Logical reasoning and programming, lab session 6

(October 30, 2023)

6.1 Show that

$$(g(f(x_1 - 2)) = x_1 + 2) \land (g(f(x_2)) = x_2 - 2) \land (x_2 + 1 = x_1 - 1)$$

is unsatisfiable by the Nelson–Oppen procedure, where x_1 and x_2 are integers and f and g uninterpreted functions.

Why does the procedure work here even though QF_LIA is non-convex?

6.2 If we want to combine theories in SMT using the Nelson–Oppen method, we require that both of them are stably infinite. Assume two theories $\mathcal{T}_1$ with the language $\{f\}$ and $\mathcal{T}_2$ with the language $\{g\}$, where f and g are uninterpreted unary function symbols. Moreover, $\mathcal{T}_1$ has only models of size at most 2 (for example, it contains $\forall X \forall Y \forall Z (X = Y \lor X = Z)$ as an axiom). Show that the Nelson–Oppen method says that

$$f(x_1) \neq f(x_2) \land g(x_2) \neq g(x_3) \land g(x_1) \neq g(x_3).$$

is satisfiable in the union of $\mathcal{T}_1$ and $\mathcal{T}_2$, but this is clearly incorrect.

6.3 Show that the following formulae are valid and provide counter-examples for the opposite implications:

(a) $\forall X p(X) \lor \forall X q(X) \to \forall X (p(X) \lor q(X))$,

(b) $\exists X (p(X) \land q(X)) \to \exists X p(X) \land \exists X q(X)$,

(c) $\exists X \forall Y p(X, Y) \to \forall Y \exists X p(X, Y)$,

(d) $\forall X p(X) \to \exists X p(X)$.

6.4 Show that the "exists unique" quantifier $\exists!$ does not commute with $\exists$, $\forall$, nor $\exists!$.

6.5 Decide whether for any formula φ holds:

(a) $\varphi \equiv \forall \varphi$,

(b) $\varphi \equiv \exists \varphi$,

(c) $\models \varphi$ iff $\models \forall \varphi$,

(d) $\models \varphi$ iff $\models \exists \varphi$,

where $\forall \varphi$ ($\exists \varphi$) is the universal (existential) closure of φ. If not, does at least one implication hold?

6.6 Show that for any set of formulae Γ and a formula φ holds if $\Gamma \models \varphi$, then $\forall \Gamma \models \varphi$, where $\forall \Gamma = \{\, \forall \psi \colon \psi \in \Gamma \,\}$. Does the opposite direction hold?

6.7 Does it hold $\Gamma \models \varphi$ iff $\forall \Gamma \models \forall \varphi$?

6.8 Find a set of formulae Γ and a formula φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$.

6.9 Produce equivalent formulae in prenex form:

(a) $\forall X(p(X) \to \forall Y(q(X,Y) \to \neg\forall Z r(Y,Z)))$,
(b) $\exists X p(X,Y) \to (q(X) \to \neg\forall Z p(X,Z))$,
(c) $\exists X p(X,Y) \to (q(X) \to \neg\exists Z p(X,Z))$,
(d) $p(X,Y) \to \exists Y(q(Y) \to (\exists X q(X) \to r(Y)))$,
(e) $\forall Y p(Y) \to (\forall X q(X) \to r(Z))$.

6.10 In **6.9** you could obtain in some cases various prefixes; the order of quantifiers can be different. Are all these variants correct?

6.11 Can we produce a formula equivalent to **6.9**e with just one quantifier?

6.12 Produce Skolemized formulae equisatisfiable with those in **6.9**. Try to produce as simple as possible Skolem functions.

6.13 Skolemize the following formula

$$\forall X(p(a) \vee \exists Y(q(Y) \wedge \forall Z(p(Y,Z) \vee \exists U \neg q(X,Y,U)))) \vee \exists W q(a,W).$$

Why is it possible in this particular case to do that without producing an equivalent formula in prenex form?