note that the eigenvalue computation must be in general approximate in general, roots of polynomials of degrees higher than 4 can't be expressed as finite formulas in coefficients $a_i$ using $\sqrt{}$.}$
System of linear polynomial equations in several unknowns

Consider the following system of 3 linear polynomial equations in 3 unknowns

\[ 2x_1 + x_2 + 3x_3 = 0 \]
\[ 4x_1 + 3x_2 + 2x_3 = 0 \]
\[ 2x_1 + x_2 + x_3 = 2 \]

and we write it in the standard matrix form

\[
\begin{bmatrix}
2 & 1 & 3 \\
4 & 3 & 2 \\
2 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
2 \\
\end{bmatrix}
\]

Using Gaussian elimination, we obtain an equivalent system

\[
\begin{bmatrix}
2 & 1 & 3 \\
0 & 1 & -4 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
-1 \\
\end{bmatrix}
\]

We see that the system has exactly one solution

\[ x_1 = \frac{9}{2}, \quad x_2 = -4, \quad x_3 = -1 \]
- The key point of this method is to produce a system in a "triangulate shape" such that there is an equation in a single unknown $x_3$, an equation in two unknowns $f_2(x_2, x_3)$ and so on.

- We can solve for $x_3$ and then transform $f_2$ by a substitution into an equation in a single unknown and solve for $x_2$ and so on.

$$x_3^4 + 1 = 0 \quad , \quad x_2 - 4x_3 = 0 \quad , \quad 2x_1 + x_2 + 3x_3 = 0$$

is a so-called Gröbner basis.

- Note that if we reorder unknowns we get a different GB.
We can go even further and compute the reduced-row Echelon form of this system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ -1 \end{bmatrix}$$

In this case we obtained so-called reduced GB.

For a linear system with one solution there is a unique reduced GB for all orderings.

In general (for general systems) for different orderings, reduced-row Echelon forms are different and also reduced GBs are different.
Example. To illustrate this for a system of linear equations we have to consider less equations than unknowns

\[
\begin{bmatrix}
2 & 4 & 2 & 1 & 2 \\
2 & 4 & 1 & 2 & 8 \\
1 & 2 & 3 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

RREF

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Different ordering

\[
\begin{bmatrix}
2 & 4 & 1 & 7 & 2 \\
1 & 2 & 8 & 2 & 4 \\
3 & 1 & 4 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4 \\
x_5 \\
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

RREF

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1/3 & -2/3 \\
0 & 0 & 1 & 2/3 & 2/3
\end{bmatrix}
\]

• The reduced-row Echelon form is unique for a given order of unknowns and it provides the reduced GB.

• Matrix of the RREF w.r.t one ordering is not equal to the matrix of the RREF w.r.t. another ordering and the corresponding reduced GB are also different.
Several non-linear polynomial eqs. in several unknowns

- Technique for transforming a system of polynomial equations with a finite number of solutions into a system that will contain a polynomial in the "last" unknown, say $x_n$

$\Rightarrow$ will allow for solving for $x_n$ and reducing the problem from $n$ to $n-1$ unknowns and so on until we solve for all unknowns.

Example:

$$f_1 = x_1^2 + x_2^2 - 1 = 0$$
$$f_2 = 25x_1x_2 - 20x_2 - 15x_1 + 12 = 0$$

Matrix form:

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 25 & -20 & 0 & -15 & 12
\end{bmatrix}
\begin{bmatrix}
x_1^2 \\
x_4x_2 \\
x_2^2 \\
x_4 \\
x_1
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}$$

$f = 0 \Rightarrow f \cdot g = 0$ for any $g \in \mathbb{Q}[x_1, x_2]$

E.g. $x_1 \cdot f_1 = 0$, $x_2 \cdot f_2 = 0$
- Adding "new equations" of the form $f_i \cdot g = 0$ to the original system produces a new system with the same solutions.

- Polynomials $f_i \cdot x^j$ are linearly independent when $f \neq 0$ since $x^j \cdot f$ has degree strictly greater than is the degree of $f$.
  
  $\Rightarrow$ by adding $x \cdot f_i \cdot (g \cdot f)$ we have a chance to add another independent row to the matrix.

- Let's add $x^1 f_1$, $x_2 f_2$ to our system and write it in the matrix form:

\[
\begin{bmatrix}
  x^1 x_2 & x_2 & x^1 x_2 & x_2 & x^3 & x_2 & x^1 & 1 \\
  f_1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
  f_2 & 0 & 0 & 25 & -20 & 0 & 0 & -15 & 12 \\
  x_1 f_1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
  x_2 f_2 & 25 & -20 & -15 & 12 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

- More rows have been added but also new monomials $x_1 x_2^2$ and $x_3^3$. 
Eliminate it by the Gaussian elimination

\[ \begin{bmatrix}
  x^4 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

The last row gives an equation in single unknown y

- We have been ordering monomials corresponding to the columns of the matrix such that we have all monomials in y at the end

- It can be shown that similar procedure works for every system of polynomial equations \( \{f_1, \ldots, f_k\} \in \mathbb{Q}[x_1, \ldots, x_n] \) with a finite number of solutions
In particular, there always are $\mathbf{a}$ finite sets $\Pi_i$, $i=1, \ldots, \mathbf{a}$ of monomials such that the extended system

$$\{ f_1, f_2, \ldots, f_a \} \cup \{ m f_j \mid m \in \Pi_j, \ j=1, \ldots, \mathbf{a} \}$$

has matrix $A$ with the following nice property:

- If the last columns of $A$ correspond to all monomials in a single unknown $x_i, (y)$ (including 1), then the last non-zero row of matrix $B$, obtained by the Gaussian elimination of $A$ produces a polynomial in single unknown $x_i, (y)$

- a very powerful technique
- a tool how to solve all systems of polynomial equations with a finite number of solutions
- In practice the main problem is how to find small sets \( I \); in acceptable time.

- The number of monomials of total degree at most \( d \) in \( n \) unknowns is given by the combination number \( \binom{n+d}{d} \).

\( \Rightarrow \) The size of the matrix \( A \) is growing very quickly.

- Practical algorithms (e.g. Fa) use many tricks how to select small sets of monomials and how to efficiently compute in exact arithmetics over \( \mathbb{Q} \).