COMPUTATIONAL GAME THEORY

Exercises

TOMÁŠ KROUPA* ONDŘEJ KUBÍČEK*
TOMÁŠ VOTROUBEK*

December 15, 2023

CONTENTS

* *AI Center, Department of Computer Science, Faculty of Electrical Engineering, Czech Technical University in Prague*

ABOUT THIS DOCUMENT

The aim of these exercises is to make you, the student of *Computational Game Theory*, think actively about the concepts introduced in the lectures. You are encouraged to work out your approach to the solution. Almost all exercises can be solved using pen & paper – write your notes and calculations on the wide margins of this document. Do not hesitate to consult the course materials available online for the basic notions and results. Most of the exercises are adapted from [2, 3, 4, 5]. Some questions are more difficult or require more extended mathematical arguments. Such items are marked with ★ .

In addition to the exercises covering the content of the lectures, the first section presents selected mathematical prerequisities necessary for understanding game theory. This is mostly based on the main concepts discussed in the undergraduate courses of linear algebra, optimization, linear programming, discrete mathematics, and probability theory.

1 MATHEMATICAL PREREQUISITIES

Exercise 1.
Duality in linear programming. The concept of duality appears in many game-theoretic problems. In this exercise we consider the linear program

$$\begin{aligned} \text{Minimize} \quad & -x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4 \\ & 2x_1 + x_2 \le 5 \\ & -x_1 + 4x_2 \ge 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Characterize the feasible set of the problem by its vertices (extreme points). Further, find the optimal solution of this linear program and formulate the dual program. What is the meaning of the dual program and its relation to the original program? What will be the optimal value of the dual problem?

Exercise 2.
Minimax/maximin values. Let X and Y be nonempty sets and consider a real function $f\colon X \times Y \to \mathbb{R}$. Show that

$$\max_{x\in X} \min_{y\in Y} f(x,y) \le \min_{y\in Y} \max_{x\in X} f(x,y), \tag{1}$$

where we assume that all maxima and minima above exist. Find an example of function f taking at least two different values such that (a) the inequality is strict and (b) the inequality becomes an equality. *Hint:* Such examples exist already when both X and Y have 2 elements.

Exercise 3.
Saddle points. Let $f\colon X \times Y \to \mathbb{R}$ be a function, where X and Y are arbitrary nonempty set. We say that $(x^*, y^*) \in X \times Y$ is a *saddle point* of f if

$$f(x,y^*) \le f(x^*,y^*) \le f(x^*,y) \qquad \text{for all } x \in X,\ y \in Y. \tag{2}$$

Show that a function f has a saddle point if, and only if,

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y), \tag{3}$$

provided that all the maxima/minima above exist.

Exercise 4.
Joining a random coalition. Assume that $N = \{1, \dots, n\}$ is a set of players. A *coalition* is any subset of N. One of the players, say $i \in N$, would like to join a coalition A of other players. What is the probability of selecting A at random? What is the probability that a coalition A is chosen in the following way. First, player i randomly picks the size of A, and then A is selected among the coalitions of such size at random?

SOLUTIONS

Solution 1.
In any linear programming problem, the feasible set is a convex polyhedron. In our case the polyhedron is bounded, so it is a convex polygon characterized by its vertices. Every vertex corresponds to a unique solution of some linear equality system associated with a subset of linear inequality constraints of the problem. For example, a point $(1,3)$ is a vertex since it is a unique solution to linear equations $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$. The remaining vertices are $(0, \frac{1}{2}), (0,4), (2,1)$. See Figure 1, which is an output of the simple online solver https://online-optimizer.appspot.com.

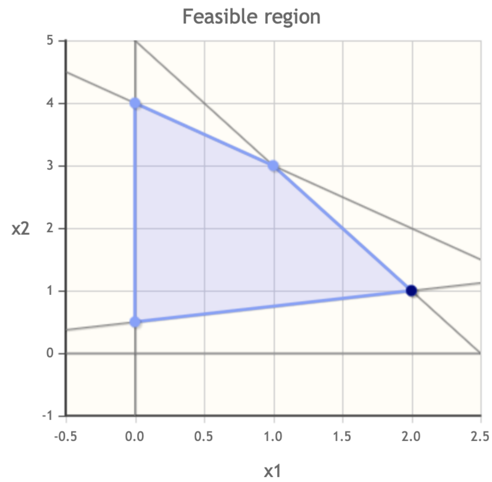


Figure 1: The feasible set corresponding to constraints (5)–(7)

What is the optimal solution to the problem? The fundamental theorem of linear programming says that, if a linear program has an optimal solution, then it is attained at some vertex of the feasible set. Clearly, at least one optimal solution exists in our case since the feasible set is bounded. Therefore, it suffices to compute the values of the objective function $f(x_1, x_2) = -x_1 + 2x_2$ for all the vertices and select the one with the minimal value. This yields the optimal solution $(2,1)$ with the optimal value is $f(2,1) = 0$.

To formulate the dual problem, it is convenient to formulate the original problem so that all inequalities are of the form $\geq$:

$$\begin{aligned} \text{Minimize} \quad & -x_1 + 2x_2 && (4)\\ \text{subject to} \quad & -x_1 - x_2 \geq -4 && (5)\\ & -2x_1 - x_2 \geq -5 && (6)\\ & -x_1 + 4x_2 \geq 2 && (7)\\ & x_1, x_2 \geq 0 && (8) \end{aligned}$$

By design, the dual program has 3 variables y_1, y_2, y_3 corresponding to the 3 primal linear constraints (5)–(7), and 2 linear constraints associated with

2 primal variables x_1, x_2. The exact form of the dual program can be found in any textbook on linear programming. Specifically the dual is

$$\begin{aligned} \text{Maximize} \quad & -4y_1 - 5y_2 + 2y_3 \\ \text{subject to} \quad & -y_1 - 2y_2 - y_3 \leq -1 \\ & -y_1 - y_2 + 4y_3 \leq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

The duality theorem immediately implies that the primal and dual problems have the same optimal value 0.

What is the interpretation of duality? The optimal value of dual problem provides the tighest lower bound on the solution of the primal problem. In fact, the dual problem controls the optimal value of the primal problem from below, and does so in the best possible way — the optimal values of both programs coincide!

We will elaborate on the idea sketched above. How to obtain a lower bound for the primal problem? Note that one possible lower bound on the values of the objective f follows from the nonnegativity constraints (8) and the constraint (5) since

$$-4 \leq -x_1 - x_2 \leq -x_1 + 2x_2.$$

Can we do better? That is, can we obtain a higher value than -4 for the lower bound? It is easy to see that by multiplying the inequalities (5) and (7) by $\frac{1}{2}$ and adding them we obtain the bound

$$-1 \leq -x_1 + \tfrac{3}{2}x_2 \leq -x_1 + 2x_2,$$

which is tighter. Can we still improve on this lower bound? Let us try to generalize the idea of combining the linear inequalities of the primal problem for estimating the optimal value from below. The goal is to find nonnegative real numbers y_1, y_2, y_3 such that

$$-4y_1 - 5y_2 + 2y_3 \leq \underbrace{y_1(-x_1 - x_2) + y_2(-2x_1 - x_2) + y_3(-x_1 + 4x_2)}_{(-y_1 - 2y_2 - y_3)x_1 + (-y_1 - y_2 + 4y_3)x_2} \leq -x_1 + 2x_2$$

and the second inequality should be as tight as possible. One way to guarantee this is to maximize the most left-hand side $-4y_1 - 5y_2 + 2y_3$ while preserving the constraints $-y_1 - 2y_2 - y_3 \leq -1$ and $-y_1 - y_2 + 4y_3 \leq 2$ expressed by the second inequality above. In other words, we have derived precisely the dual problem.

Solution 2.

To prove the inequality (1), define

$$F(x) = \min_{y \in Y} f(x, y), \qquad x \in X,$$

and let $x^* \in X$ be the maximizer of F over X. Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \max_{x \in X} F(x) = F(x^*) = \min_{y \in Y} f(x^*, y).$$

For every $y \in Y$, we have

$$f(x^*, y) \le \max_{x \in X} f(x, y)$$

by the definition of maximum, which implies

$$\min_{y \in Y} f(x^*, y) \le \min_{y \in Y} \max_{x \in X} f(x, y).$$

Hence, (1) is proved.

We will construct two examples (a)-(b) under the assumption that $X = Y = \{1, 2\}$. First, consider a function f given by the matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & 1 & 2 \\ \cline{3-4} X & 1 & 0 & 1 \\ \cline{3-4} & 2 & 1 & 0 \\ \cline{3-4} \end{array}$$

Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = 0 < 1 = \min_{y \in Y} \max_{x \in X} f(x, y).$$

For the second example (b), take a matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & 1 & 2 \\ \cline{3-4} X & 1 & 3 & 0 \\ \cline{3-4} & 2 & 2 & 2 \\ \cline{3-4} \end{array}$$

Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = 2 = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Solution 3.

We define two functions: $F(x) = \min_{y \in Y} f(x, y)$, for all $x \in X$, and $G(y) = \max_{x \in X} f(x, y)$, for every $y \in Y$. Assume that f has a saddle point (x^*, y^*). Then the saddle point definition (2) and the definition of maxima/minima imply

$$\min_{y \in Y} G(y) \le G(y^*) = \max_{x \in X} f(x, y^*) = f(x^*, y^*) = \min_{y \in Y} f(x^*, y) = F(x^*) \le \max_{x \in X} F(x).$$

We obtained the inequality

$$\min_{y \in Y} \max_{x \in X} f(x, y) \le \max_{x \in X} \min_{y \in Y} f(x, y),$$

and since the converse inequality always holds (Exercise 2), we proved the minimax equality (3).

To prove the converse implication, assume that (3) holds. Let x^* and y^* be such that $F(x^*) = \max_{x \in X} F(x)$ and $G(y^*) = \min_{y \in Y} G(y)$, respectively. Then the assumption (3) gives

$$F(x^*) = \max_{x \in X} F(x) = \max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y) = \min_{y \in Y} G(y) = G(y^*). \tag{9}$$

Further, from the definition of F, G and x^*, y^* we get

$$F(x^*) \leq f(x^*, y^*) \leq G(y^*)$$

and (9) implies that even $F(x^*) = f(x^*, y^*) = G(y^*)$. Let $x \in X$ and $y \in Y$. Then

$$f(x, y^*) \leq G(y^*) = f(x^*, y^*) = F(x^*) \leq f(x^*, y).$$

Solution 4.

There are 2^{n-1} coalitions A to which player i doesn't belong, $i \notin A$. Therefore $p(A) = 2^{1-n}$ is the probability of picking such a coalition randomly. Consider now the second variant of random choice of the coalition. Let A be a random coalition selected in this way. We can compute the probability $q(A)$ using the definition of conditional probability,

$$q(A) = q(A \mid \text{the size is } |A|) \cdot q(\text{the size is } |A|) = \frac{1}{\binom{n-1}{|A|}} \cdot \frac{1}{n},$$

where we used the fact that $q(A) = q(A \wedge \text{the size is } |A|)$. We can easily verify that q is a probability distribution:

$$\sum_{A \subseteq N \setminus \{i\}} q(A) = \frac{1}{n} \sum_{A \subseteq N \setminus \{i\}} \frac{1}{\binom{n-1}{|A|}} = \frac{1}{n} \sum_{a=0}^{n-1} \binom{n-1}{a} \frac{1}{\binom{n-1}{a}} = 1.$$

Note that both p and q depend only on the size of each coalition. Specifically, if A and B are coalitions such that $|A| = |B|$, then $p(A) = p(B) = 2^{1-n}$ and $q(A) = q(B)$. Probability distributions p and q appear in the Banzhaf and Shapley values in coalitional game theory. We can see that there is a several order magnitude difference between their values (Figure 2). For example, in case that there are $n = 60$ players, the probabilities of coalition A with $|A| = 10$ are $p(A) \approx 1.7 \times 10^{-18}$ and $q(A) \approx 2.6 \times 10^{-13}$, respectively.

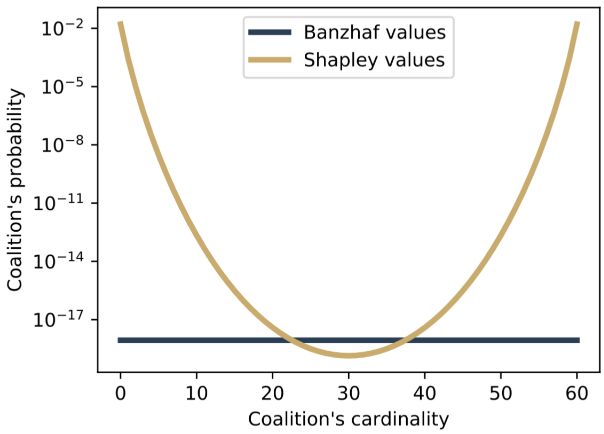


Figure 2: Values of p and q.

Part I. Game Theory

2 NORMAL-FORM GAMES

Exercise 5.
Single attacker is about to attack one of the 4 military bases (denoted $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$). However, due to pay cuts, the military can only spend resources to defend one of these bases. If the attacker attacks the defended bases, it gets a utility of -1, while the military gets 2. On the other hand, when the attack is succesful, the military does not receive anything, but the attacker gets the following rewards based on the base it attacked $\mathbf{T}_1 \to 3$, $\mathbf{T}_2 \to 7$, $\mathbf{T}_3 \to 1$ and $\mathbf{T}_4 \to 5$. (a) Formalize this as a Normal-Form Game. (b) Let us assume that when the attacker attacks base, the alarm is triggered. If attacked base is near the protected one, the military may dispatch striking team, which has 50% chance to interrupt the attack. For base $\mathbf{T}_i$, the nearby bases are $\mathbf{T}_{i-1}$ and $\mathbf{T}_{i+1}$. Bases $\mathbf{T}_1$ and $\mathbf{T}_4$ are also close to each other. (c) Find all pure Nash Equilibria in those games.

Exercise 6.
The two-player normal-form game with the payoff matrix

		Player 2		
		V	**S**	**R**
Player 1	**U**	$1,3$	$4,2$	$-1,2$
	C	$1,0$	$2,-2$	$0,-1$
	D	$1,2$	$-1,1$	$3,3$

(a) Find all Pareto optimal outcomes. (b) Find all pure Nash equilibria. (c) Find all dominated pure strategies and apply iterative removal of these strategies.

Exercise 7.
Is there a game, that has at least one pure Nash Equilibrium, but during the proces of iterative removal, it is removed? Either show that there cannot exist such a game, or find an example

Exercise 8.
Alice and Bob play one round of a zero-sum game captured by the payoff matrix of Alice:

		Bob		
		e	f	g
Alice	a	6	0	-1
	b	5	4	9
	c	9	-3	-1
	d	-1	1	-1

Alice reveals publicly that she will be using strategy b. After making her choice public she must stick to it. Can Bob take advantage of knowing the strategic choice of Alice compared to the standard situation when Alice's strategic choice wouldn't be known a priori?

Exercise 9.

Show that the following two-player zero-sum game doesn't have an equilibrium in pure strategies. The strategy space of each player is the set $X = \{0, \frac{1}{10}, \ldots, \frac{9}{10}, 1\}$ and the payoff function of Player 1 is

$$u(s_1, s_2) = \frac{1}{1 + (s_1 - s_2)^2}, \qquad s_1, s_2 \in X.$$

SOLUTIONS

Solution 5.
(a) We use the military as a row player and the attacker as a column player. The first value corresponds to the utility of the defender, while the second corresponds to the attacker's utility. The utility matrix in the Normal-Form Game is then

		Attacker			
		$\mathbf{T}_1$	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$
Military	$\mathbf{T}_1$	$2,-1$	$0,7$	$0,1$	$0,5$
	$\mathbf{T}_2$	$0,3$	$2,-1$	$0,1$	$0,5$
	$\mathbf{T}_3$	$0,3$	$0,7$	$2,-1$	$0,5$
	$\mathbf{T}_4$	$0,3$	$0,7$	$0,1$	$2,-1$

(b) In the changed game, when the attacker attacks defended base, the result is still the same. But when the base is nearby the protected base, the utility changes in the following way.

$$u_1(\mathbf{T}_i, \mathbf{T}_{i+1}) = \frac{1}{2}u_1(\mathbf{T}_{i+1}, \mathbf{T}_{i+1}) + \frac{1}{2} \cdot 2$$
$$u_2(\mathbf{T}_i, \mathbf{T}_{i+1}) = \frac{1}{2}u_2(\mathbf{T}_{i+1}, \mathbf{T}_{i+1}) + \frac{1}{2} \cdot (-1)$$

The utility matrix in changed game is

		Attacker			
		$\mathbf{T}_1$	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$
Military	$\mathbf{T}_1$	$2,-1$	$1,3$	$0,1$	$1,2$
	$\mathbf{T}_2$	$1,1$	$2,-1$	$1,0$	$0,5$
	$\mathbf{T}_3$	$0,3$	$1,3$	$2,-1$	$1,2$
	$\mathbf{T}_4$	$1,1$	$0,7$	$1,0$	$2,-1$

(c) Neither game has any pure Nash Equilibrium.

Solution 6.
(a) Pareto optimal strategy is such a strategy profile in which neither player can improve its utility without decreasing the utility of any other player. Strategy. For example $(\mathbf{U}, \mathbf{V})$ is not Pareto optimal strategy, because when choosing strategy $(\mathbf{D}, \mathbf{R})$, the row player improves its utility from 1 to 3, while columns player utility remains unchanged. The only Pareto optimal strategies are $(\mathbf{D}, \mathbf{R})$ and $(\mathbf{U}, \mathbf{S})$ with utilities

$$u_1(\mathbf{D}, \mathbf{R}) = 3 \quad u_2(\mathbf{D}, \mathbf{R}) = 3$$
$$u_1(\mathbf{U}, \mathbf{S}) = 4 \quad u_2(\mathbf{U}, \mathbf{S}) = 2$$

(b) Nash Equilibrium is a strategy profile where neither player can improve its utility by changing its strategy. For example $(\mathbf{U}, \mathbf{S})$ is not a Nash Equilibrium, because column player may choose action $\mathbf{V}$ and its utility would improve from 2 to 3. Nash equilibria are $(\mathbf{D}, \mathbf{R})$, $(\mathbf{U}, \mathbf{V})$, $(\mathbf{C}, \mathbf{V})$.
(c) Strategy s_i of a player dominates different strategy if, regardless of the opponent's strategy, the utility of s_i is always greater than that of s_j. Similarly, for weak domination, the utility of s_i is always greater or equal than the utility of s_j, and it is strictly greater for at least one opponent's strategy. Pure strategy $\mathbf{V}$ strictly dominates $\mathbf{S}$ (also $\mathbf{R}$ weakly dominates $\mathbf{S}$).

