Computational Game Theory

Weighted Voting Games

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Voting and simple games

A coalitional game $v\colon \mathcal{P}(N) \to \mathbb{R}$ is called simple if it is monotone, $v(A) \in \{0,1\}$ for each $A \subseteq N$, and $v(N) = 1$.

- A simple game models voting or the completion of a task
- A coalition $A \subseteq N$ is called
  - winning if $v(A) = 1$
  - loosing if $v(A) = 0$

We are seeking computationally efficient representations of simple games and methods for the computation of voting power.

The voting system of the EU in 2011

Example

- 27 states in the Council of the EU
- A law requires the support of
  1. 50% of the countries
  2. 62% of the population of the EU
  3. 74% of the “commissioners” of the EU

It is a simple game with 27 players and $2^{27} \approx 134 \cdot 10^6$ coalitions.

How to represent it?

Weighted voting

Definition

A weighted voting game is a simple game v such that there exist $\mathbf{w} \in \mathbb{R}^n_+$ and $q > 0$ with $\sum_{i \in N} w_i \geq q$, such that

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i \geq q, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

- The quota q and weights $w_1, \ldots, w_n$ can be chosen integral
- There are simple games which are not weighted voting games

Vector weighted voting games

Definition

A vector weighted voting game is a simple game v such that there exist k weighted voting games represented by weight vectors $\mathbf{w}^1, \ldots, \mathbf{w}^k \in \mathbb{R}_+^n$ and quotas $q^1, \ldots, q^k > 0$, such that

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i^j \geq q^j \text{ for each } j = 1, \ldots, k, \\ 0 & \text{otherwise,} \end{cases} \qquad A \subseteq N.$$

- Every simple game is a vector weighted voting game
- The dimension of a simple game v is the minimal number k making v a vector weighted voting game

This is a vector weighted voting game with $k = 3$ where

G^1 :	29	29	29	29	27	27	14	13	12	12	12	12	12	10	10	10	7	7	7	7	7	4	4	4	4	4	3	255
G^2 :	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
G^3 :	170	123	122	120	82	80	47	33	22	21	21	21	21	18	17	17	11	11	11	8	8	5	4	3	2	1	1	620

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Types of players in simple games

A player $i \in N$ in a simple game v is

- vetoer if $\forall A \subseteq N$: A is winning $\Rightarrow i \in A$
- dictator if $\forall A \subseteq N$: A is winning $\Leftrightarrow i \in A$
- null if $\forall A \subseteq N$: A and $A \cup i$ are loosing
- pivotal to some $A \subseteq N$ if A is loosing and $A \cup i$ is winning (such coalition A is called a swing for i)

Example

The permanent members of UNSC are vetoers. Towns 5 and 6 in the Nassau County Board are null players. A single-player coalition $\{i\}$ is winning in a game with a dictator i.

Finding vetoers and null players in weighted voting games

Facts

1. A player i is a vetoer if, and only if, $N \setminus i$ is loosing.
2. A player i is null if, and only if, $\varphi_i^S(v) = 0$.

- Fact 1 means that checking i is a vetoer amounts to decide if the inequality $\sum_{j \in N \setminus i} w_j < q$ is satisfied
- On the one hand, it is known that deciding whether a player i is null is co-NP complete
- On the other, Fact 2 implies that any algorithm for computing the Shapley value can be used to identify null players

Power indices

The Shapley value of simple games

Definition

The Shapley–Shubik index of player i in a simple game v is

$$\begin{aligned} \varphi_i^S(v) &= \frac{1}{n!} \cdot \sum_{\substack{A \subseteq N \setminus i \\ A \text{ swing for } i}} |A|!(n - |A| - 1)! \\ &= \frac{1}{n!} \cdot |\{\pi \in \Pi \mid i \text{ pivotal to } A_i^\pi\}| \end{aligned}$$

Example 1

A weighted majority game

$N = \{1, 2, 3\}$

Each player $i \in N$ has i votes. This is a weighted voting game with $\mathbf{w} = (1, 2, 3)$ and $q = 4$ such that

$$v(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, 12 \\ 1 & A = 13, 23, 123 \end{cases}$$

For each permutation π the player i pivotal to A_i^π is highlighted:

12**3** 1**3**2 21**3** 2**3**1 3**1**2 3**2**1

$$\varphi^S(v) = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{4}{6}\right)$$

Example 2

Simple majority game

$N = \{1, \dots, n\}$

Weighted voting game with $\mathbf{w} = (\frac{1}{n}, \dots, \frac{1}{n})$ and $q = \lfloor \frac{n}{2} \rfloor + 1$,

$$v(A) = \begin{cases} 1 & |A| > \frac{n}{2}, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

- $\varphi_i^S(v) = \varphi_j^S(v)$ for each $i, j \in N$ — Symmetry
- $\varphi_1^S(v) + \cdots + \varphi_n^S(v) = 1$ — Efficiency

$$\boldsymbol{\varphi}^S(v) = \left(\tfrac{1}{n}, \dots, \tfrac{1}{n}\right) = \mathbf{w}$$

Example 3

UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A binary decision is approved by all the permanent members and at least four non-permanent members
- Weighted voting game with $w_i = 20$ for $i = 1, \dots, 5$, $w_i = 1$ for $i = 6, \dots, 15$ and $q = 104$

$$v(A) = \begin{cases} 1 & \text{if } A \supseteq \{1, \dots, 5\} \text{ and } |A| \geq 9, \\ 0 & \text{otherwise.} \end{cases}$$

