Inverse Kinematics of 6R Manipulator by Newton's Method

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Newton's Method (Example)

Task: find a root of $f(x) = x^2 - 2$

Given: starting point $x_0 = 1$, number of steps $m = 3$, tolerance $e = 10^{-5}$

Newton's step:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Steps:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2 \cdot 1} = 1.5$$

$$x_2 = 1.5 - \frac{0.25}{3} = 1.416667$$

$$x_3 = \ldots = 1.414216$$

The norm $\|f(x_3)\| = 6 \cdot 10^{-6} \leq e \Rightarrow x_3$ is a good approximation of a root

Newton's Method (Algorithm)

Algorithm 1: Newton's Method

Input: $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_s(\mathbf{x})), \mathbf{x}_0 \in \mathbb{R}^n, m \in \mathbb{R}, e \in \mathbb{R}_+$

Output: $(s, \mathbf{x}^*)$, where s denotes the state of convergence and $\mathbf{x}^* \in \mathbb{R}^n$. If the algorithm converged in m steps with tolerance e, then $s =$ True and $\mathbf{x}^*$ is the approximated solution. Otherwise, $s =$ False and $\mathbf{x}^*$ is the point obtained in the last iteration.

$\mathbf{x}^* \leftarrow \mathbf{x}_0$

for ($k \leftarrow 0;\ k < m;\ k \leftarrow k+1$)

  if $\|\mathbf{f}(\mathbf{x}^*)\| \leq e$ **then**

    return $(\text{True}, \mathbf{x}^*)$

  $\mathbf{J} \leftarrow \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*}$

  $\mathbf{x}^* \leftarrow \mathbf{x}^* - \mathbf{J}^+ \mathbf{f}(\mathbf{x}^*)$

return $(\text{False}, \mathbf{x}^*)$

Equations for R_e and t_e

$$\mathbf{M}_e = \mathbf{M}_1^0 \mathbf{M}_2^1 \mathbf{M}_3^2 \mathbf{M}_4^3 \mathbf{M}_5^4 \mathbf{M}_6^5$$

$$\underbrace{\begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\substack{\text{pose of the} \\ \text{end effector}}} = \prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i + \underbrace{\theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i}_{\text{DH parameters}})$$

Hence,

$$\mathbf{R}_e = \underbrace{\prod_{i=1}^{6} \mathbf{R}_i^{i-1}(\theta_i)}_{\mathbf{R}(\boldsymbol{\theta})}, \quad \overline{\mathbf{t}_e} = \begin{bmatrix} \mathbf{t}_e \\ 1 \end{bmatrix} = \underbrace{\prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i) \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}}_{\bar{\mathbf{t}}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{t}(\boldsymbol{\theta}) \\ 1 \end{bmatrix}},$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_6)$.

IKT Equations

The IKT equations are:

$$\mathbf{f}(\boldsymbol{\theta}) = \begin{bmatrix} \text{vec}\left(\mathbf{R}(\boldsymbol{\theta}) - \mathbf{R}_e\right) \\ \mathbf{t}(\boldsymbol{\theta}) - \mathbf{t}_e \end{bmatrix} = \mathbf{0}$$

where

$$\text{vec}(\mathbf{R}) = \begin{bmatrix} r_{11} & r_{21} & r_{31} & r_{12} & r_{22} & r_{32} & r_{13} & r_{23} & r_{33} \end{bmatrix}^\top$$

The Jacobian is:

$$\mathbf{J}(\boldsymbol{\theta}) = \frac{\partial \mathbf{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial \text{vec}(\mathbf{R}(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}} \\ \frac{\partial \mathbf{t}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \end{bmatrix} \in C(\boldsymbol{\theta}, \mathbb{R})^{12 \times 6}$$

since $\mathbf{R}_e$ and $\mathbf{t}_e$ are fixed and don't depend on $\boldsymbol{\theta}$.

Jacobian

By taking partial derivative of $\mathbf{R}(\boldsymbol{\theta})$ w.r.t. θ_k we get:

$$\frac{\partial \mathbf{R}(\boldsymbol{\theta})}{\partial \theta_k} = \prod_{i=1}^{k-1} \mathbf{R}_i^{i-1}(\theta_i) \cdot \frac{\partial \mathbf{R}_k^{k-1}(\theta_k)}{\partial \theta_k} \cdot \prod_{i=k+1}^{6} \mathbf{R}_i^{i-1}(\theta_i) \in C(\boldsymbol{\theta}, \mathbb{R})^{3\times 3}$$

By taking partial derivative of $\bar{\mathbf{t}}(\boldsymbol{\theta})$ w.r.t. θ_k we get:

$$\frac{\partial \bar{\mathbf{t}}(\boldsymbol{\theta})}{\partial \theta_k} = \prod_{i=1}^{k-1} \mathbf{M}_i^{i-1}(\theta_i) \cdot \frac{\partial \mathbf{M}_k^{k-1}(\theta_k)}{\partial \theta_k} \cdot \prod_{i=k+1}^{6} \mathbf{M}_i^{i-1}(\theta_i) \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \in C(\boldsymbol{\theta}, \mathbb{R})^{4}$$