		Player 2	
		V	**R**
Player 1	**U**	1,3	−1,2
	C	1,0	0,−1
	D	1,2	3,3

After removing **S**, both **U** and **C** are weakly dominated by **D**.

		Player 2	
		V	**R**
Player 1	**D**	1,2	3,3

Finally now **R** strictly dominates **V**. This leaves only strategy profile $(\mathbf{D}, \mathbf{R})$, which is also a Nash equilibrium.

Solution 7.

In order to remove a strategy in iterative removal, it has to be dominated. Let us start by checking if it may happen that the Nash Equilibrium can be strongly dominated.

Let us start by reiterating the definition of strong dominance.

Strategy s_i' is strictly dominated by s_i if following condition holds

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \;\; \forall s_{-i} \in \mathcal{S}_{-i}$$

Now let us reiterate the definition of Nash Equilibrium.

Strategy profile s^* is a Nash Equilibrium if following condition holds

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \;\; \forall i \in \mathcal{N} \;\; \forall s_i \in \mathcal{S}_i$$

Now let us suppose that there exist Nash Equilibrium s^* in which player i plays strategy s_i^* that is strongly dominated by s_i. This means that for other players strategy s^*_i following holds

$$u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$$

This is in direct contradiction with the definition of a Nash Equilibrium, therefore if the iterative removal removes only strongly dominated strategies, it cannot remove Nash Equilibrium.

Now let us focus on weak domination.

Strategy s_i' is is weakly dominated by strategy s_i if following condition holds

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \;\; \forall s_{-i} \in \mathcal{S}_{-i}$$

and for at least a single strategy s_{-i} following holds

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

Now let us suppose that there exist Nash Equilibrium s^* in which player i plays strategy s_i^* that is weakly dominated by s_i and for single opponents strategy s'_{-i} following hold

$$u_i(s_i, s'_{-i}) > u_i(s_i^*, s'_{-i})$$

However, since it does not have to hold that $s'_{-i} = s^*_{-i}$, then it does not immidiately violate the Nash Equilibrium condition. But it may help us with finding the counterexample, because now we know that it can only hold in a case where $s'_{-i} \neq s^*_{-i}$

One such a game is

Player 2

		d	e
	a	$0,1$	$1,0$
Player 1	b	$-1,0$	$1,2$
	c	$1,0$	$0,1$

Solution 8.

First, note that the analyzed game has a unique pure strategy equilibrium (b, f) and the value of game is equal to 4:

Bob

		e	**f**	g
Alice	a	6	0	-1
	b	5	$\boxed{4}$	9
	c	9	-3	-1
	d	-1	1	-1

Knowing that the choice of Alice is b, her equilibrium strategy, Bob plays strategy f since this is his best response strategy. However, note that revealing publicly the equilibrium strategy doesn't help the opposite player in any zero-sum game. Indeed, if Bob doesn't know the strategy of Alice, he would play the strategy that guarantees him the minimal loss for every possible strategic choice of Alice, which is precisely the minmax strategy f.

Solution 9.

It suffices to show that the function u has no saddle point. This is equivalent to the fact that the minmax value $\overline{v}$ of Player 2 is strictly greater than maxmin value $\underline{v}$ of Player 1. Specifically, these values are

$$\underline{v} = \max_{s_1 \in X} \underline{f}(s_1),$$
$$\overline{v} = \min_{s_2 \in X} \overline{f}(s_2),$$

where $\underline{f}$ and $\overline{f}$ are the functions defined by

$$\underline{f}(s_1) = \min_{s_2 \in X} u(s_1, s_2),$$
$$\overline{f}(s_2) = \max_{s_1 \in X} u(s_1, s_2),$$

for all $s_1, s_2 \in X$. It is easy to check that

$$\underline{f}(s_1) = \begin{cases} \frac{1}{1+(s_1-1)^2} & 0 \leq s_1 \leq \frac{1}{2}, \\ \frac{1}{1+s_1^2} & \frac{1}{2} < s_1 \leq 1, \end{cases} \qquad \text{for all } s_1 \in X,$$

and

$$\overline{f}(s_2) = 1, \ s_2 \in X.$$

Hence $\underline{v}_1 = \underline{f}(\frac{1}{2}) = \frac{4}{5} < \overline{v}_1 = 1$, so u has no saddle point. Thus, the game has no equilibrium in pure strategies. However, note that it must have at least one equilibrium in mixed strategies by von Neumann's theorem.

3 SOLVING NORMAL-FORM GAMES

Exercise 10.
Alice and Bob play Rock–Paper–Scissors, but Bob's fingers hurt preventing him from signalling "Scissors". Model this scenario as a zero-sum game and find its equilibrium.

Exercise 11.
We present the game called *Battle of the Sexes*. Its name is derived from the situation where a couple (Alice and Bob) is trying to plan what to do on Saturday. The alternatives are going to a concert (C) or watching a football match (F). Bob prefers football and Alice prefers the concert, but both prefer being together to being alone, even if that means agreeing to the less-preferred recreational activity.

Alice \ Bob	C	F
C	$2,1$	$0,0$
F	$0,0$	$1,2$

Find equilibrium strategies of this game.

Exercise 12.
The two-player zero-sum game with the payoff matrix for the first player

Player 1 \ Player 2	0	1
0	1	-1
1	-1	1

is called *Matching Pennies*. In this game, each player chooses one bit (or a side of the coin), 0 or 1, in the following way: each player inserts into an envelope a slip of paper on which his choice is written. The envelopes are sealed and submitted to a referee. If both players have selected the same bit, Player 2 pays one dollar to Player 1. If they have selected opposite bits, Player 1 pays one euro to Player 2. Find an equilibrium of this game.

Exercise 13.
Consider a two-person zero-sum game with the payoff matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{R}\,.$$

Find equilibrium strategies of the row and column player.

SOLUTIONS

Solution 10.
This is a matrix game which can be described by the payoff matrix for Alice:

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Since this is a two-person zero-sum game, its solution can be recovered by two dual linear programming problems. For Alice we solve the problem with variables x_0 and $\mathbf{x} = (x_1, x_2, x_3)^\intercal$:

$$\begin{aligned} \text{Maximize} \quad & x_0 \\ \text{subject to} \quad & \mathbf{A}^\intercal \mathbf{x} - \mathbf{1}x_0 \geq \mathbf{0}, \\ & \sum_{i=1}^{3} x_i = 1, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

The problem for Bob has variables y_0 and $\mathbf{y} = (y_1, y_2)^\intercal$:

$$\begin{aligned} \text{Minimize} \quad & y_0 \\ \text{subject to} \quad & \mathbf{A}\mathbf{y} - \mathbf{1}y_0 \leq \mathbf{0}, \\ & \sum_{i=1}^{2} y_i = 1, \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Equivalently, we can write:

$$\begin{aligned} \text{Maximize} \quad & x_0 \\ \text{subject to} \quad & x_2 - x_3 - x_0 \geq 0, \\ & -x_1 + x_3 - x_0 \geq 0, \\ & x_1 + x_2 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0, \end{aligned}$$

and

$$\begin{aligned} \text{Minimize} \quad & y_0 \\ \text{subject to} \quad & -y_2 - y_0 \leq 0, \\ & y_1 - y_0 \leq 0, \\ & -y_1 + y_2 - y_0 \leq 0, \\ & y_1 + y_2 = 1, \\ & y_1, y_2 \geq 0. \end{aligned}$$

The solutions of those two problems are

$$\mathbf{x}^* = (0, \tfrac{1}{3}, \tfrac{2}{3})^\intercal \quad \text{and} \quad \mathbf{y}^* = (\tfrac{1}{3}, \tfrac{2}{3})^\intercal.$$

The value of game is equal to the common value in the optima, $x_0^* = y_0^* = \frac{1}{3}$.

Solution 11.
It is easy to verify that both strategy profiles (C, C) and (F, F) are equilibria in pure strategies. We will show that the game has a mixed strategy equilibrium, too.

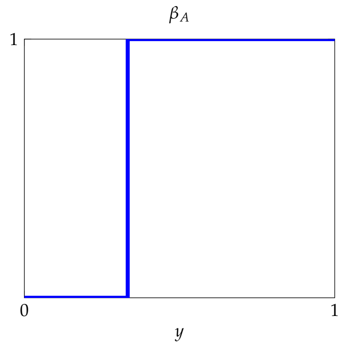
Let $x \in [0,1]$ and $y \in [0,1]$ be the probabilities of playing C for Alice and Bob, respectively. Since each player has only two pure strategies, the set of all mixed strategies can be viewed as the unit interval $[0,1]$. Thus, the expected utility of Alice is $U_A(x,y) = 2xy + (1-x)(1-y)$ and the expected utility of Bob is $U_B(x,y) = xy + 2(1-x)(1-y)$, for all $x, y \in [0,1]$.

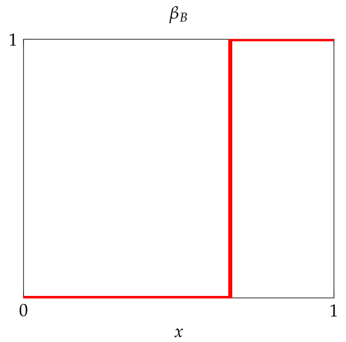
Now, we compute the best responses of both players. For Alice this is the mapping defined by

$$\beta_A(y) = \operatorname*{arg\,max}_{x \in [0,1]} U_A(x,y), \qquad y \in [0,1],$$

and analogously for Bob. We get

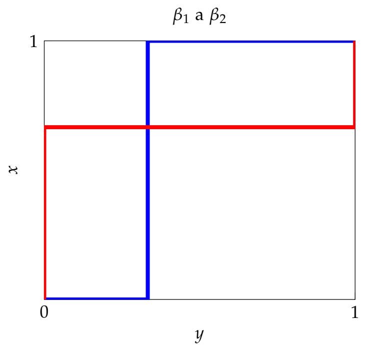
$$\beta_A(y) = \begin{cases} 0 & 0 \le y < \frac{1}{3}, \\ [0,1] & y = \frac{1}{3}, \\ 1 & \frac{1}{3} < y \le 1, \end{cases} \qquad \beta_B(x) = \begin{cases} 0 & 0 \le x < \frac{2}{3}, \\ [0,1] & x = \frac{2}{3}, \\ 1 & \frac{2}{3} < x \le 1. \end{cases}$$





We know that $(x^*, y^*) \in [0,1]^2$ would correspond to an equilibrium in mixed strategies if, and only if,

$$x^* \in \beta_A(y^*) \text{ and } y^* \in \beta_B(x^*). \tag{10}$$



The geometric interpretation of the condition (10) is that $(x^*, y^*) \in [0,1]^2$ is the point of the common intersection of the graphs of β_A a β_B. Therefore we obtain a mixed strategy equilibrium in which Alice plays the mixed strategy $(\frac{2}{3}, \frac{1}{3})$ and Bob uses the mixed strategy $(\frac{1}{3}, \frac{2}{3})$.

Solution 12.
Matching Pennies is a two-player zero-sum game, so we can formulate the equilibrium problem as a linear programming problem, analogously to Exercise 10. However, we will take a different approach and find the solution in a more elementary way.

First, it can be easily checked that the game has no solution in pure strategies. Let $x \in [0,1]$ and $y \in [0,1]$ denote the probabilities of selecting zero bit for Player 1 and 2, respectively. The expected utility of Player 1 is then the function given by $U(x,y) = 4xy - 2x - 2y + 1$, for every $x, y \in [0,1]$. For any choice $x \in [0,1]$ of Player 1, Player 2 will select the most harmful strategy for Player 1. This implies that Player 1 gets in this worst case scenario

$$\ell(x) := \min_{y \in [0,1]} U(x,y) = \begin{cases} 2x-1 & 0 \leq x < \frac{1}{2}, \\ 0 & x = \frac{1}{2}, \\ 1-2x & \frac{1}{2} < x \leq 1, \end{cases} \qquad x \in [0,1].$$

Player 1 can secure the maxmin value

$$\max_{x \in [0,1]} \ell(x) \tag{11}$$

which is equal to the minmax value of Player 2 by von Neumann's theorem. The equilibrium strategy of Player 1 is then any solution to the optimization problem (11). The only such solution is $x^* = \frac{1}{2}$. Repeating this analysis for Player 2 and the minmax value, we arrive at the same solution $y^* = \frac{1}{2}$ for Player 2. In conclusion, the only equilibrium strategy for each player is to randomize uniformly between the two choices.

Solution 13.
Let $\mathbf{x} = (x, 1-x)^\intercal$ a $\mathbf{y} = (x, 1-y)^\intercal$ be the vectors of mixed strategies for the row and column player, respectively, where $x, y \in [0,1]$. Note that each such vector is fully determined by its first coordinate since every player has only two strategies. The expected payoff of the row player is then given by the function $U\colon [0,1]^2 \to \mathbb{R}$ such that

$$U(x,y) = \mathbf{x}^\intercal \mathbf{A} \mathbf{y} = (a+d-b-c)xy + (b-d)x + (c-d)y + d.$$

By von Neumann's minimax theorem, an equilibrium mixed strategy profile exists, and it necessarily corresponds to the saddle point (x^*, y^*) of U. By elementary analysis, we know that (x^*, y^*) must satisfy the condition

$$\frac{\partial U}{\partial x}(x^*, y^*) = \frac{\partial U}{\partial y}(x^*, y^*) = 0,$$

which reads as

$$(a+d-b-c)y + b - d = (a+d-b-c)x + c - d = 0.$$

Assume that $a+d-b-c \neq 0$. Then the only solution is

$$\begin{aligned} \mathbf{x}^* &= (x^*, 1-x^*)^\intercal = \tfrac{1}{a+d-b-c}(d-c, a-b)^\intercal, \\ \mathbf{y}^* &= (y^*, 1-y^*)^\intercal = \tfrac{1}{a+d-b-c}(d-b, a-c)^\intercal. \end{aligned}$$

Since the Hessian of U

$$\begin{bmatrix} 0 & a+d-b-c \\ a+d-b-c & 0 \end{bmatrix}$$

is indefinite, (x^*, y^*) is indeed the saddle point.

Now, let $a + d - b - c = 0$. The function U becomes

$$U(x, y) = (b - d)x + (c - d)y + d.$$

First, suppose $b \geq d$ and $c \geq d$. Since $a = b + c - d$ we obtain $a \geq c$ and $a \geq b$. For example, the matrix $\mathbf{A}$ can be

$$\mathbf{A} = \begin{bmatrix} 10 & 4 \\ 5 & 1 \end{bmatrix}.$$

This matrix has a saddle point in the first row and the second column. Thus, the game will have an equilibrium in pure strategies. We can proceed analogously in the three remaining cases: $b \geq d$ and $c < d$, $b < d$ and $c \geq d$, $b < d$ and $c < d$.

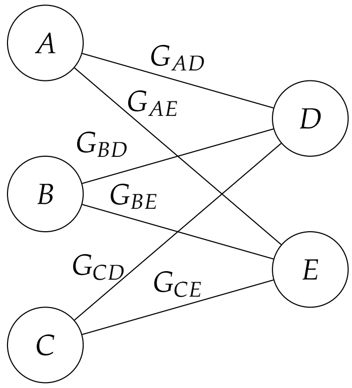
4 TRACTABLE CLASSES OF GAMES

Exercise 14.
Three inmates, Aaron, Bob, and Carl, are trying to escape prison. They know there are 5 possible ways to escape. Either they may escape through the infirmary or from the prison courtyard. In the infirmary, the prisoners may escape by sneaking into the van that is bringing the resources (V) or climbing down from the windows (W). Out of the courtyard, the inmates can either escape through the old warehouse (O), climb over the fence (F), or risk their luck and run through the main gate (G). Denise is guarding the infirmary. However, Denise can only guard the van or the windows, not both. Similarly, Elijah is guarding the courtyard and can only guard a single escape option simultaneously. On the other hand, each inmate can risk any route out. All of them may use the same route but do not have to. Formulate this as a constant-sum polymatrix game.

SOLUTIONS

Solution 14.
The underlying graph of this game is a complete bipartite graph, where one set is made out of inmates and the second out of the guards. This means that each inmate and each guard play a two-player normal-form game against each other. This means there are a total of 6 games being played.

1. Aaron against Denise G_{AD}
2. Bob against Denise G_{BD}
3. Carl against Denise G_{CD}
4. Aaron against Elijah G_{AE}
5. Bob against Elijah G_{BE}
6. Carl against Elijah G_{CE}



Now let us focus on some single game between prisoner p and guard g. If the guard guards the escape route that p chose, then p receives reward 0, and the guard receives reward 1. If p chooses an escape route that is not guarded by g, but the guard can guard it, then the inmate receives reward 1, and the guard receives 0. If the guard cannot guard the escape route of the prisoner, then neither player receives any reward.

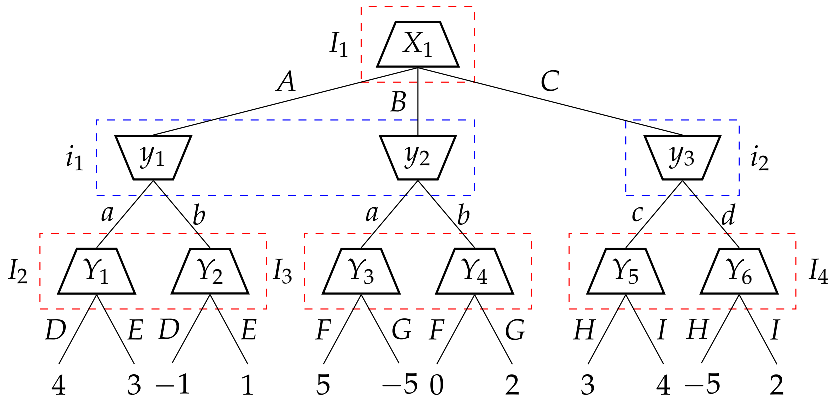
These games are all similar, so let us show only a single one for G_{AD}.