If $i \in \{6, \dots, 15\}$, then $\varphi_i^S(v) = \binom{9}{3} \cdot \frac{8! \cdot 6!}{15!} \approx 0.0019$
If $j \in \{1, \dots, 5\}$, then symmetry and efficiency give

$$\varphi_j^S(v) = \frac{1}{5}(1 - 10 \cdot \varphi_i^S(v)) \approx 0.1963$$

Banzhaf value of simple games

Let $s_i(v)$ be the number of swings for player i in a simple game v,

$$\begin{aligned} s_i(v) &= |\{A \subseteq N \setminus i \mid A \text{ is a swing for } i\}| \\ &= |\{A \subseteq N \setminus i \mid v(A \cup i) - v(A) = 1\}| \end{aligned}$$

Definition

The Banzhaf index of player i in a simple game v is

$$\varphi_i^B(v) = \frac{s_i(v)}{2^{n-1}}$$

The Banzhaf index is not efficient: $\sum_{i \in N} \varphi_i^B(v) \neq 1$

The Banzhaf index – example

Example (Weighted majority voting, cont'd)

Each player $i \in \{1, 2, 3\}$ has i votes. The swings for players are:

Player i	*Coalitions*	$s_i(v)$
1	$\emptyset, 2, \boxed{3}, 23$	1
2	$\emptyset, 1, \boxed{3}, 13$	1
3	$\emptyset, \boxed{1}, \boxed{2}, \boxed{12}$	3

$$\varphi^B(v) = \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}\right)$$

Making the Banzhaf index efficient

Definition

The normalized Banzhaf index of player i in a simple game v is

$$\beta_i(v) = \frac{s_i(v)}{s_1(v) + \cdots + s_n(v)}$$

The two Banzhaf indices preserve the power ratios of players:

$$\beta_i(v) = \frac{2^{n-1}}{s_1(v) + \cdots + s_n(v)} \cdot \varphi_i^B(v)$$

Example

UN Security Council – The old and the new voting system

O 11 members, approval by at least 7 votes

N 15 members, approval by at least 9 votes

Shapley–Shubik indices

O $\varphi_1^S(v) = 0.1974$, $\varphi_6^S(v) = 0.0022$ 90 : 1

N $\varphi_1^S(v) = 0.1963$, $\varphi_6^S(v) = 0.0019$ 100 : 1

Normalized Banzhaf indices

O $\beta_1(v) = \frac{19}{105}$, $\beta_6(v) = \frac{1}{63}$ 11 : 1

N $\beta_1(v) = \frac{106}{635}$, $\beta_6(v) = \frac{21}{1270}$ 10 : 1

Weights and power

Weights and the Shapley-Shubik index

Fact

Let v be a weighted voting game with weight vector $\mathbf{w}$ and quota q. If $w_i \leq w_j$ for players $i, j \in N$, then $\varphi_i^S(v) \leq \varphi_j^S(v)$.

This fact doesn't exclude possibility that voters with radically different weights have identical voting power!

Example

2010 elections in the UK

- Conservative Party 307 seats
- Labour party 258 seats
- Liberal Democrats 57 seats
- Other parties 28 seats (the most powerful among them has 8)

The minimal winning coalitions are $\{\text{Cons}, \text{Lab}\}$ and $\{\text{Cons}, \text{Lib}\}$. To form a winning coalition, Lab and Lib need each other and a few smaller parties, which implies

$$\varphi^S_{\text{Lab}}(v) = \varphi^S_{\text{Lib}}(v).$$

Voting power as a function of quota $q = 1, \ldots, \sum_{i \in N} w_i$

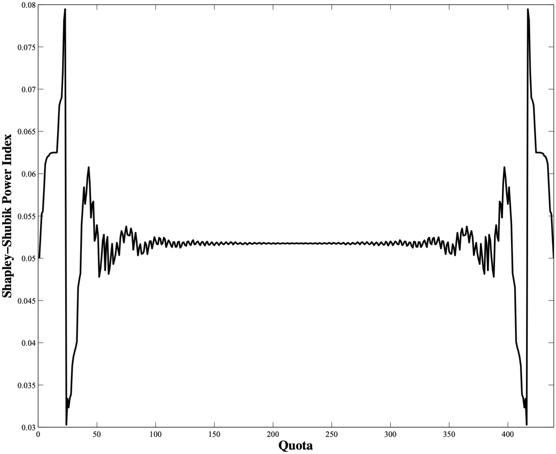

Figure 4.1: The Shapley value of player 10 (weight 23) for weight vector (1, 2, 4, 5, 16, 17, 20, 21, 21, 23, 24, 24, 27, 28, 28, 33, 33, 36, 36, 40).

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Voting power and quota

Proposition

Let v be a weighted voting game with weights $\mathbf{w}$ and quota q and v' be a weighted voting game with weights $\mathbf{w}$ and quota $\sum_{j \in N} w_j + 1 - q$. Then $\varphi_i^S(v) = \varphi_i^S(v')$ for each player i.

Moreover:

- The graph has peaks at $q = w_i$ and $q = \sum_{j \in N \setminus i} w_j + 1$
- In a significant proportion of randomly generated weighted voting games, the Shapley-Shubik index of player i is minimized at $q = w_i + 1$