Tractable classes of games. Learning.

Lecture 3

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Solving normal-form games

- *Nash equilibrium* is very difficult to compute even in a two-player general-sum game
- *Maxmin*/*minmax* strategies in a two-player zero-sum game are optimal solutions to dual linear programs

What to do?

1. Find tractable classes of games in-between
2. Introduce online learning to recover equilibria
3. Design tractable solution concepts

Polymatrix games

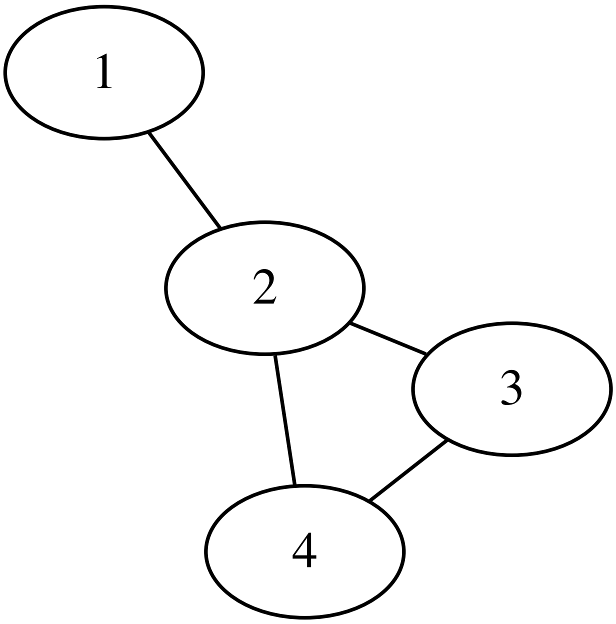
A normal-form game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a *polymatrix game* if there is an undirected graph (N, E) without loops and, for each $\{i, j\} \in E$, pairwise utility functions

$$u_{ij} \colon S_i \times S_j \to \mathbb{R}, \quad u_{ji} \colon S_j \times S_i \to \mathbb{R}$$

such that the utility of player $i \in N$ is

$$u_i(\mathbf{s}) = \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j), \qquad \mathbf{s} \in \mathbf{S}.$$

Example of polymatrix game



$$u_1(\mathbf{s}) = u_{12}(s_1, s_2)$$
$$u_2(\mathbf{s}) = u_{21}(s_2, s_1) + u_{23}(s_2, s_3) + u_{24}(s_2, s_4)$$

Size of normal-form games

Assumptions: n players and $|S_i| = k$ for each $i \in N$

- A two-player zero-sum game has the size k^2
- A normal-form game has the size $n \cdot k^n$
- A polymatrix game has the size at most

$$\binom{n}{2} \cdot 2k^2 = n(n-1)k^2$$

Zero-sum polymatrix games

A polymatrix game is *zero-sum* if, for each $\mathbf{s} \in \mathbf{S}$,

$$\sum_{i \in N} u_i(\mathbf{s}) = 0.$$

- For example, pairwise games may be zero-sum:

$$u_{ij} + u_{ji} = 0$$

- But the last property is not necessary for a polymatrix game to be zero-sum

Solving zero-sum polymatrix games

Minimize $\sum_{i \in N} w_i$ subject to the *constraints*

$$
\begin{aligned}
U_i(s_i, p_{-i}) &\leq w_i, & \forall i \in N, \forall s_i \in S_i \\
w_i \in \mathbb{R},\ p_i &\in \Delta_i & \forall i \in N
\end{aligned}
$$

Claim. The following are equivalent for $\mathbf{p}^* \in \Delta$.

1. $\mathbf{p}^*$ is a Nash equilibrium.
2. $(\mathbf{p}^*, \mathbf{w}^*)$ is an optimal solution to the LP above with

$$w_i^* = \max_{s_i \in S_i} U_i(s_i, \mathbf{p}_{-i}^*), \qquad i \in N.$$

Potential games

A normal-form game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a *potential game* if there exists a function $P \colon \mathbf{S} \to \mathbb{R}$ such that for all $i \in N$, every $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$, and every $s_i, t_i \in S_i$,

$$u_i(s_i, \mathbf{s}_{-i}) - u_i(t_i, \mathbf{s}_{-i}) = P(s_i, \mathbf{s}_{-i}) - P(t_i, \mathbf{s}_{-i}).$$

Claim

Every potential game has a pure Nash equilibrium

$$\mathbf{s}^* \in \arg\max_{\mathbf{s} \in \mathbf{S}} P(\mathbf{s}).$$

Learning in normal-formal games

- The algorithms discussed so far are *offline* in the sense that the entire game is processed at once
- The players must compute equilibria first and only then they can play optimally
- It seems natural to explore *dynamics* defining iterative methods that converge to the equilibria

Best response dynamics for pure NE

1. *Initialization*: an arbitrary strategy profile $\mathbf{s} \in \mathbf{S}$
2. If $s_i \in \mathbf{BR}(\mathbf{s}_{-i})$ for each player $i \in N$, then $\mathbf{s}$ is a pure NE
3. If $s_i \notin \mathbf{BR}(\mathbf{s}_{-i})$ for some player $i \in N$, then pick $t_i \in \mathbf{BR}(\mathbf{s}_{-i})$, update $\mathbf{s} := (t_i, \mathbf{s}_{-i})$, and go to 2.

The BR dynamics fail to terminate for most games.

Claim

In a potential game, BR dynamics converge to a pure NE starting from an arbitrary initial strategy profile.

Fictitious Play for two-player games

An *iterative method* for approximating a mixed strategy NE:

- In the step k the history is $(s_1^1, s_2^1), \ldots, (s_1^{k-1}, s_2^{k-1})$
- Player 1 believes that Player 2 is using the mixed strategy

$$\hat{p}_2^k := \frac{1}{k-1} \sum_{j=1}^{k-1} \delta_{s_2^j}$$

  and plays the best response
- Player 2 behaves analogously

FP: Algorithm

1. *Initialization*: any strategy profile (s_1^1, s_2^1) and $k \leftarrow 2$
2. In the round k
   i. Player 1 plays $s_1^k \in \mathbf{BR}(\hat{p}_2^k)$
   ii. Player 2 plays $s_2^k \in \mathbf{BR}(\hat{p}_1^k)$
3. $k \leftarrow k + 1$ and go to 2.

Claim

If the sequences $\hat{p}_1^1, \hat{p}_1^2, \ldots$ and $\hat{p}_2^1, \hat{p}_2^2, \ldots$ converge, then their limit is a Nash equilibrium.

FP: Failure of convergence

In the bimatrix game

$$\begin{bmatrix} 0,0 & 1,0 & 0,1 \\ 0,1 & 0,0 & 1,0 \\ 1,0 & 0,1 & 0,0 \end{bmatrix}$$

the unique NE consists of the uniform distributions. However, the empirical frequencies fail to converge when FP starts at strategy profile $(1, 2)$.

FP: Convergence

The empirical frequencies of play in FP *converge* in the following classes of games:

1. Two-player zero-sum games.
2. Potential games.
3. The games solvable by iterated elimination of strictly dominated strategies.

FP: Summary

Every player

- observes only the history of play and its utility values
- assumes that the opponent is playing according to the observed empirical frequencies, yet this strategy is not used by the player itself
- is focusing only on the opponent's actions