		Denise: V	Denise: W
Aaron	V	0, 1	1, 0
	W	1, 0	0, 1
	O	0, 0	0, 0
	F	0, 0	0, 0
	G	0, 0	0, 0

Now, let us check that this is a constant-sum game. Guard receives reward 1 for each caught inmate, and prisoner receives reward 1 if it manages to escape. This means that the sum of rewards is the number of caught and escaped prisoners. This is always the same as the number of prisoners in the game, in our case 3.

5 EXTENSIVE FORM GAMES

Exercise 15.
Consider following Extensive Form Game



Transform this game into a normal-form

Exercise 16.
Formulate the following game of a small poker as an EFG

- There is an ante of $1
- Deck is composed of these card J, J, Q, Q all with same color
- Each player receives a card at the beginning
- Player 1 either folds or bets $2
- Player 2 either calls or folds
- Player with the higher card wins

Exercise 17.
Consider the following two-player zero-sum game. Player 1 has the oppening move, in which he chooses action in the set A, B. A lottery is then conducted with either α or β selected, where α occurs with probability 25%. Finally, Player 2 chooses either a or b. If the outcome of the lottery was α, Player 1 receives reward 1 if both players selected the same action (either A, a or B, b). Otherwise Player 2 receives reward 1. If the outcome of the lottery was β, Player 2 receives reward 3 if both players selected the same action, otherwise Player 1 receives reward 3. Draw a game tree for a perfect information version of this game. Then visualize infosets for each of the following situations
(a) Player 2 knows the action taken by Player 1, but does not know the outcome of the lottery.
(b) Player 2 knows the outcome of the lottery, but does not know the action taken by Player 1.
(c) Player 2 knows the outome of the lottery only if Player 1 has selected A.
(d) Player 2 knows the action taken by Player 1 only if the outcome of the lottery is α.

(e) Player 2 does not know both the outcome of the lottery and action chosen by the Player 1.

★ **Exercise 18.**
Consider following map of Counter-Strike map Dust 2. There are 2 players, Terrorist (T) and Counter Terrorist (CT). The goal of the T is either to kill the CT or plant a bomb, which explodes after some time. Goal of the CT is to either kill the T and if the bomb was planted, the other goal is to defuse the bomb.

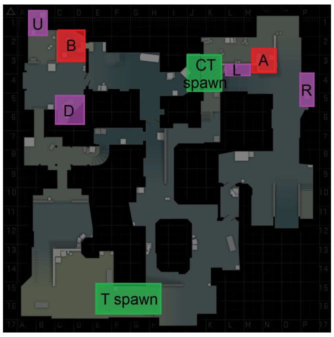


Figure 3: Counter-Strike map Dust 2

This map has 2 bomb sites A and B. Site B has two camping sites U and D, while A has camping sites L and R. The game then have following rules

- CT picks which site to defend at the beginning.
- T observes with probability p_{see} the CT, if the CT goes to the site B. If the CT is observed, then the T has p_{sn} probability that it will kill the CT.
- If T observed that CT goes to B, then it goes to the site A, otherwise it makes choice between A and B.
- When CT arrives at a site it camps at locations L or R on A or U or D on B
- If both CT and T picks the same site they engage in combat, in which one of them dies. If T knows which location CT picked it always wins the combat otherwise the CT is killed with probability p_c
- If T picked undefended site it freely plants a bomb, which informs CT that the bomb has been planted
- When CT is informed that bomb has been planted it runs to the other site and kills T with probability p_{st} and then he tries to defuse a bomb, otherwise he is killed.
- If CT ran from R to the B or from U to the A, he cannot defuse a bomb in time even when he kills the T. If CT ran from L to the B or from D to the A and kills the T, it defuses the bomb in time.

- T always gets reward 1 if it kills the CT. If CT kills T before he plants the bomb, then he gets reward of 1. If the bomb has been planted and CT kills T and is able to defuse the bomb it again gets reward of 1, but when the bomb explodes he gets reward of $-\frac{1}{2}$

SOLUTIONS

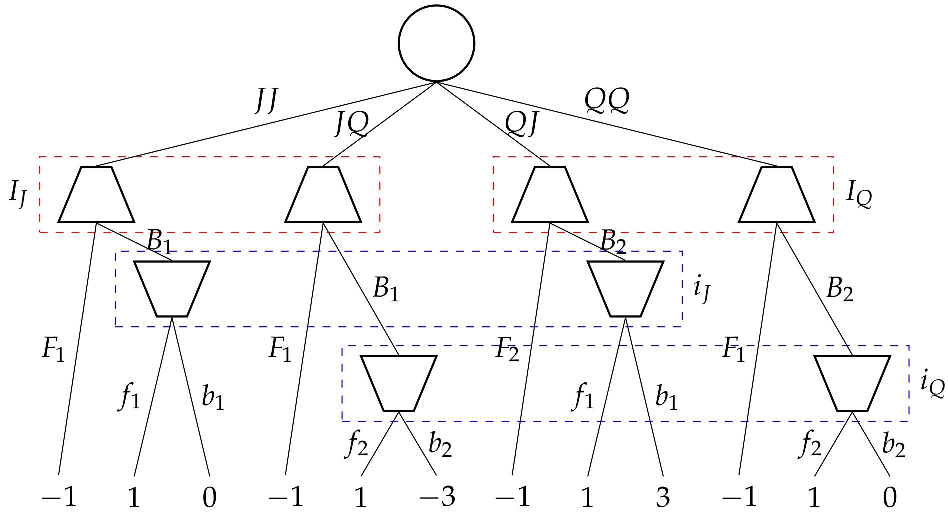
Solution 15.
When transforming a game from extensive-form into the normal-form, each pure action for player must take into account what the player would play in each infoset. Therefore the pure actions in normal-form game are a Cartesian product across infoset actions.

$$\mathcal{A}_j^{\mathrm{NFG}} = \bigtimes_{\mathcal{I}_i \in \mathcal{I}} \mathcal{A}_j(\mathcal{I}_i)$$

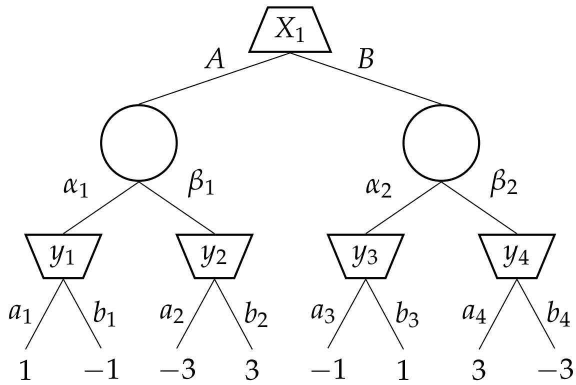
We will name the pure actions in NFG by concatenating the names of all actions, which create this action. For example first player actions, which is composed out of actions A, D, F, H will be called $ADFH$.
Pure actions of the player 1 are *ADFH, ADFI, ADGH, ADGI, AEFH, AEFI, AEGH, AEGI, BDFH, BDFI, BDGH, BDGI, BEFH, BEFI, BEGH, BEGI, CDFH, CDFI, CDGH, CDGI, CEFH, CEFI, CEGH, CEGI.*
Pure action of the player 2 are ac, ad, bc, bd. The utility matrix is then

P1 \ P2	ac	ad	bc	bd
$ADFH$	4	4	−1	−1
$ADFI$	4	4	−1	−1
$ADGH$	4	4	−1	−1
$ADGI$	4	4	−1	−1
$AEFH$	3	3	1	1
$AEFI$	3	3	1	1
$AEGH$	3	3	1	1
$AEGI$	3	3	1	1
$BDFH$	5	5	0	0
$BDFI$	5	5	0	0
$BDGH$	−5	−5	2	2
$BDGI$	−5	−5	2	2
$BEFH$	5	5	0	0
$BEFI$	5	5	0	0
$BEGH$	−5	−5	2	2
$BEGI$	−5	−5	2	2
$CDFH$	3	−5	3	−5
$CDFI$	4	2	4	2
$CDGH$	3	−5	3	−5
$CDGI$	4	2	4	2
$CEFH$	3	−5	3	−5
$CEFI$	4	2	4	2
$CEGH$	3	−5	3	−5
$CEGI$	4	2	4	2

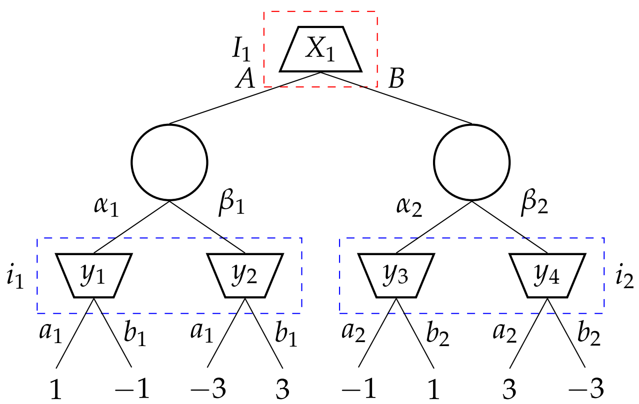
Solution 16.
The game starts with dealing a card. We can either create 2 chance nodes (one for each card), or just a single one, with 4 possible outcomes. We will show the second variant.



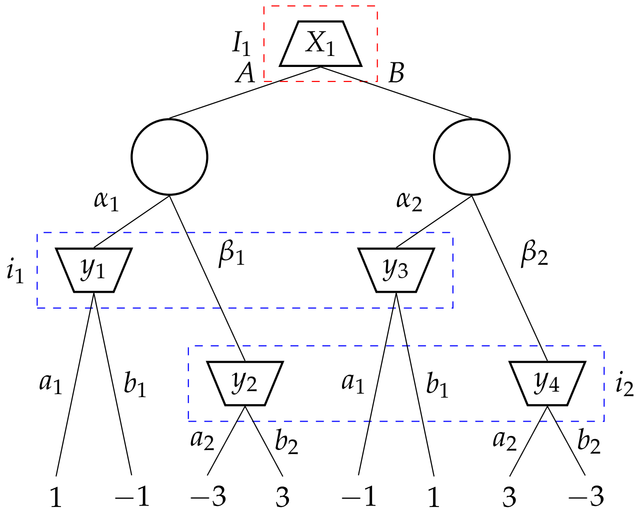
Solution 17.
The game tree consists of three levels. On first the Player 1 chooses it's action, on second, the chance node representing lottery occurs and on the last level the Player 2 chooses it's action. Note that each chance node and each decision node of Player 2 have different actions.



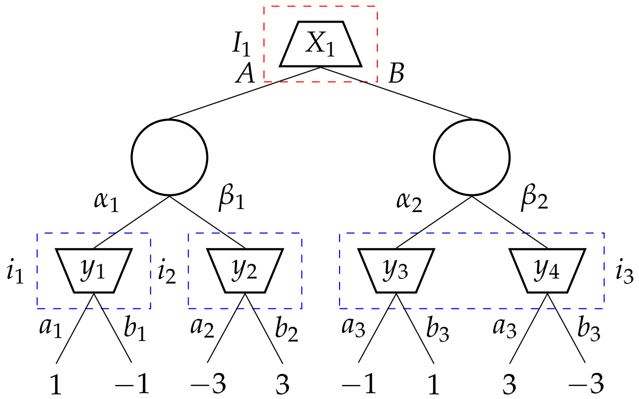
(a) Player 2 knows the action played by Player 1. Therefore, neither y_1 or y_2 cannot be in the same infoset as any of y_3 and y_4. But since it does not know outcome of the lottery, then y_1, y_2 has to be in the same infoset. Similarly y_3, y_4 has to be in the same infoset. Notice the action names of Player 2. In the same infoset all the actions have to be the same for a player.



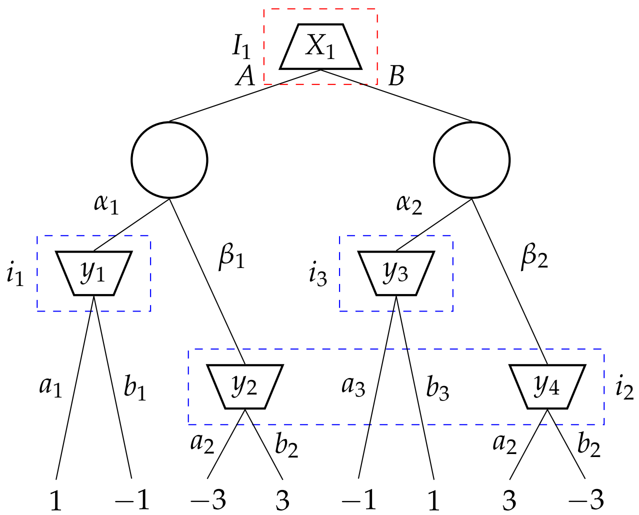
(b) By applying similair approach as in (a) we see that y_1 and y_3 are in the same infoset and y_2, y_4 are in the same infoset.



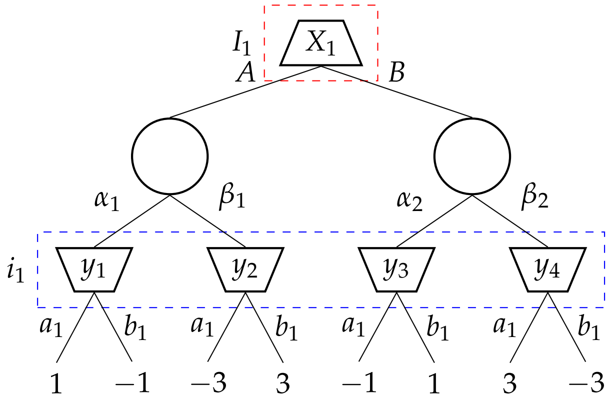
(c) Player 2 knows the outcome of the lottery if Player 1 has selected A. Therefore it always knows the action of Player 1, meaning that neither y_1 or y_2 cannot be in the same infoset as both y_3 and y_4. Furthermore y_1 and y_2 cannot be in the infoset, since Player 1 played A, so the outcome of lottery is known.



(d) By applying the same reasoning as in (c) we know that y_1 and y_3 are in separate infosets, while y_2 and y_4 are in the same infoset.

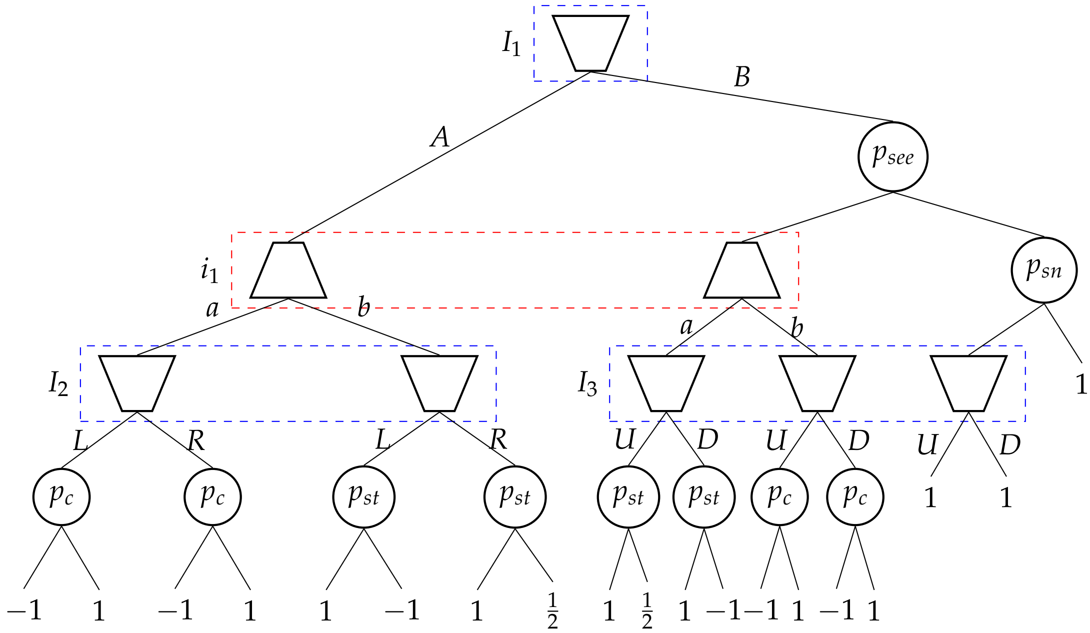


(e) Since Player 2 does not know anything about actions taken by Player 1 or outcome of the lottery, then all of his decision nodes are in the same infoset.



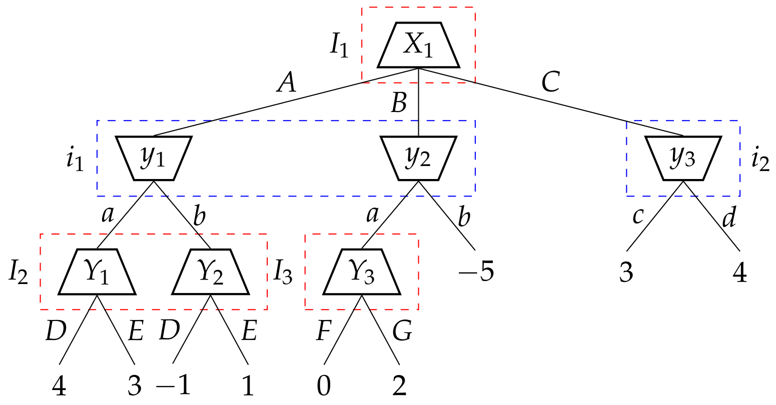
★ **Solution 18.**

This game may be modelled in a multiple ways and we will use just one of them. In this representation we consider the game as a zero-sum and we show only the reward to the T. Left subtree under chance node p is taken with probability $1 - p$ and the right with p.

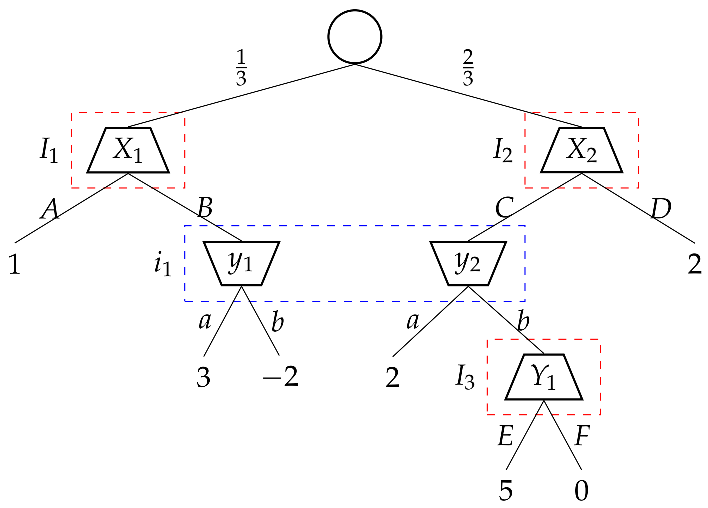


6 SOLVING IMPERFECT INFORMATION EXTENSIVE-FORM GAMES

Exercise 19.
Consider the following game. Write down a sequence-form linear program for both players



Exercise 20.
Consider the following game. Write down a sequence-form linear program for both players



Exercise 21.
Consider the small poker from Exercise 16. Wirte down a sequence-form linear program for both players in this game.

SOLUTIONS

Solution 19.
First let us remind the Sequence Form LP for the maximizing player

$$
\begin{aligned}
&\max_{r_1,v} v(root) \\
&\text{s.t. } r_1(\emptyset) = 1 \\
&r_1(\sigma_1) \geq 0 && \forall \sigma_1 \in \Sigma_1 \\
&\sum_{a \in \mathcal{A}(I_1)} r_1(\sigma_1 a) = r_1(\sigma_1) && \forall I_1 \in \mathcal{I}_1, \sigma_1 = \text{seq}_1(I_1) \\
&\sum_{I' \in \mathcal{I}_2 : \sigma_2 a = seq_2(I')} v(I') + \sum_{\sigma_1 \in \Sigma_1} u(\sigma_1, \sigma_2 a) r_1(\sigma_1) \geq v(I) && \forall I \in \mathcal{I}_2, \sigma_2 = \text{seq}_2(I) \forall a \in \mathcal{A}(I)
\end{aligned}
$$

where Σ_1 contains all posible sequences, which player 1 can take, r_1 variables are called reach and it is probability of playing given sequence $\sigma_1 \in \Sigma_1$. v variable represents expected value for given information set $I_2 \in \mathcal{I}_2$. $\text{seq}_i(I)$ gives sequence of actions, which player i has to take to reach infoset $I \in \mathcal{I}_i$, because of perfect recall, then only single sequence leads into any infoset. $\sigma_i a$ represents sequence σ_i prolonged by single action a.
Let us first find all possible sequences and info sets for both player

Σ_1	Σ_2
$\emptyset$	$\emptyset$
A	a
B	b
C	c
AD	d
AE	
BF	
BG	

$\mathcal{I}_1$	$\mathcal{I}_2$
I_1	i_1
I_2	i_2
I_3	

Now we are ready to state the linear programs for both players

$$
\begin{aligned}
&\max_{r_1,v} v(i_1) + v(i_2) \\
&\text{s.t. } r_1(\emptyset) = 1 \\
&\quad r_1(A) + r_1(B) + r_1(C) = r_1(\emptyset) \\
&\quad r_1(AD) + r_1(AE) = r_1(A) \\
&\quad r_1(BF) + r_1(BG) = r_1(B) \\
&\quad r_1(AD)u(AD,a) + r_1(AE)u(AE,a) + r_1(BF)u(BF,a) + r_1(BG)u(BG,a) \geq v(i_1) \\
&\quad r_1(AD)u(AD,b) + r_1(AE)u(AE,b) + r_1(B)u(B,b) \geq v(i_1) \\
&\quad r_1(C)u(C,c) \geq v(i_2) \\
&\quad r_1(C)u(C,d) \geq v(i_2) \\
&\quad r_1(A) \geq 0 \quad r_1(B) \geq 0 \quad r_1(C) \geq 0 \\
&\quad r_1(AD) \geq 0 \quad r_1(AE) \geq 0 \\
&\quad r_1(BF) \geq 0 \quad r_1(BG) \geq 0
\end{aligned}
$$

Note that the objective function is not $v(root)$, but rather sum of 2 infosets. The variable v symbolizes expected value for given infoset, therefore the expectation in root is sum across all infosets directly under root node.

$$
\begin{aligned}
&\min_{r_2,v} v(I_1) \\
&\text{s.t. } r_2(\emptyset) = 1 \\
&\quad r_2(a) + r_2(b) = r_2(\emptyset) \\
&\quad r_2(c) + r_2(d) = r_2(\emptyset) \\
&\quad v(I_2) \leq v(I_1) \\
&\quad v(I_3) + r_2(b)u(B,b) \leq v(I_1) \\
&\quad r_2(c)u(C,c) + r_2(d)u(C,d) \leq v(I_1) \\
&\quad r_2(a)u(AD,a) + r_2(b)u(AD,a) \leq v(I_2) \\
&\quad r_2(a)u(AE,a) + r_2(b)u(AE,a) \leq v(I_2) \\
&\quad r_2(a)u(BF,a) \leq v(I_3) \\
&\quad r_2(a)u(BG,a) \leq v(I_3) \\
&\quad r_2(a) \geq 0 \quad r_2(b) \geq 0 \quad r_2(c) \geq 0 \quad r_2(d) \geq 0
\end{aligned}
$$

Linear program for second player uses minimization instead of maximiziation and also it uses opposite inequality for expected value conditions. This is due to zero-sum property, where we used negative of each reward, different approach would be to use negative values explicitly and then the objective and conditions be the same.
The result of this LP is that $r_1(C) = 1, r_2(b) = 1, r_2(c) = 1$, the other reach probabilities are 0. Expected values are $v(I_1) = 3, v(I_2) = 3, v(I_3) = 2, v(i_1) = 0, v(i_2) = 3$. Therefore the value of the game is 3.

Solution 20.
We will first find all possible sequences and info sets for both players

Σ_1	Σ_2
$\emptyset$	$\emptyset$
A	a
B	b
C	
D	
CE	
CF	

$\mathcal{I}_1$	$\mathcal{I}_2$
I_1	i_1
I_2	
I_3	

Sequence-form linear programs for both players are

$$
\begin{aligned}
&\max_{r_1,v} v(i_1) + \tfrac{1}{3}r(A)u(A,\emptyset) + \tfrac{2}{3}r(D)u(D,\emptyset) \\
&\text{s.t. } r_1(\emptyset) = 1 \\
&\quad r_1(A) + r_1(B) = r_1(\emptyset) \\
&\quad r_1(C) + r_1(D) = r_1(\emptyset) \\
&\quad r_1(CE) + r_1(CF) = r_1(C) \\
&\quad \tfrac{1}{3}r_1(B)u(B,a) + \tfrac{2}{3}r_1(C)u(C,a) \geq v(i_1) \\
&\quad \tfrac{1}{3}r_1(B)u(B,b) + \tfrac{2}{3}r_1(CE)u(CE,b) + \tfrac{2}{3}r_1(CF)u(CF,b) \geq v(i_1) \\
&\quad r_1(A) \geq 0 \quad r_1(B) \geq 0 \quad r_1(C) \geq 0 \quad r_1(D) \geq 0 \\
&\quad r_1(CE) \geq 0 \quad r_1(CF) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\min_{r_2,v} v(I_1) + v(I_2) \\
&\text{s.t. } r_2(\emptyset) = 1 \\
&\quad r_2(a) + r_2(b) = r_2(\emptyset) \\
&\quad \tfrac{1}{3}u(A) \leq v(I_1) \\
&\quad \tfrac{1}{3}r_2(a)u(B,a) + \tfrac{1}{3}r_2(b)u(B,b) \leq v(I_1) \\
&\quad \tfrac{2}{3}r_2(a)u(C,a) + v(I_3) \leq v(I_2) \\
&\quad \tfrac{2}{3}u(D) \leq v(I_2) \\
&\quad \tfrac{2}{3}r_2(b)u(BE,b) \leq v(I_3) \\
&\quad \tfrac{2}{3}r_2(b)u(BF,b) \leq v(I_3) \\
&\quad r_1(a) \geq 0 \quad r_1(b) \geq 0
\end{aligned}
$$

Note that in both Linear Programs we propagate the chance node as close to the values as possible. The resulting reaches from these LPs are $r_1(B) = 1, R_1(C) = 1, r_1(CE) = 1, r_2(b) = 1$, other reaches are 0. Expected values are $v(i_1) = \frac{7}{3}, v(I_1) = 1, v(I_2) = \frac{4}{3}, v(I_3) = 0$. Value of the game is therefore $\frac{7}{3}$

Solution 21.
We will first find all possible sequences and info sets for both players

Σ_1	Σ_2
$\emptyset$	$\emptyset$
F_1	f_1
B_1	b_1
F_2	f_2
B_2	b_2

$\mathcal{I}_1$	$\mathcal{I}_2$
I_J	i_J
I_Q	i_Q

$$
\begin{aligned}
\max_{r_1, v} & v(i_1) + v(i_2) + \tfrac{1}{6} r(F_1) u(F_1, JJ) + \tfrac{2}{6} r(F_1) u(F_1, JQ) + \tfrac{2}{6} r(F_2) u(F_2, QJ) + \tfrac{1}{6} r(F_2) u(F_2, QQ) \\
\text{s.t. } & r_1(\emptyset) = 1 \\
& r_1(F_1) + r_1(B_1) = r_1(\emptyset) \\
& r_1(F_2) + r_1(B_2) = r_1(\emptyset) \\
& \tfrac{1}{6} r_1(B_1) u(B_1, f_1, JJ)) + \tfrac{2}{6} r_1(B_2) u(B_2, f_1, QJ) \geq v(i_1) \\
& \tfrac{1}{6} r_1(B_1) u(B_1, b_1, JJ)) + \tfrac{2}{6} r_1(B_2) u(B_2, b_1, QJ) \geq v(i_1) \\
& \tfrac{2}{6} r_1(B_1) u(B_1, f_2, JQ)) + \tfrac{1}{6} r_1(B_2) u(B_2, f_2, QQ) \geq v(i_2) \\
& \tfrac{2}{6} r_1(B_1) u(B_1, b_2, JQ)) + \tfrac{1}{6} r_1(B_2) u(B_2, b_2, QQ) \geq v(i_2) \\
& r_1(F_1) \geq 0 \quad r_1(B_1) \geq 0 \quad r_1(F_2) \geq 0 \quad r_1(B_2) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{r_2, v} & v(I_1) + v(I_2) \\
\text{s.t. } & r_2(\emptyset) = 1 \\
& r_2(f_1) + r_2(b_1) = r_2(\emptyset) \\
& r_2(f_2) + r_2(b_2) = r_2(\emptyset) \\
& \tfrac{1}{6} u(F_1, JJ) + \tfrac{2}{6} u(F_1, JQ) \leq v(I_1) \\
& \tfrac{1}{6} r(f_1) u(B_1, f_1, JJ) + \tfrac{1}{6} r(b_1) u(B_1, b_1, JJ) + \tfrac{2}{6} r(f_2) u(B_1, f_2, JQ) \\
& \ + \tfrac{2}{6} r(b_2) u(B_1, b_2, JQ) \leq v(I_1) \\
& \tfrac{2}{6} u(F_2, QJ) + \tfrac{1}{6} u(F_2, QQ) \leq v(I_2) \\
& \tfrac{2}{6} r(f_1) u(B_2, f_1, QJ) + \tfrac{2}{6} r(b_1) u(B_2, b_1, QJ) + \tfrac{1}{6} r(f_2) u(B_2, f_2, QQ) \\
& \ + \tfrac{1}{6} r(b_2) u(B_2, b_2, QQ) \leq v(I_2) \\
& r_2(f_1) \geq 0 \quad r_2(b_1) \geq 0 \quad r_2(f_2) \geq 0 \quad r_2(b_2) \geq 0
\end{aligned}
$$

The resulting reaches from these LPs are $r_1(F_1) = 1, r_1(B_2) = 1, r_2(f_1) = 1, r_2(b_2) = 1$, other reaches are 0. This means that player should always bet if he gets Queen and always folds if he gets Jack. Expected values are $v(I_1) = -\frac{1}{2}, v(I_2) = \frac{1}{3}, v(i_1) = \frac{1}{3}, v(i_2) = 0$. The value of the game is $-\frac{1}{6}$.

7 ALTERNATIVES TO NASH EQUILIBRIUM

Exercise 22.

The concept of correlated equilibrium is a generalization of Nash equilibrium in the following sense. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be an n-player strategic game, $(p_1, \ldots, p_n)$ be its Nash equilibrium in mixed strategies, and define

$$p(s_1, \ldots, s_n) := \prod_{i\in N} p_i(s_i), \qquad (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n.$$

Then p is a correlated equilibrium of G. Prove this claim.

Exercise 23.

Imagine a situation, where 2 drivers are driving the same road in the opposite direction. First driver is in the correct lane, while the second one is in the opposite. If they both continue in the same lane, they will crash into each other, which results in the reward 0, similarly if they both change the lane. However, if only one of them switches lane they will miss each other, which is beneficial for both of them and they will receive reward 3. Moreover, if the driver is in the wrong lane, it will be penalized with reward -1, regardless of the outcome of the game. Model this as a normal-form game and find all of it's correlated equilibria.

Exercise 24.

Show that there exists a unique correlated equilibrium in the following game

		Player 2	
		L	R
Player 1	T	$1, 0$	$c, 1+d$
	B	$0, 1$	$1+a, b$

, in which $a, b, c, d \in (-\frac{1}{4}, \frac{1}{4})$. Find this correlated equilibrium. What is the limit of the correlated equilibrium payoff as a, b, c, d approach 0?

Exercise 25.

Consider following utility matrix two-person normal-form game, where row player is a leader, which publicly announces its strategy

		F				
		a	b	c	d	e
L	T	$2, 4$	$6, 4$	$9, 0$	$1, 2$	$7, 4$
	B	$8, 4$	$0, 4$	$3, 6$	$1, 5$	$0, 0$

Find Strong and Weak Stackelberg Equilibrium

SOLUTIONS

Solution 22.
Clearly, p is a probability distribution on the joint strategy space S by the definition. We know that p is a correlated equilibrium if the inequality

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}). \tag{12}$$

holds for each player i and all strategies $s_i, s_i' \in S_i$. Since $(p_1, \ldots, p_n)$ is a Nash equilibrium, it follows that for each player $i \in N$ and all strategies $s_i, s_i' \in S_i$ such that $p_i(s_i) > 0$,

$$U_i(s_i, p_{-i}) \geq U_i(s_i', p_{-i}), \tag{13}$$

where U_i denotes the expected utility (payoff) of player i. Since

$$p_i(s_i) \cdot U_i(s_i, p_{-i}) = \sum_{s_{-i}\in S_{-i}} p_i(s_i) \cdot \prod_{j\neq i} p_j(s_j) \cdot u_i(s_i, s_{-i}) = \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i, s_{-i})$$

and

$$p_i(s_i) \cdot U_i(s_i', p_{-i}) = \sum_{s_{-i}\in S_{-i}} p_i(s_i) \cdot \prod_{j\neq i} p_j(s_j) \cdot u_i(s_i', s_{-i}) = \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}),$$

from (13) we get (12). In case that $p_i(s_i) = 0$, necessarily $p(s_i, s_{-i}) = 0$ for every $s_{-i} \in S_{-i}$, so that boths sides of (12) are zero.

Solution 23.
The normal-form representation of this game is

		Driver 2	
		Stay	*Change*
Driver 1	*Stay*	$0, -1$	$3, 3$
	Change	$2, 2$	$-1, 0$

Now let us denote all of the 4 possible joint strategies followingly

$$\alpha = p(Stay, Stay) \tag{14}$$
$$\beta = p(Stay, Change) \tag{15}$$
$$\gamma = p(Change, Stay) \tag{16}$$
$$\delta = p(Change, Change) \tag{17}$$

The inequalities for the Correlated equilibrium are then

$$0\alpha + 3\beta \geq 2\alpha - 1\beta \tag{18}$$
$$2\gamma - 1\delta \geq 0\gamma + 3\delta \tag{19}$$
$$-1\alpha + 2\gamma \geq 3\alpha + 0\gamma \tag{20}$$
$$3\beta + 0\delta \geq -1\beta + 2\delta \tag{21}$$

This may be simplified to

$$2\beta \geq \alpha \tag{22}$$
$$\gamma \geq 2\delta \tag{23}$$
$$\gamma \geq 2\alpha \tag{24}$$
$$2\beta \geq \delta \tag{25}$$

Some interesting correlated equilibria are that if $\alpha = \delta = 0$, then any mixed strategy with support β, γ is a Correlated equilibrium. If $\delta = 0$, then $\alpha \in [0, \frac{2}{7}]$, similarly if $\alpha = 0$, then $\delta \in [0, \frac{2}{7}]$

Solution 24.
Let us start by writing all the correlated equilibrium conditions

$$\begin{aligned} p(T,L) + (c-1-a)p(T,R) &\geq 0 \\ -p(B,L) + (1+a-c)p(B,R) &\geq 0 \\ (-1-d)p(T,L) + (1-b)p(B,L) &\geq 0 \\ (1+d)p(T,R) + (b-1)p(B,R) &\geq 0 \end{aligned}$$

Now let us introduce following substitions

$$\begin{aligned} X &= 1+a-c \\ Y &= 1-b \\ Z &= 1+d \end{aligned}$$

If we rewrite the conditions with following substitions we will end up with

$$\begin{aligned} p(T,L) &\geq Xp(T,R) \\ Xp(B,R) &\geq p(B,L) \\ Yp(B,L) &\geq Zp(T,L) \\ Zp(T,R) &\geq Yp(B,R) \end{aligned}$$

Since we know that each of $a, b, c, d \in (-\frac{1}{4}, \frac{1}{4})$, we can find the domain for our substitions

$$\begin{aligned} X &\in (\frac{1}{2}, \frac{3}{2}) \\ Y &\in (\frac{3}{4}, \frac{5}{4}) \\ Z &\in (\frac{3}{4}, \frac{5}{4}) \end{aligned}$$

This means that we can multiply or divide with them without changing the inequality. Now let us use this to get following inequalities. First we use the first inequality, than the fourth, then second and lastly the third

$$p(T,L) \geq Xp(T,R) \geq \frac{XY}{Z}p(B,R) \geq \frac{XY}{XZ}p(B,L) \geq \frac{XYZ}{XYZ}p(T,L) = p(T,L)$$

We see that in order for this to hold, all of the inequalities have to be equalities. This means that we can express each probability by $p(T,L)$

$$\begin{aligned} p(T,R) &= \frac{p(T,L)}{X} \\ p(B,R) &= \frac{Zp(T,L)}{XY} \\ p(B,L) &= \frac{Zp(T,L)}{Y} \end{aligned}$$

Now we can find the actual prbabilities from the condition that all of these has to sum up to 1

$$\begin{aligned} p(T,L) + \frac{p(T,L)}{X} + \frac{Zp(T,L)}{XY} + \frac{Zp(T,L)}{Y} &= 1 \\ (XY + Y + Z + ZX)p(T,L) &= XY \\ p(T,L) &= \frac{XY}{(XY + Y + Z + ZX)} \end{aligned}$$

Out of this we can get all other probabilities

$$\begin{aligned} p(T,R) &= \frac{Y}{XY + Y + Z + ZX} \\ p(B,R) &= \frac{Z}{XY + Y + Z + ZX} \\ p(B,L) &= \frac{ZX}{XY + Y + Z + ZX} \end{aligned}$$

The value of this equilibrium is than

$$\begin{aligned} u_1 &= p(T,L) + cp(T,R) + (1+a)p(B,R) = \frac{XY + cY + (1+a)Z}{XY + Y + Z + ZX} \\ u_2 &= p(B,L) + (1+d)p(T,R) + bp(B,R) = \frac{ZX + (1+d)Y + bZ}{XY + Y + Z + ZX} \end{aligned}$$

The only thing that remains is expressing X, Y, Z with the original variables. In order to compute what happens when all of the original variables a, b, c, d goes to 0, we will keep the substitions. In such a case

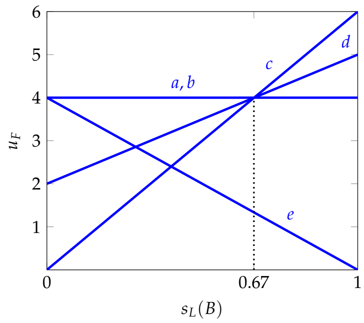
$$\begin{aligned} X &= 1 \\ Y &= 1 \\ Z &= 1 \end{aligned}$$

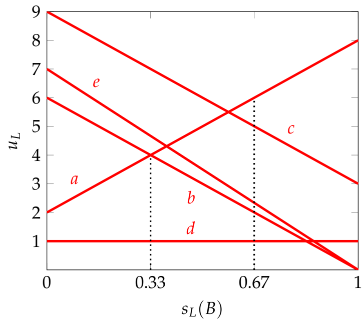
In this case the resulting values are

$$\begin{aligned} u_1 &= 0.5 \\ u_2 &= 0.5 \end{aligned}$$

, which is the same as Nash equilibrium. If looking closely, one would notice that this game is symmetrical and is similair to the Matching Pennies from exercise 12.

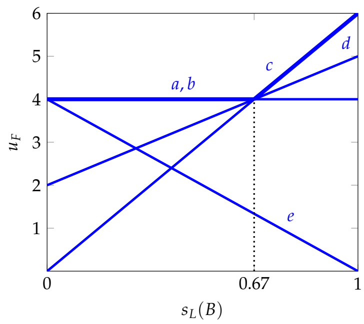
Solution 25.
Since the leader has only two actions, we may visualize the expected utility based on the policy announced by the leader.





When computing the Stackelberg equilibrium, the follower always takes a best response to the leader's policy. Best responses are as follows

$$\mathrm{BR}(s_L) = \begin{cases} a, b, e & s_L(B) = 0, \\ a, b & 0 < s_L(B) < \frac{2}{3}, \\ a, b, c, d & s_L(b) = \frac{2}{3}, \\ c & \frac{2}{3} < s_L(b) \leq 1, \end{cases}$$



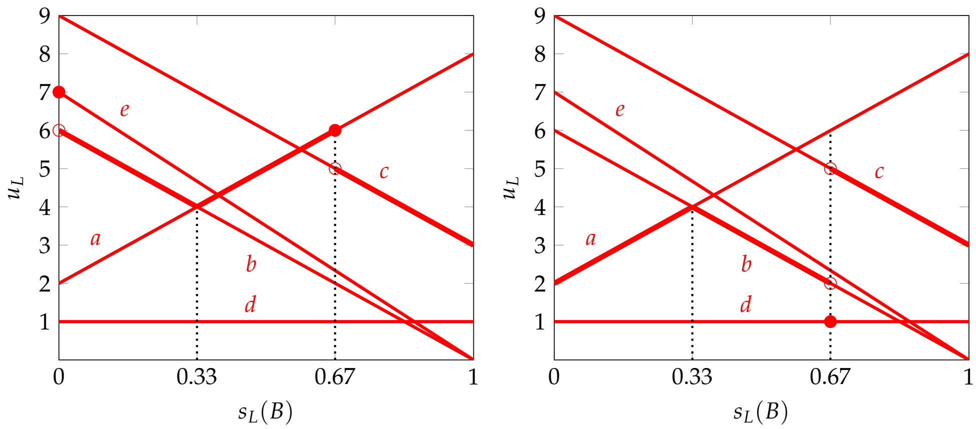
Now we know, based on the policy of the leader, what are the follower best responses. Now we have to figure out which best response the follower would take.

Let us define functions $\overline{\beta}(s_L)$ and $\underline{\beta}(s_L)$, which, based on the leader's policy, give pure action that the follower should play to maximize or minimize the leader's utility. Similarly we define function $\overline{\beta}(s_L)$ and $\underline{\beta}(s_L)$ that corresponds to the expected utility for leader when it fixes its policy.

$$\overline{\beta}(s_L) = \arg\max_{a_F \in BR(s_L)} u_L(s_L, a_F) \qquad \underline{\beta}(s_L) = \arg\min_{a_F \in BR(s_L)} u_L(s_L, a_F)$$

The policies are then

$$\overline{\beta}(s_L) = \begin{cases} e & s_L(B) = 0, \\ b & 0 < s_L(B) \leq \frac{1}{3}, \\ a & \frac{1}{3} < s_L(B) \leq \frac{2}{3}, \\ c & \frac{2}{3} < s_L(B) \leq 1, \end{cases} \qquad \underline{\beta}(s_L) = \begin{cases} a & 0 \leq s_L(B) < \frac{1}{3}, \\ b & \frac{1}{3} \leq s_L(B) < \frac{2}{3}, \\ d & s_L(B) = \frac{2}{3}, \\ c & \frac{2}{3} < s_L(B) \leq 1, \end{cases}$$



Strong Stackelberg equilibrium from the left plot is strategy $s_L(B) = 0$ because this maximizes the leader's utility. Weak Stackelberg equilibrium does not exist, because if we take $\varepsilon > 0$ and policy $s_L(B) = \frac{2}{3} + \varepsilon$. When lowering the value of ε, we always get better utility, and when $\varepsilon = 0$, the value of action c is 5, which is the best we can get. But when $\varepsilon = 0$, the utility drops to 1, because optimal action changes to d. Therefore we cannot set such ε, which would maximize the value, so the Weak Stackelberg equilibrium does not exist.

Part II. Incomplete-Information Games

8 BAYESIAN GAMES

Exercise 26.
In the following game, player 1 has types 1_t and 1_b, and player 2 has types 2_l and 2_r. Players decide on the following behavior strategies: 1_t plays the mix $(1/3, 2/3)$, 1_b plays $(1/2, 1/2)$, 2_l plays $(1/4, 3/4)$ and 2_r plays $(1/2, 1/2)$. Nature chooses the types 1_t and 2_r for the players. The payoff matrices and their joint probabilities are given below.

1/3

	L	R
T	1,3	0,3
B	3,0	1,2

1/4

	L	R
T	2,3	1,1
B	0,2	2,0

1/4

	L	R
T	0,3	3,2
B	3,2	1,0

1/6

	L	R
T	2,3	1,0
B	1,3	2,2

Find the interim expected utility of player 1.

Exercise 27.
In the following game of rock-paper-scissors you might be playing a simple opponent who is unaware that he can play strategies other than paper.

2/3

		R	P	S
	R	0	-1	1
a_1	P	1	0	-1
	S	-1	1	0

1/3

		P
	R	-1
a_1	P	0
	S	1

(26)

Which of the following strategy profiles are Bayesian equilibria?

1. Player 1: R; Player 2 normal: P, simple: P
2. Player 1: P; Player 2 normal: S, simple: P
3. Player 1: S; Player 2 normal: R, simple: P
4. None of the above

Exercise 28.

Consider a the following Bayesian simultaneous-move game involving two armies fighting for an island. Army 1 can be of type *weak*, or *strong*, both with equal probability. Army 2 is always *weak*. Both players can chose to *attack* or *not to attack*. Neither army can observe the other's action. Army 2 does not know which type of army it is fighting against.

The following matrices capture the payoffs. Army 1 is the row player.

50%

a_1^w	A	N
A	$-3,-3$	$5,2$
N	$0,5$	$0,0$

50%

a_1^s	A	N
A	$3,-3$	$5,0$
N	$0,5$	$0,0$

Which of the following strategy profiles are Bayesian equilibria?

1. Army 1: (Weak: Not attack, Strong: Attack);
   Army 2: Attack
2. Army 1: (Weak: Not attack, Strong: Attack);
   Army 2: Not attack
3. Army 1: (Weak: Attack, Strong: Attack);
   Army 2: Attack
4. There is no Bayesian equilibrium.

Exercise 29.

Consider a car trade between a single seller and a single buyer. The car in question is in good condition with probability 70%; however only the seller knows its condition. The market price is P and is non-negotiable. Both players can either accept or decline the trade.

A car in good condition is worth 12 to the buyer, and 11 to the seller; a poor condition car is worth 6 to the buyer, but is worthless to the seller.

70%

		T	N
$\text{seller}_{\text{good}}$	T	$P, 12-P$	$11, 0$
	N	$11, 0$	$11, 0$

30%

		T	N
$\text{seller}_{\text{poor}}$	T	$P, 6-P$	$0, 0$
	N	$0, 0$	$0, 0$

1. Convert the game to strategic form and find the pure bayesian equilibria.
2. Compute the interim expected utility for both players in case of a good-quality car with a market price $P = 10$ and the strategies (TT, T).

Exercise 30.
Player 2 has two types: left with probability 2/3 and right with probability 1/3. Find the Bayesian Nash equilibrium of the game.

2/3

		L	R
a_1	U	$1, 0$	$0, 2$
	D	$0, 3$	$3, 0$

1/3

		L	M	R
a_1	U	$0, 2$	$2, 1$	$0, 3$
	D	$2, 0$	$1, 3$	$0, 2$

(27)

Exercise 31.
Consider the following Bayesian simultaneous-move game with players a_1 and a_2. Player a_2 has two types a_2^l with probability $1/3$ and a_2^r with prob. $2/3$.

1/3

		L	R
a_1	U	$1, 2$	$1, 3$
	D	$2, 2$	$0, 1$

2/3

		L	R
a_1	U	$3, 2$	$1, 1$
	D	$2, 1$	$1, 0$

(28)

1. Convert the game to strategic form.
2. Find a pure Bayesian equilibrium of the game.

Exercise 32.
Consider the following Bayesian simultaneous-move game with players a_1 and a_2. Player a_1 has two types a_1^t and a_1^b, and two actions U and D. Player a_2 has only one type and actions L and R. Both types of player a_1 are equally likely.

50%

a_1^l		L	R
	U	$1,0$	$0,2$
	D	$0,3$	$1,0$

50%

a_1^r		L	R
	U	$0,2$	$1,1$
	D	$1,0$	$0,2$

1. Draw Game 32 in extensive form.
2. Calculate Bayesian equilibria of the game.

SOLUTIONS

Solution 26.

First, we compute the ex-post utilities in the top two games. We get $1/12 + 0 + 2/4 + 2/4 = 13/12$ and $1/3 + 1/6 + 0 + 1/3 = 5/6$. Since player 1 only knows his type, we need to weigh these values by his belief of actually ending up in one of these state games. The probability of the top left game is $1/3 \times 12/7 = 12/21$ and the top right is $1/4 \times 12/7 = 12/28$. The interim utility is then $41/42$.

Solution 27.

None of the profiles are equilibria. Actual equilibria are mixed e.g. player 1 uniformly mixes between R, P and S, while player 2 mixes uniformly between RP and SP.

Solution 28.

Army 1 knows the payoff matrix; thus, it suffices to compare the payoff to the other strategy. Army 2 has to compare the expected payoff, instead.

The first strategy is a Bayesian equilibrium. Other answers are not correct.

Solution 29.

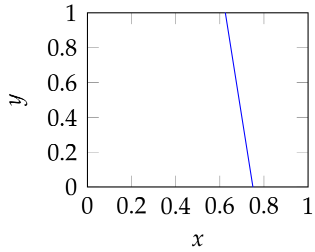
1. While we do not know the exact value of P, strategies where seller does not sell the bad leave $0.3P$ on the table. Comparing the remaining strategies, we find that always trading is worth it if P is at least 11 – in which case the buyer declines everything above 10.2. If P is below 11, trading only the bad car is the weakly dominant strategy; furthermore, if P is below 6, the buyer will accept.

	buyer T	N
TT	$P, 0.7(12-P) + 0.3(6-P)$	$7.7, 0$
TN	$0.7P, 0.7(12-P)$	$7.7, 0$
NT	$0.7 * 11 + 0.3P, 0.3(6-P)$	$7.7, 0$
NN	$0.7 * 11, 0$	$7.7, 0$

2. The seller knows the situation exactly and his utility is simply the market price. The buyer only has the prior probabilities to rely on and his expected utility is $0.7(12-10) + 0.3(6-10)$ or 0.2. If we compare the interim utilities of the cases when both players would unilaterally deviate, we would find that the seller would wish to switch, while the buyer would not.

Solution 30.

There is a interim-dominated strategy for the right type of player 2. Trying pure strategies for player 1 yields nothing, which means that player 2 plays a strategy such that player 1 is indifferent between up and down. Equating the utilities gets us the following picture where x is between 5/8 and 6/8.



As x must be mixed, the utilities from both actions for the left type of player 2 must be equal just like in the case of player 1, which happens when the action up is played with probability 3/5. If we check the best response of the right type, we notice he always plays right. Therefore the only equilibrium is for player 1 to play up with prob 3/5, for the left type of player 2 to play left with prob 6/8, and the right type always plays the right action.

Solution 31.

	a_2: LL	LR	RL	RR
a_1: U	7,6	3,4	7,7	3,5
D	6,4	4,2	4,3	2,1

(29)

Equilibrium is (U, RL). You can find it by looking for best responses, or eliminating dominated strategies.

Solution 32.

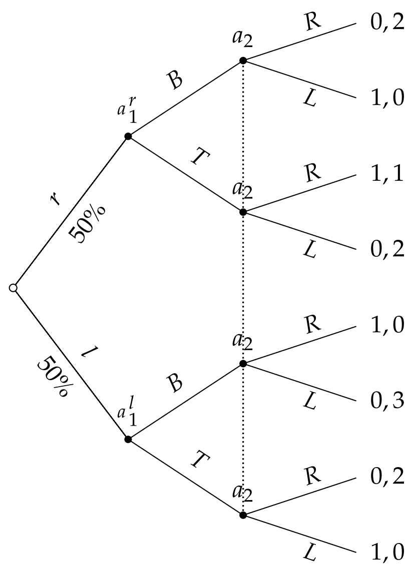


Figure 4: Game 32 in extensive form.

EQUILIBRIA : Start by checking pure strategies for player a_2. Guessing R does not work; Player a_1^r's best response would be T, while a_1^l's would be B. Thus, R is not a best response as it has a lower expected payoff than L. Playing L does not work for the same reason.

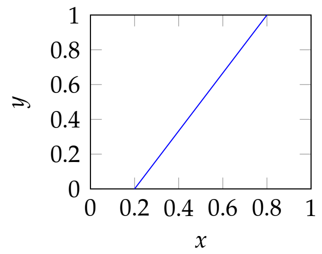
Player a_2 must, therefore, play a mixed strategy, and hence his expected payoff for both actions must be equal. Let x be the probability that a_1^l plays T, and y that a_1^r plays T.

$$\frac{3(1-x)}{2} + \frac{2y}{2} = \frac{3x}{2} + \frac{1y}{2} + \frac{2(1-y)}{2}$$

which simplifies to

$$x = \frac{1+3y}{5}.$$

Notice, that valid x's that satisfy the equation lie within $\frac{1}{5} \leq x \leq \frac{4}{5}$.



Now we can find the fully mixed strategy for player a_2. Let q be the probability that a_2 plays L. Given that player a_1^l plays a mixed strategy, we equate the payoffs for T and B, as we did before. We find that $q = \frac{1}{2}$.

All strategies satisfying $x = \frac{1+3y}{5}$ and $q = \frac{1}{2}$ are Bayesian equilibria.

9 SIMPLE AUCTIONS

Exercise 33.
BASIC DEFINITIONS AND TERMINOLOGY Suppose there are two bidders a_1 and a_2 with private values drawn i.i.d. from a uniform distribution on $[0, 1]$. For each of the examples, find the equilibrium, then answer the following:

- b_i: What are the bids of each player $i \in \{a_1, a_2\}$.
- $x_i(\mathbf{b}), p_i(\mathbf{b})$: What is the allocation to each bidder and what do they pay for it?
- $u_i(\mathbf{b})$: What is the utility of each player.
- R: What revenue does the auction generate.

---

1. The private values of the bidders are $v_1 = 0.3$ and $v_2 = 0.9$. Suppose the auction is run as a second-price sealed-bid auction.
2. The private values remain $v_1 = 0.3$ and $v_2 = 0.9$, but the auction is now run as a *first-price* auction.
3. The private values are $v_1 = 0.6$ and $v_2 = 0.9$, and the auction is run as a second-price auction.
4. As before, the private values are $v_1 = 0.6$ and $v_2 = 0.9$, but the auction is now run as a first-price auction.

Exercise 34.
Suppose there are two bidders a_1 and a_2, whose independent private values are either 1 or 3 with equal probability. Assume ties are broken randomly.

1. Find the expected revenue using second-price auction rules.
2. ★ What would be the revenue of a first-price auction?
3. Now suppose there were three bidders instead of two. How does the revenue change?

Exercise 35.
Suppose there are two bidders in a first price auction, whose IPVs are drawn from an exponential distribution $F(x) = x^a$. Find the optimal bidding strategy.

HINT: Assume that bidder i wins with a private value of v, let Y be the random variable denoting the highest value among the other players with a distribution function F_Y. For a symmetric IPV (drawn from a continuous distribution) auction, the optimal bidding strategy β_i of a player i is

$$\beta_i(v) = \mathbf{E}[Y|Y < v] = v - \int_0^v \frac{F_Y(x)}{F_Y(v)} dx$$

Exercise 36.

SANITY CHECK FOR HW3: Suppose there are two bidders in a first price auction, whose IPVs are drawn from an normal distribution with a mean of 10 and a standard deviation of 1. ★ Find the optimal bid of a player with a private value of 11.

Exercise 37.

Consider a second-price auction involving two bidders a_1 and a_2 whose private values are either 0 or 1 with equal probability. Bidder a_1 sometimes makes a mistake about his value for the object: when his value is 1, he knows it is 1; however, when his value is 0, half of the time, he believes it is actually 1 by mistake. Assume that ties are broken randomly and that bids must be integers. Draw the full game tree of this situation.

Exercise 38.

Using the tree from the last exercise, calculate a_1's expected utility from bidding 0 compared to bidding 1, when a_1 thinks his value is 1. What is a_1's optimal strategy?

SOLUTIONS

Solution 33.

- Second price auctions are *DSIC* — bidding your private value is a (weakly) dominant strategy. In a first-price auction, bidders have to take into account their beliefs of other players values: for our setting of two bidders with private values drawn i.i.d. from a uniform distribution on $[0, 1]$ the Bayesian Nash equilibrium strategy is to bid a half of your value (we will see why later).
- Allocations denote which bidder wins which items, or even how much of each. In single item auctions, allocations are just 0-1 vectors with at most one 1. In both the first-price and the second-price auctions, the highest bidder wins the item. In a first-price auction, the highest bidder pays his own bid, in a second-price auction the highest bidder pays the second highest bid.
- While more complex models exist, we will consider only the quasilinear utility $u_i(\mathbf{b}) = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$. Your utility is "what you get minus what you pay for it."
- The reveune is the sum of the payments.

---

1. $\mathbf{b} = (0.3, 0.9)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.3)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.6$; and $R = 0.3$.
2. $\mathbf{b} = (0.15, 0.45)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.45)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.45$; and $R = 0.45$.
3. $\mathbf{b} = (0.6, 0.9)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.6)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.3$; and $R = 0.6$.
4. $\mathbf{b} = (0.3, 0.45)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.45)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.45$; and $R = 0.45$.

Solution 34.
The simplest thing to do here is to enumerate all of the realizations:

	v_1	v_2	R
25 %	1	1	1
25 %	1	3	1
25 %	3	1	1
25 %	3	3	3

which add up to an expected revenue of 1.5. The same method can be used to show that the three-player auction yields a revenue of 2.

In the case of the first-price auction, the task is not so simple. There is no formula to calculate the equilibrium strategies like there was for the continuous distribution. In fact, due to the discreteness, equilibria will be in mixed strategies – add asymmetry and there may be no equilibrium at all!

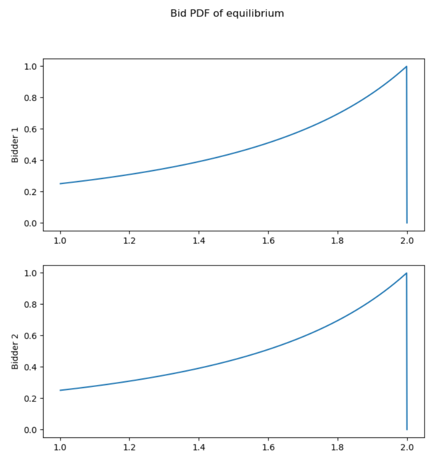


Figure 5: Mixed strategy equilibrium of the first price auction.

Solution 35.
Due to the revenue equivalence theorem, we know that the optimal strategy in a first-price auction is to bid the same amount as you would expect to pay as a winner in a second-price auction. Thus, we need to compute the distribution function the highest IPV below v as that is what the winner would pay.

In general, the CDF of the k'th order statistic of a continuous distribution is

$$F_k(x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1 - F(x))^{n-j}$$

We can equivalently say that all other bids must be smaller than the bid v, the probability of which is just $F(v)^{n-1}$. In our case $F_Y(v) = F(v)^(2-1) = x^a$. The optimal strategy is then

$$\beta(v) = v - \frac{1}{v^a}\int_0^v x^a dx = v - \frac{1}{v^a}\left[\frac{x^{a+1}}{a+1}\right]_0^v = v - \frac{v^{a+1}}{(a+1)v^a} = v - \frac{v}{a+1}$$

Solution 36.
We can use the same approach as in the last exercise; although, due to the complexity of the normal distribution, it will be easier to use a computer. The optimal bid is approximately 9.7124.

Solution 37.
One possible solution is drawn in Figure 6.

Solution 38.
Start by assuming that a_2 plays rationally and bids truthfully (if you drew branches for other possibilities at all), then for each of a_1's bidding strategies, find the possible outcomes and what their probabilities are.

The expected payoff of bidding 0 is 0.125, compared to a payoff of 0.1875 for bidding 1, when a_1 thinks his value is 1. Bidding truthfully (with respect to observed values) is still the optimal strategy.

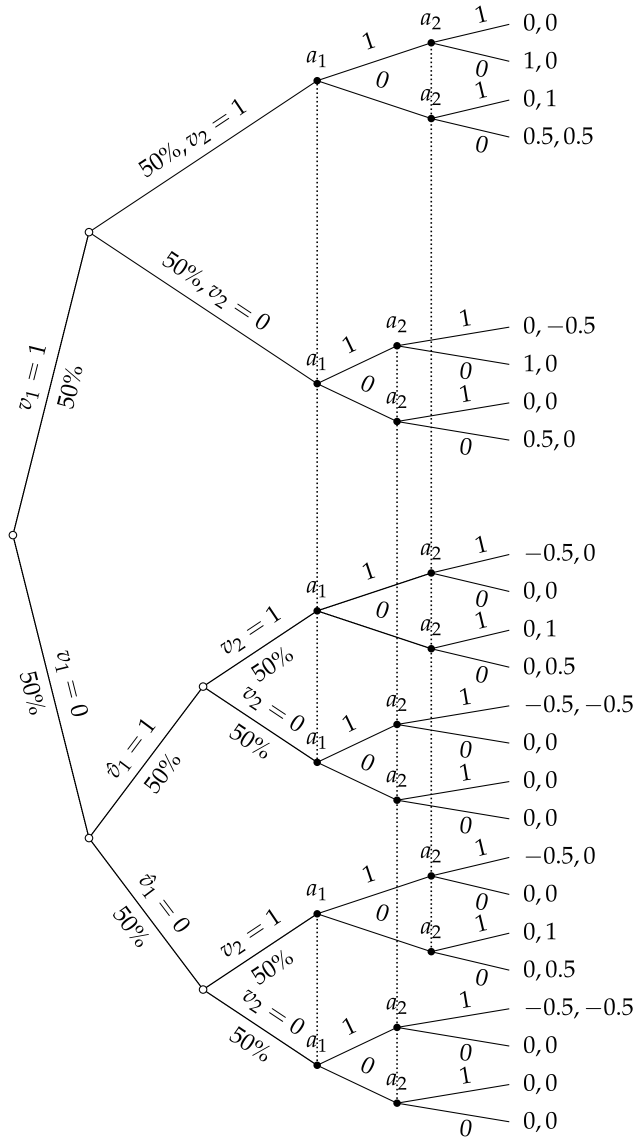


Figure 6: The game tree with expected payoffs instead of an explicit tie-breaking round. There are multiple correct solutions.

10 OPTIMAL AUCTIONS AND VCG

Exercise 39.
Suppose you had a single good to sell to a single agent with a private valuation which is drawn from an exponential distribution.

$$F(x) = 1 - e^{-x}$$
$$f(x) = e^{-x}$$

The agent knows her value and you must post a price that she can either take or leave. This is a monopoly question that is equivalent to an optimal auction design (setting a reserve price) with a single bidder. What price should you set to maximize your profit?

Hint: calculate the agent's virtual valuation.

Exercise 40.
Consider an optimal auction with two bidders A and B who have independent private values, but where A's valuation is drawn from uniform $[0, 1]$ while B's valuation is drawn from uniform $[0, 3]$. If A's valuation realized valuation is 0.8 and B's realized valuation is 1.6, who wins and what does she pay?

Hint: Use Meyerson's optimal auction and virtual valuations.

Exercise 41.
Suppose there are two items A and B to be sold in simultaneous English auction in which you wish to participate. You have no use for the items separately, but value winning both at \$10. Describe the possible issues with such an auction format.

Exercise 42.
Group of homeowners are deciding whether to rebuild their access road. To make it fair, they will use VGC. According to the collected utilities, two homeowners agree with the project and one does not.

Figure out the outcome and the payments under VCG rules.

	build	do not	p
a_1	20	0	
a_2	10	0	
a_3	0	25	

Exercise 43.
What happens if two of the homeowners cooperate and try to cheat the system by increasing their bids?

	build	do not	p
a_1	25	0	
a_2	15	0	
a_3	0	25	

Exercise 44.
What happens if an agent manages to submit multiple bids?

	build	do not	p
a_1	2	0	
a_1	2	0	
a_2	0	1	

Exercise 45.
There are two items A and B to be sold using a VCG auction. The private values of the participants a_1 and a_2 are listed in the table below. What should be the allocation of items and the payments of the bidders?

	v_i			
	A	B	AB	p
a_1	10	5	15	
a_2	1	6	12	

Exercise 46.
What should the bidders pay in the following VCG auction?

	v_i			
	A	B	AB	p
a_1	10	5	15	
a_2	1	10	12	

SOLUTIONS

Solution 39.
We will solve this using virtual valuations and then derive the same answer using calculus.

VIRTUAL VALUATIONS: The formula for virtual valuations is $\varphi(v) = v - \frac{1-F(V)}{f(v)}$. For our exponential distribution we get

$$\varphi(v) = v - \frac{1-(1-e^{-v})}{e^{-v}} = v - 1.$$

Revenue is maximized at $\varphi(R) = 0$.

$$\begin{aligned} \varphi(R) &= 0 \\ R - 1 &= 0 \\ R &= 1 \end{aligned}$$

CALCULUS: start by writing the formula for the expected revenue of the seller using a reserve price of R.

$$E_v[u_R(v)] = \int_0^\infty u_R(v) f(v) dv$$

As there is only a single bidder the utility only depends on whether he wins or not.

$$u_R(v) = \begin{cases} R & v \geq R \\ 0 & v < R \end{cases}$$

Adjust the interval of integration, and solve by substitution.

$$\begin{aligned} E_v[u_R(v)] &= \int_R^\infty R f(v) dv \\ &= R \int_R^\infty e^{-v} dv \\ &= -R \left[e^u\right]_{-R}^{-\infty} \\ &= Re^{-R} \end{aligned}$$

Maximal expected revenue is located at an inflection point.

$$\begin{aligned} (Re^{-R})' &= 0 \\ e^{-R} - Re^{-R} &= 0 \\ e^{-R}(1-R) &= 0 \\ R &= 1 \end{aligned}$$

Using both methods, we found that the reserve price of one maximizes the seller's revenue.

Solution 40.

The winner is the agent that with the highest virtual valuation, as long as his value is above his reserve price. The winner's payment is the lowest amount he could have declared and still won.

Using the distributions of private values

$$f_A(x) = 1 \qquad f_B(x) = 1/3$$
$$F_A(X) = x \qquad F_B(x) = x/3$$

we can compute the virtual valuations

$$\varphi_A(v) = v - (1 - v) = 2v - 1$$
$$\varphi_B(v) = v - \frac{1 - v/3}{1/3} = 2v - 3$$

Plugging in the actual private values we get $\varphi_A(0.8) = 0.6$ and $\varphi_B(1.6) = 0.2$.

Agent A's payment is the lowest value y that is both higher than A's reserve price

$$\varphi_A(y) \geq 0$$
$$2y - 1 \geq 0$$
$$y \geq 1/2$$

and is still higher than the virtual valuations of other players $\varphi_A(y) \geq \varphi_B(1.6)$.

$$\varphi_A(y) \geq \varphi_B(1.6)$$
$$2y - 1 \geq 1/5$$
$$y \geq 6/10$$

Agent A's bid is higher than his reserve price, so he wins the auction and pays 0.6 for the item.

Solution 41.

Participating in such auctions may result in negative values for the bidders. Outcomes of such auctions may be sub-optimal from the point of both efficiency and revenue at the same time.

Solution 42.

	build	do not	p
a_1	20	0	15
a_2	10	0	5
a_3	0	25	0

Solution 43.

	build	do not	p
a_1	25	0	10
a_2	15	0	0
a_3	0	25	0

Solution 44.

	build	do not	p
a_1	2	0	0
a_2	2	0	0
a_3	0	1	0

Solution 45.

	v_i			
	A	B	AB	p
a_1	10	5	15	6
a_2	1	6	12	5

Solution 46.

	v_i			
	A	B	AB	p
a_1	10	5	15	6
a_2	1	10	12	5

Part III.
Coalitional Games

11 THE CORE

Notation. We will often adopt the standard practice of omitting curly brackets and commas in applications of coalitional function if the clarity is not impaired. For example, we may write $v(237)$ in place of $v(\{2,3,7\})$, when this improves readability. We also use the following notation for the sum of coordinates of vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\mathbf{x}(A) := \sum_{i \in A} x_i, \qquad A \subseteq \{1, \dots, n\}. \tag{30}$$

Exercise 47.
Let v be a superadditive game: $v(A \cup B) \geq v(A) + v(B)$ for every disjoint pair of coalitions A and B. Show that this implies the following condition:

$$v(N) \geq v(A_1) + \cdots + v(A_k),$$

for every coalitional structure $\{A_1, \dots, A_k\}$.

Exercise 48.
Five friends want to sell gin & tonic cocktails at a party. Three of them have a bottle of gin apiece and each of the other two friends has five bottles of tonic. A price of cocktails made from one gin bottle and five tonic bottles is 2000 CZK. Model this situation as a coalitional game, decide if it is superadditive, and compute its core.

Exercise 49.
A coalitional game $v\colon \mathcal{P}(N) \to \mathbb{R}$ over the player set $N = \{1,2,3\}$ is

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ 1 & A = \{1\}, \{2\}, \\ 2 & A = \{3\}, \\ 4 & |A| = 2, \\ 5 & A = N. \end{cases}$$

Is v superadditive? What is the core of v?

Exercise 50.
Let $N = \{1,2,3\}$. Describe the core of the game

$$v(A) = \begin{cases} 3 & \text{if } A = N, 13 \\ 1 & \text{if } A = 12, 23 \\ 0 & \text{if } A = 1, 2, 3, \emptyset \end{cases}$$

and decide if v is superadditive.

Exercise 51.
Describe the core of a game v over the player set $N = \{1,2,3\}$, where

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ |A| - 1 & A \neq \emptyset. \end{cases}$$

Exercise 52.
Alice has a left glove. Bob and Cyril have one right glove each. The number of pairs of gloves collected by a coalition is its worth. Define the resulting coalitional game, decide if it is superadditive or supermodular, and describe its core.

Exercise 53.
A *simple game* is a coalitional game $v\colon \mathcal{P}(N) \to \{0,1\}$ that is monotone and $v(N) = 1$. We call a player $i \in N$ in a simple game v a *veto player*, if $v(A \setminus i) = 0$ holds for each coalition $A \subseteq N$. Show that the following is true for any simple game v:

(a) Player i is veto in game v if, and only if, $v(N \setminus i) = 0$.

(b) There is a veto player in game v if, and only if, $\mathcal{C}(v) \neq \emptyset$.

(c) Let $W \neq \emptyset$ be the set of veto players. Then the core of v is

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x}(W) = 1 \text{ and } x_i = 0 \text{ for all } i \notin W\}.$$

Exercise 54.
Argue that the following game v is superadditive and its core is empty:

$$v(A) = \begin{cases} 0 & |A| \leq 1 \\ 1 & |A| \geq 2 \end{cases} \qquad A \subseteq \{1,2,3\}.$$

Exercise 55.
Minimum spanning tree game. The costs of connecting the cities denoted as $1, 2$, and 3 to the supplier of energy 0 are depicted in Figure 7. The minimum cost spanning tree game is defined as a coalitional game c over player set $N = \{1,2,3\}$, in which the worth of each coalition $A \subseteq N$ is the cost $c(A)$ associated with the minimum spanning tree over vertices $A \cup \{0\}$. Describe the cost game c and show that the core of c is nonempty.

★ **Exercise 56.**
Coalitional game v is *supermodular* if, for all $A, B \subseteq N$ the inequality

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$$

holds. Prove that the following assertions are equivalent.

(a) v is supermodular.

(b) For all $A, B \subseteq N$ with $A \subseteq B$ and every $C \subseteq N \setminus B$,

$$v(A \cup C) - v(A) \leq v(B \cup C) - v(B).$$

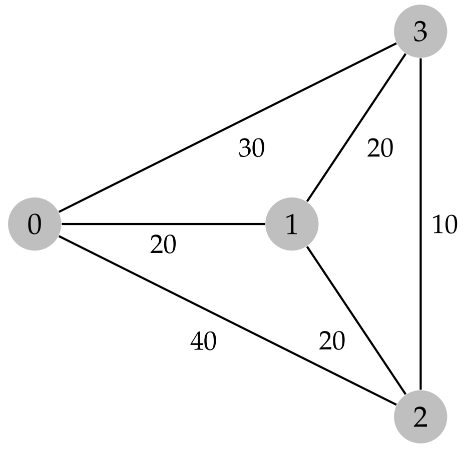


Figure 7: Graph from Exercise 55

(c) For all $A, B \subseteq N$ with $A \subseteq B$ and every $i \in N \setminus B$,

$$v(A \cup i) - v(A) \leq v(B \cup i) - v(B).$$

(d) $\mathbf{x}^{\pi} \in \mathcal{C}(v)$, for every $\pi \in \Pi$, where $x_i^{\pi} := v(A_i^{\pi} \cup i) - v(A_i^{\pi})$ and $A_i^{\pi} := \{j \in N \mid \pi(j) < \pi(i)\}$.

(e) The core of v is the convex hull of the marginal vectors in v, that is,

$$\mathcal{C}(v) = \text{conv}\{\mathbf{x}^{\pi} \mid \pi \in \Pi\}.$$

(f) Vertices of $\mathcal{C}(v)$ are precisely marginal vectors $\mathbf{x}^{\pi}$.

Exercise 57.
One might think that non-superadditive games have empty cores. This is not true, in general. For example, take the game v on the n-player set $N = \{1, \dots, n\}$, where $n > 2$:

$$v(A) = \begin{cases} 1 & |A| < n, \\ n & |A| = n. \end{cases}$$

Show that v is not superadditive and $\mathcal{C}(v) \neq \emptyset$.

SOLUTIONS

Solution 47.
Since coalitions $A_1, \dots, A_k$ are pairwise disjoint, we can apply superadditivity to A_1 and A_2 to obtain

$$v(A_1) + v(A_2) + v(A_3) + \cdots + v(A_k) \le v(A_1 \cup A_2) + v(A_3) + \dots v(A_k).$$

Proceeding analogously in the next $k-2$ steps, we get

$$v(A_1) + \cdots + v(A_k) \le v(A_1 \cup \cdots \cup A_k) = v(N),$$

where the last equality follows from $N = A_1 \cup \cdots \cup A_k$.

Solution 48.

$$G = \{1,2,3\}, \ \ T = \{4,5\}, \ \ N = G \cup T$$
$$v(A) = 2000 \cdot \min\{|A \cap G|, |A \cap T|\}, \qquad A \subseteq N$$

It is easy to prove that v is superadditive. Let $A \cap B = \emptyset$. We can neglect the multiplicative constant 2000. Then we want to show that

$$v(A \cup B) \ge v(A) + v(B).$$

Using the definition of v, we get

$$\min\{|(A \cup B) \cap G|, |(A \cup B) \cap T|\} \ge \min\{|A \cap G|, |A \cap T|\} + \min\{|B \cap G|, |B \cap T|\},$$

which is the same as

$$\min\{\underbrace{|A \cap G| + |B \cap G|}_{a_1+b_1}, \underbrace{|A \cap T| + |B \cap T|}_{a_2+b_2}\} \ge$$
$$\min\{\underbrace{|A \cap G|}_{a_1}, \underbrace{|A \cap T|}_{a_2}\} + \min\{\underbrace{|B \cap G|}_{b_1}, \underbrace{|B \cap T|}_{b_2}\}.$$

This can be written as $a_i + b_i \ge a_j + b_k$, where i, j, k are the indices of the corresponding minimizers. The definition of minimum yields $a_i \ge a_j$ and $b_i \ge b_k$, from which the superadditive inequality follows.

The core contains a unique allocation,

$$\mathcal{C}(v) = \{(0, 0, 0, 2\,000, 2\,000)\}.$$

We can again neglect the multiplicative constant 2000. From

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_4 + x_5 &\ge 2 \\ x_1 + x_3 + x_4 + x_5 &\ge 2 \\ x_2 + x_3 + x_4 + x_5 &\ge 2 \end{aligned}$$

derive $x_4 + x_5 = 2 - x_1 - x_2 - x_3$, which implies together with nonnegativity constraints that $x_1 = x_2 = x_3 = 0$, and also that $x_4 \ge 1$ and $x_5 \ge 1$. Hence, necessarily $x_4 = x_5 = 1$.

Solution 49.
Game v is superadditive, if the inequality $v(A \cup B) \geq v(A) + v(B)$ holds for all $A, B \subseteq N$, $A \cap B = \emptyset$. Since $v(N) < v(12) + v(3)$, game v is not superadditive. It is easy to see that $\mathcal{C}(v)$ is empty. Indeed, every allocation $\mathbf{x} \in \mathcal{C}(v)$ must satisfy the conditions $x_1 + x_2 + x_3 = 5$, $x_1 + x_2 \geq 4$, and $x_3 \geq 2$. But adding the last two inequalities together yields $5 = x_1 + x_2 + x_3 \geq 6$, a contradiction.

Solution 50.
By the definition,

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 = 3, \quad x_1 + x_2 \geq 1, \quad x_1 + x_3 \geq 3, \quad x_2 + x_3 \geq 1\}.$$

We will show that $\mathcal{C}(v) = \operatorname{conv}\{(1,0,2),(2,0,1)\}$. Since the coalition $\{1,3\}$ accepts only total payoffs ≥ 3 and since $x_1 + x_2 + x_3 = 3$, it is reasonable to think that player 2's payoff should be 0. Indeed, from $x_1 + x_3 \geq 3$ and from $x_1 + x_3 = 3 - x_2$, we get $3 - x_2 \geq 3$, which gives $0 \geq x_2$, hence $x_2 = 0$. Then

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 1, x_2 = 0, x_3 \geq 1, x_1 + x_3 = 3\},$$

which is the line segment with vertices $(1,0,2),(2,0,1)$. It can be checked that this is a superadditive game. Observe that Player 2 cannot block the formation of grand coalition, since he accepts payoff 0. However, if Player 2's worth is changed to $v(\{2\}) = 1$, then $\mathcal{C}(v) = \emptyset$.

Solution 51.
Using the identity $|A \cup B| = |A| + |B| - |A \cap B|$ we can easily verify that v is supermodular, that is, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. This implies that its core $\mathcal{C}(v)$ coincides with the convex hull of its marginal vectors $\mathbf{x}^\pi$, where π is a permutation of N. For example, permutation $\pi(1) = 3, \pi(2) = 1$, $\pi(3) = 2$ determine the order of players

$$2\,3\,1$$

together with the marginal vector $\mathbf{x}^\pi$ whose coordinates are

$$\begin{aligned} x_2^\pi &= v(2) - v(\emptyset) = 0, \\ x_3^\pi &= v(23) - v(2) = 1, \\ x_1^\pi &= v(123) - v(23) = 1. \end{aligned}$$

The remaining marginal vectors are computed analogously. This shows that the core is a triangle with vertices $(0,1,1),(1,0,1)$, and $(1,1,0)$, which is located in the plane given by the equation $x_1 + x_2 + x_3 = 2$.

Solution 52.
The glove game over the player set $N = \{1,2,3\}$ is

$$v(A) = \begin{cases} 1 & A = \{1,2\},\{1,3\},N, \\ 0 & \text{otherwise.} \end{cases}$$

The glove game v is monotone and superadditive, but not supermodular. The core of v is

$$\mathcal{C}(v) = \{(1,0,0)\}.$$

Indeed, we can argue as follows. The inequalities $x_1 + x_2 \geq 1$ and $x_1 + x_3 \geq 1$ combined together give $2x_1 + x_2 + x_3 \geq 2$. Since $x_2 + x_3 = 1 - x_1$, the last inequality says that $x_1 \geq 1$. Since x_2 and x_3 must be nonnegative, we get $x_1 = 1$. Thus, the only core allocation is $(1, 0, 0)$.

Solution 53.
(a) The first implication is trivial. Assume that $v(N \setminus i) = 0$. Then monotonicity gives $v(A \setminus i) = 0$ for every $A \subseteq N$.

(b) Let $k \in N$ be a veto player in game v. We define an allocation vector $\mathbf{x} \in \mathbb{R}^n$ as follows:

$$x_i = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

Since v is non-constant, $v(N) = 1 = \sum_{i \in N} x_i = \mathbf{x}(N)$. Choose $A \subseteq N$. If $k \in A$, then $\mathbf{x}(A) = 1 \geq v(A)$. If $k \notin A$, then $\mathbf{x}(A) = 0 = v(A)$, since k is veto. We have shown that $\mathbf{x} \in \mathcal{C}(v)$.

Conversely, assume that v has no veto players. We want to conclude that v has empty core. By way of contradiction, let $\mathbf{x} \in \mathcal{C}(v)$. Then the condition $\mathbf{x}(N) = 1$ implies that there exists $i \in N$ such that $x_i > 0$, hence $\mathbf{x}(N \setminus i) = 1 - x_i < 1$. Since i is not veto, $v(N \setminus i) = 1 > \mathbf{x}(N \setminus i)$, which contradicts our assumption $\mathbf{x} \in \mathcal{C}(v)$.

(c) Let $\mathbf{x} \in \mathbb{R}^n_+$ be such that $\mathbf{x}(W) = 1$ and $x_i = 0$ for all $i \notin W$. We want to show that $\mathbf{x} \in \mathcal{C}(v)$. Clearly, $\mathbf{x}(N) = \mathbf{x}(W) = 1$. If $A \subseteq N$ is loosing, that is, $v(A) = 0$, then $\mathbf{x}(A) \geq 0$. Let $v(A) = 1$. This implies that $A \supseteq W$, which gives

$$\mathbf{x}(A) \geq \mathbf{x}(W) = 1 = v(A).$$

Therefore, $\mathbf{x} \in \mathcal{C}(v)$.

Conversely, let $\mathbf{x} \in \mathcal{C}(v)$. Then $x_i \geq 0$ for all $i \in N$ and $\mathbf{x}(N) = 1$. We need to show that $x_i = 0$ for all $i \in N \setminus W$. Pick $i \in N \setminus W$. Player i is not veto and, hence,

$$1 = \mathbf{x}(N) \geq \mathbf{x}(N \setminus i) \geq v(N \setminus i) = 1,$$

which implies $\mathbf{x}(N) = \mathbf{x}(N \setminus i)$, so that $x_i = 0$.

Solution 54.
The game v is in fact a simple majority game. It is easy to see that v is superadditive. We show that $\mathcal{C}(v) = \emptyset$. By contradiction, assume that $\mathbf{x} \in \mathcal{C}(v)$. Then $x_1 + x_2 + x_3 = 1$ and $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$. Combining the last three inequalities together gives

$$2 \cdot \underbrace{(x_1 + x_2 + x_3)}_{1} \geq 3,$$

a contradiction. Hence, $\mathcal{C}(v) = \emptyset$. The same conclusion follows immediately from Exercise 53, since there are no veto players in the game.

Solution 55.
We can easily compute

$$c(A) = \begin{cases} 0 & A = \emptyset, \\ 20 & A = 1, \\ 30 & A = 3, \\ 50 & A = 123, \\ 40 & \text{otherwise.} \end{cases}$$

The core of c is the set of allocations $\mathbf{x} \in \mathbb{R}^3_+$ such that $x_1 + x_2 + x_3 = 50$, $x_1 \leq 20$, $x_2 \leq 40$, $x_3 \leq 30$, and

$$x_1 + x_2 \leq 40, \quad x_1 + x_3 \leq 40, \quad x_2 + x_3 \leq 40.$$

We can easily see that the cost allocation vector $\mathbf{x} = (20, 20, 10)$ is in the core of c, since it measures the costs of individual players with respect to the minimum spanning tree over the full player set N.

★ **Solution 56.**
First, we prove (a) ⇒ (b). Let (a) be true. Choose $A, B \subseteq N$, where $A \subseteq B$ and $C \subseteq N \setminus B$. Then

$$v(A \cup C) + v(B) \leq v(\underbrace{(A \cup C) \cup B}_{B \cup C}) + v(\underbrace{(A \cup C) \cap B}_{A}).$$

Implication (b) ⇒ (c) is trivial. We show that (c) ⇒ (a). Select $A, B \subseteq N$. If $A \subseteq B$, then (a) is true. Therefore, assume that $A \not\subseteq B$ and let $P := A \cap B$, $R := A \setminus B$. The assumption implies $R \neq \emptyset$ and we may write $R = \{i_1, \ldots, i_k\}$, where $k = |R|$. Since $B \supseteq P$ and for any $\ell = 1, \ldots, k-1$,

$$B \cup \{i_1 \ldots, i_\ell\} \supseteq P \cup \{i_1 \ldots, i_\ell\},$$

item (c) gives the following inequalities:

$$\begin{gathered} v(B \cup i_1) - v(B) \geq v(P \cup i_1) - v(P) \\ v(B \cup i_1 \ldots i_{\ell+1}) - v(B \cup i_1 \ldots i_\ell) \geq v(P \cup i_1 \ldots i_{\ell+1}) - v(P \cup i_1 \ldots i_\ell) \end{gathered}$$

Summing all the inequalities, we get

$$v(\underbrace{B \cup R}_{A \cup B}) - v(B) \geq v(\underbrace{P \cup R}_{A}) - v(P),$$

which proves (a).

Further, we need to prove (a) ⇔ (d). We can use already proved equivalences (a) ⇔ (b) ⇔ (c). We will show implication (c) ⇒ (d). Let (c) holds. We want to show that $\mathbf{x}^\pi \in \mathcal{C}(v)$ for any permutation $\pi \in \Pi$. We obtain

$$\mathbf{x}^\pi(N) = \sum_{i \in N} x_i^\pi = \sum_{i \in N} (v(A_i^\pi \cup i) - v(A_i^\pi)) = v(N) - v(\emptyset) = v(N).$$

We show that $\mathbf{x}^\pi$ is coalitionally rational, that is, $\mathbf{x}^\pi(A) \geq v(A)$, for every nonempty coalition $A \subseteq N$. Let $a := |A|$. The players in A can be enumerated as follows: $A = \{i_1, \ldots, i_a\}$, where $\pi(i_1) < \cdots < \pi(i_a)$. Write $B_k := \{i_1, \ldots, i_k\}$ for each $k = 1, \ldots, a$. Then

$$B_k = A \cap \left(A_{i_k}^\pi \cup i_k\right).$$

Define $B_0 := \emptyset$. By assumption (c), this inequality is satisfied for all $k = 1, \ldots, a$:

$$v(B_k) - v(B_{k-1}) \leq v\left(A^{\pi}_{i_k} \cup i_k\right) - v\left(A^{\pi}_{i_k}\right) = x^{\pi}_{i_k}.$$

This implies

$$v(A) = v(B_a) = \sum_{k=1}^{a} (v(B_k) - v(B_{k-1})) \leq \sum_{k=1}^{a} x^{\pi}_{i_k} = \mathbf{x}^{\pi}(A),$$

which finishes the proof of (d).

In the next step we check that implication (d) $\Rightarrow$ (a) is true. Let v be a coalitional game fulfilling $\mathbf{x}^{\pi} \in \mathcal{C}(v)$ for all $\pi \in \Pi$. Supermodular inequality (a) holds trivially, when at least one of the sets $A, B \subseteq N$ is empty. Therefore, assume that $A, B \neq \emptyset$. Put $r := |A \cap B|$, $q := |A \cup B|$, $t := |B|$, and write

$$\begin{aligned} A \cap B &= \{i_1, \ldots, i_r\}, \\ B \setminus A &= \{i_{r+1}, \ldots, i_t\}, \\ A \setminus B &= \{i_{t+1}, \ldots, i_q\}, \\ N \setminus (A \cup B) &= \{i_{q+1}, \ldots, i_n\}. \end{aligned}$$

Define permutation π by $\pi(i_j) := j$, for all $j \in N$. It follows from (d) that

$$v(A) \leq \mathbf{x}^{\pi}(A) = \sum_{\substack{j \in N \\ i_j \in A}} x^{\pi}_{i_j} = \sum_{\substack{j \in N \\ i_j \in A}} \left(v(A^{\pi}_{i_j} \cup i_j) - v(A^{\pi}_{i_j})\right).$$

The last sum can be split into two sums,

$$\begin{aligned} &\sum_{j=1}^{r} \left(v(i_1 \ldots i_j) - v(i_1 \ldots i_{j-1})\right) \\ &+ \sum_{j=t+1}^{q} \left(v(B \cup i_{t+1} \ldots i_j) - v(B \cup i_{t+1} \ldots i_{j-1})\right) = \\ &v(A \cap B) - v(\emptyset) + v(A \cup B) - v(B), \end{aligned}$$

which shows (a).

Clearly, item (e) implies (d) immediately. We show that implication (d) $\Rightarrow$ (e) holds. From (d) we obtain the inclusion $\mathcal{C}(v) \supseteq \operatorname{conv}\{\mathbf{x}^{\pi} \mid \pi \in \Pi\}$, by convexity of the core. The converse inclusion $\mathcal{C}(v) \subseteq \operatorname{conv}\{\mathbf{x}^{\pi} \mid \pi \in \Pi\}$ requires an involved proof, for which we refer the reader to [2, Theorem 5.18].

The proof is finished after we show the equivalence (e) $\Leftrightarrow$ (f). The implication (f) $\Rightarrow$ (e) is a direct consequence of the characterization of convex polytope $\mathcal{C}(v)$ as the convex hull of its vertices. Suppose that (e) holds. Then every vertex of $\mathcal{C}(v)$ is necessarily a marginal vector $\mathbf{x}^{\pi}$, for some $\pi \in \Pi$. It remains to prove that every marginal vector is a vertex of $\mathcal{C}(v)$. Let $\pi \in \Pi$. It follows from the definition of marginal vectors that

$$\mathbf{x}^{\pi}(A^{\pi}_i) = v(A^{\pi}_i), \qquad i = 1, \ldots, n.$$

This is a linear system whose matrix is triangular with nonzero elements on the diagonal. Therefore, the matrix is nonsingular and $\mathbf{x}^{\pi}$ is a vertex of $\mathcal{C}(v)$.

Solution 57.
The game v is not superadditive as $v(12) < v(1) + v(2)$. It is easy to see that $(1, \ldots, 1)$ is the only allocation in the core $\mathcal{C}(v)$.

12 THE SHAPLEY VALUE

Exercise 58.
Prove that the Shapley value $\boldsymbol{\varphi}^S(v)$ of any supermodular game v is the center of gravity of the core of v.

Exercise 59.
Let Γ be the set of all coalitional games over the player set $N = \{1, \ldots, n\}$. Consider a solution mapping $\boldsymbol{\psi}\colon \Gamma \to \mathbb{R}^n$ defined by

$$\psi_i(v) := v(1 \ldots i) - v(1 \ldots i-1), \qquad i \in N.$$

Show that $\boldsymbol{\psi}$ is efficient, additive, and it has the null player property, but fails symmetry.

★ **Exercise 60.**
Prove that the Shapley value $\boldsymbol{\varphi}^S$ is efficient, additive, it has the null player property, and it is symmetric.

Exercise 61.
Consider the coalitional game v on the player set $N = \{1, 2, 3\}$, where

$$v(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, \\ 8 & A = 12, \\ 5 & A = 13, 23, \\ 10 & A = N. \end{cases}$$

Show that the game is superadditive and not supermodular. Further, prove that its core is nonempty and the Shapley value is not a core allocation.

Exercise 62.
Let $\mathcal{G} = (N, E)$ be an undirected weighted graph without self-loops, where the vertex set $N = \{1, \ldots, n\}$ is identified with the set of players, elements of E are $\{i, j\}$ with $i, j \in N$ and $i \neq j$, and the weights are $w_{i,j} \in \mathbb{R}$ for $\{i, j\} \in E$. The *induced subgraph game* is the coalitional game

$$v(A) = \sum_{\substack{i,j \in A \\ \{i,j\} \in E}} w_{i,j}.$$

Show that the Shapley value of a player is the half of total weight of the edges to adjacent players.

Exercise 63.
Decide if the assertions below are true or false.

1. If the core is nonempty, it contains the Shapley value.

2. If marginal contributions of players i and j to every coalition are equal, then their Shapley values coincide.

3. The core of every monotone coalitional game is nonempty.

4. The Shapley value $\boldsymbol{\varphi}^S(v) = (\varphi_1^S(v), \ldots, \varphi_n^S(v))$ of every n-player coalitional game v is uniquely determined by the Shapley values of the first $n-1$ players $(\varphi_1^S(v), \ldots, \varphi_{n-1}^S(v))$.

5. The Shapley value is *individually rational*, that is, $\varphi_i^S(v) \geq v(i)$ for every game v and each player i.

SOLUTIONS

Solution 58.
Let $N = \{1, \ldots, n\}$ and v be a supermodular game over N. By supermodularity and Exercise 56, the core of v is the convex hull of its marginal vectors,

$$\mathcal{C}(v) = \text{conv}\,\{\mathbf{x}^\pi \mid \pi \in \Pi\},$$

where Π is the set of all permutations of the player set N. Therefore, it suffices to show that $\boldsymbol{\varphi}^S(v)$ can be written as $\boldsymbol{\varphi}^S(v) = \sum_{\pi\in\Pi} a_\pi \cdot \mathbf{x}^\pi$, where $a_\pi = \frac{1}{n!}$. But this is immediate, since an equivalent formula for the Shapley value is

$$\boldsymbol{\varphi}^S(v) = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot \mathbf{x}^\pi.$$

Solution 59.
First, we check *efficiency*:

$$\sum_{i\in N} \psi_i(v) = \sum_{i\in N} (v(1\ldots i) - v(1\ldots i-1)) = v(N) - v(\emptyset) = v(N).$$

Aditivity: For all $v, w \in \Gamma$ we get

$$\begin{aligned} \psi_i(v+w) &= (v+w)(1\ldots i) - (v+w)(1\ldots i-1) \\ &= v(1\ldots i) - v(1\ldots i-1) \\ &+ w(1\ldots i) - w(1\ldots i-1) \\ &= \psi_i(v) + \psi_i(w). \end{aligned}$$

Null player property: Let $i \in N$ be the null player. This means that

$$v(A \cup i) = v(A)$$

for each coalition $A \subseteq N$. Then, putting $A := \{1, \ldots, i-1\}$ yields $\psi_i(v) = 0$.

We show that $\boldsymbol{\psi}$ fails symmetry. Letting $N = \{1,2,3\}$ we define a game

$$v(A) = \begin{cases} 1 & A = \{2,3\}, N \\ 0 & \text{otherwise,} \end{cases} \qquad A \subseteq N.$$

Then $\boldsymbol{\psi}(v) = (0,0,1)$. However, players 2 and 3 have the same contributions to one-player coalition $\{1\}$, that is, $v(12) = v(13)$. This implies that $\boldsymbol{\psi}$ fails symmetry.

★ **Solution 60.**
The Shapley value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^\pi.$$

Efficiency:

$$\sum_{i\in N} \varphi_i^S(v) = \sum_{i\in N} x_i^\pi = \sum_{i\in N} \left(v(A_i^\pi \cup i) - v(A_i^\pi)\right) = v(N).$$

Aditivity: Let u, v be coalitional games. By $\mathbf{x}^{u,\pi}$ and $\mathbf{x}^{v,\pi}$ we denote the corresponding marginal vectors in $\mathbb{R}^n$. It can easily be checked that $\mathbf{x}^{u+v,\pi} = \mathbf{x}^{u,\pi} + \mathbf{x}^{v,\pi}$. Then, for each player $i \in N$,

$$\begin{aligned}\varphi_i^S(u+v) &= \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{u+v,\pi} = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot (x_i^{u,\pi} + x_i^{v,\pi}) = \\ &\sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{u,\pi} + \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{v,\pi} = \varphi_i^S(u) + \varphi_i^S(v).\end{aligned}$$

Null player property: If i is a null player, then necessarily $x_i^\pi = 0$ for any $\pi \in \Pi$. Then $\varphi_i^S(v) = 0$.

Symmetry: Let $i, j \in N$ be players such that

$$v(A \cup i) = v(A \cup j), \qquad \text{for each coalition } A \subseteq N \setminus ij. \tag{31}$$

We want to show that $\varphi_i(v) = \varphi_j(v)$. We will show that for symmetric players i and j, we can transform the formula for $\varphi_i(v) = \frac{1}{n!} \sum_{\pi\in\Pi} x_i^\pi$ to $\varphi_j(v)$ just by reordering the summation terms. Define a bijective (permutation) function ξ, which swaps the order of elements i and j.

$$\begin{aligned} \pi &= [\ \overbrace{\ldots}^{A_i^\pi}\ i\ \ldots\ j\ \ldots\ ] \\ \xi(\pi) &= [\ \ldots\ j\ \ldots\ i\ \ldots\ ] \end{aligned} \qquad \text{or} \qquad \begin{aligned} \pi &= [\ \overbrace{\ldots\ j\ \ldots}^{A_i^\pi}\ i\ \ldots\ ] \\ \xi(\pi) &= [\ \ldots\ i\ \ldots\ j\ \ldots\ ] \end{aligned}$$

Let $\pi \in \Pi$ and $\tau = \xi(\pi)$, then:

1. If i precedes j in π, then $x_i^\pi = x_j^\tau$ since $A_i^\pi = A_j^\tau$.
2. If j precedes i, then $A_i^\pi = A_j^\tau \setminus i \cup j$. So we have

   $$v(A_i^\pi \cup i) = v(A_j^\tau \setminus i \cup j \cup i) = v(A_j^\tau \cup j).$$

   If we replace i in A_i^π by j we get A_j^τ. Thus, by the symmetry of i and j we also have $v(A_i^\pi) = v(A_j^\tau)$.

Putting it together, for every π we get

$$x_i^\pi = v(A_i^\pi \cup i) - v(A_i^\pi) = v(A_j^\tau \cup j) - v(A_j^\tau) = x_j^\tau.$$

Solution 61.

The game is obviously superadditive since $v(ij) \geq v(i) + v(j)$ for all $i \neq j$ and $v(123) \geq v(ij) + v(k)$ for all pairwise different i, j, k. However, it is not supermodular:

$$v(12) + v(23) > v(123) + v(2).$$

A core allocation is, for example, $(4, 4, 2)$. The Shapley value is $\frac{1}{6}(23, 23, 14)$, but it doesn't belong to the core:

$$v(12) = 8 > \tfrac{1}{6}(23 + 23) = 7\tfrac{2}{3}.$$

Solution 62.
For each edge $\{i,j\} \in E$ we define a coalitional game $v_{i,j}(A) = w_{i,j}$ whenever $i,j \in A$ and $v_{i,j}(A) = 0$ otherwise. Then

$$\sum_{\{i,j\}\in E} v_{i,j}(A) = \sum_{\substack{i,j\in A \\ \{i,j\}\in E}} w_{i,j} = v(A).$$

The Shapley value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{\{i,j\}\in E} \varphi_i^S(v_{i,j}) = \frac{1}{2} \sum_{\{i,j\}\in E} w_{i,j},$$

where the last equality follows from efficiency and symmetry of the Shapley value in game $v_{i,j}$.

Solution 63.

1. False. For example, the Shapley value of the glove game (Exercise 52) is not an element of the core.

2. True. This is exactly the symmetry of Shapley value. Or, it follows immediately from the formula for Shapley value.

3. False. For example, take a two-player game

$$v(1) = v(2) = v(12) = 1.$$

4. True. By efficiency of the Shapley value,

$$\varphi_n^S(v) = v(N) - \sum_{i=1}^{n-1} \varphi_i^S(v).$$

5. False. Consider a 2-player game v that is not superadditive. Such a game satisfies the inequality $v(1,2) < v(1) + v(2)$, which implies $\varphi_1^S(v) < v(1)$.

13 WEIGHTED VOTING GAMES

Exercise 64.
A company has 3 shareholders whose shares are distributed in the following way. The first has 50 % shares and the remaining two have 25 % shares each. The three shareholders vote by using a weighted majority of votes. Define the resulting coalitional game. Compute the Shapley-Shubik index using the random order approach and then calculate the normalized Banzhaf index.

Exercise 65.
Weighted majority game. Four members of a committee decide on a proposition by weighted majority. The voting weights are $2, 1, 1, 1$, and the decision is approved when the weighted sum of votes is ≥ 3. Describe this situation as a coalitional game, calculate the Shapley-Shubik index, and discuss how the Shapley-Shubik indices correspond to the individual weights. What is the normalized Banzhaf index?

Exercise 66.
Let v be a weighted majority game with a weight vector $\mathbf{w}$ and quota q. Show that if $w_i \leq w_j$ for players $i, j \in N$, then $\varphi_i^S(v) \leq \varphi_j^S(v)$. Is the same conclusion true also for the Banzhaf index?

Exercise 67.
Find an example of a weighted voting game with 3 players in which players 1 and 2 have different weights, but their Shapley values are equal.

Exercise 68.
The paradox of new members. Find an example of a simple game with a null player whose Shapley value becomes positive after the player set is enlarged.

Exercise 69.
The paradox of size I. Consider an n-player simple game v with weights $(2, 1, \ldots, 1)$ and quota $q = n + 1$. What happens with the voting power of player 1 if this player splits into two players with equal weights 1?

Exercise 70.
The paradox of size II. Consider an n-player simple game v with weights $(2, \ldots, 2)$ and quota $q = 2n - 1$. Now, what happens with the voting power of player 1 if this player splits into two players with equal weights 1?

Exercise 71.
Vector weighted voting games. A company has CEO, CFO, CTO, and the separate Board containing director, the deputy of director, and 8 other members. A decision is approved if

1. CEO+CFO or CEO+CTO agrees with it and
2. the director and deputy vote for it together with at least 4 other Board members.

Model this situation as a vector weighted voting game. Which players are vetoers?

SOLUTIONS

Solution 64.
The player set is $N = \{1, 2, 3\}$. The coalitional game is

$$v(A) = \begin{cases} 1 & A = N, \{1,2\}, \{1,3\}, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

For the calculation of the Shapley-Shubik index of i we enumerate all the permutations such that i makes the preceding coalition winning:

$$1\boxed{2}3 \quad 1\boxed{3}2 \quad 2\boxed{1}3 \quad 23\boxed{1} \quad 3\boxed{1}2 \quad 32\boxed{1}$$

Then

$$\varphi_1^S(v) = \tfrac{4}{6}, \quad \varphi_2^S(v) = \varphi_3^S(v) = \tfrac{1}{6}.$$

In order to compute the normalized Banzhaf index $\beta(v)$, we enumerate the number of swings for each player:

$$\boxed{1}\boxed{2} \quad \boxed{1}\boxed{3} \quad \boxed{1}23$$

Hence, $s_1(v) = 3$, $s_2(v) = s_3(v) = 1$. These numbers are divided by the total number of swings:

$$\beta_1(v) = \frac{3}{5}, \quad \beta_1(v) = \beta_1(v) = \frac{1}{5}.$$

Solution 65.
This situation is captured by a weighted majority game v defined as follows. Let $w_1 = 2$, $w_2 = w_3 = w_4 = 1$, and $q = 3$. Define

$$v(A) = \begin{cases} 1 & \sum\limits_{i \in A} w_i \geq q, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for all } A \subseteq \{1, 2, 3, 4\}.$$

The Shapley-Shubik index of player i is

$$\varphi_i^S(v) = \sum_{\substack{A \subseteq N \setminus \{i\} \\ i \text{ pivotal to } A}} \frac{1}{n\binom{n-1}{|A|}}.$$

To compute $\boldsymbol{\varphi}^S(v) = \left(\varphi_1^S(v), \varphi_2^S(v), \varphi_3^S(v), \varphi_4^S(v)\right)$, realize that

$$\varphi_1^S(v) + \varphi_2^S(v) + \varphi_3^S(v) + \varphi_4^S(v) = 1$$

by efficiency. Moreover, players 2, 3, and 4 are symmetric in this game, since their individual contribution to each coalition is equal. By symmetry of the Shapley value, this means that

$$\varphi_2^S(v) = \varphi_3^S(v) = \varphi_4^S(v).$$

Hence, we need to compute only one Shapley-Shubik index, say $\varphi_2^S(v)$. Clearly, player 2 is pivotal to coalitions $\{1\}$ and $\{3, 4\}$. Then

$$\varphi_2^S(v) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6},$$

and

$$\varphi_1^S(v) = 1 - 3 \cdot \varphi_2^S(v) = \frac{1}{2}.$$

We obtain the Shapley-Shubik index $\boldsymbol{\varphi}^S(v) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, which is different from the vector of relative weights $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. In conclusion, the Shapley-Shubik index of player 1 indicates that the voting power of player 1 is slightly higher than the relative voting weight $\frac{2}{5}$. The normalized Banzhaf index is in this case the same as the Shapley-Shubik index, since the swings for players are $s_1(v) = 6$ and $s_2(v) = s_3(v) = s_4(v) = 2$.

Solution 66.

Let $w_i \le w_j$ for some players $i, j \in N$. We want to show that

$$\varphi_i^S(v) = \frac{|\{\pi \in \Pi \mid i \text{ pivotal to } A_i^\pi\}|}{n!} \le \frac{|\{\pi \in \Pi \mid j \text{ pivotal to } A_j^\pi\}|}{n!} = \varphi_j^S(v).$$

To this end, it suffices to show that for each permutation π in which i is pivotal there exists a permutation σ making j pivotal. Take such π and define σ as the transposition:

$$\sigma(k) = \begin{cases} j & k = i, \\ i & k = j, \\ k & \text{otherwise,} \end{cases} \qquad k \in N.$$

This definition implies that $A_i^\pi = A_j^\sigma$. By the hypothesis A_i^π is loosing and $A_i^\pi \cup i$ is winning. This means that $\sum_{k \in A_j^\sigma} w_k = \sum_{k \in A_i^\pi} w_k < q$ and $\sum_{k \in A_j^\sigma} w_k + w_j \ge \sum_{k \in A_i^\pi} w_k + w_i \ge q$. In other words, A_j^σ is loosing and $A_j^\sigma \cup j$ is winning, which was to be proved.

The same inequality holds for the Banzhaf index. Indeed, it suffices to prove that $s_i(v) = |\{A \subseteq N \mid A \text{ swing for } i \text{ in } v\}| \le s_j(v)$. Equivalently, we want to show that for each swing A for i there exists a swing B for j. Let A be a swing for i, that is, A is loosing ($\sum_{k \in A} < q$) and $A \cup i$ is winning ($\sum_{k \in A \cup i} \ge q$). We will distinguish two cases. First, if $j \notin A$, then we may take $B = A$, since A is loosing and $A \cup j$ is winning by the assumption $w_j \ge w_i$. Second, let $j \in A$ and consider now $B = (A \setminus j) \cup i$. Then B is loosing,

$$\sum_{k \in A \setminus j} w_k + w_i \le \sum_{k \in A \setminus j} w_k + w_j < q,$$

and $B \cup j = A \cup i$ is winning.

Solution 67.

This is, for example, the simple game in which the winning coalitions are 12, 13, and 123. The weights are $w_1 = 5$, $w_2 = 2$, $w_3 = 3$, and the quota is $q = 6$.

Solution 68.

Let v be a weighted voting game with $n = 3$, weights $\mathbf{w} = (2, 2, 1)$, and quota $q = 4$. Clearly, player 3 is null in this game, so $\varphi_3^S(v) = 0$. Now, take the game v' with 4 players and weights $(2, 2, 1, 1)$ and the same quota. In this game player 3 is pivotal for coalition $\{1, 4\}$, for example, which means that $\varphi_3^S(v') > 0$.

Solution 69.

Since N is the only winning coalition in v, each player has the same Shapley value $\frac{1}{n}$ by symmetry. If player 1 decides to split into two new players, the resulting simple game v' has $n+1$ players, the weight vector $(1,\ldots,1)$, and the same quota. This means that the Shapley value of each player is $\frac{1}{n+1}$. Therefore, the combined voting power of two new players is $\frac{2}{n+1}$, which is almost two-times greater than the original voting power of player 1.

Solution 70.

Since N is the only winning coalition in game v, each player has the same Shapley value $\frac{1}{n}$ by symmetry. After the split the game becomes the weighted voting game v' with the player set $N = \{1,\ldots,n+1\}$, weights $(1,1,2,\ldots,2)$ and the same quota $q = 2n-1$. It is easy to see that player 1 is pivotal only to coalition $\{3,\ldots,n+1\}$ in game v'. Then $\varphi_1^S(v') = \frac{(n-2)!}{n!} = \frac{1}{n(n+1)}$ and similarly for player 2. Consequently, the joint voting power of those two players is $\frac{2}{n(n+1)}$, which is lower than the Shapley value of player 1 in the original game v.

Solution 71.

The original simple game v can be modelled as a vector weighted voting game with two component games v^1, v^2 on 13 players as follows. We define the player set $N = \{1,\ldots,13\}$, where $\{1,\ldots,10\}$ is the Board with director 1 and deputy 2, and players $11, 12, 13$ are the CEO, CFO, and CTO, respectively. The weighted voting game v^1 has weights $\mathbf{w}^1 = (10,10,1,\ldots,1,0,0,0)$ and quota $q^1 = 24$. The weighted voting game v^2 has weights $\mathbf{w}^2 = (0,\ldots,0,2,1,1)$ and quota $q^2 = 3$. By the definition a coalition $A \subseteq N$ is winning in v whenever it is winning in both v^1 and v^2.

Clearly, the only vetoers are $1, 2, 11$ (director, deputy, and CEO).

REFERENCES

[1] Chalkiadakis G., Elkind E., and Wooldridge M. *Computational Aspects of Cooperative Game Theory*. Morgan & Claypool, 2012.

[2] J. González-Díaz, I. García-Jurado, and M. G. Fiestras-Janeiro. *An Introductory Course on Mathematical Game Theory*, volume 115 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.

[3] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.

[4] G. Owen. *Game theory*. Academic Press Inc., San Diego, CA, third edition, 1995.

[5] B. Peleg and P. Sudhölter. *Introduction to the theory of cooperative games*, volume 34 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Springer, Berlin, second edition, 2007.