Normal-form games

Lecture 1

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Game theory

- Mathematical theory of interactive decision-making
- **Game** involves multiple *players* such that the choice of *strategy* of each player determines the *outcome*
- The seminal work:

J. von Neumann, O. Morgenstern. *Theory of Games and Economic Behavior.* Princeton University Press, 1944.

Problems in game theory

- Compute optimal strategies in extremely large games
- Design optimal auctions
- Allocate the cost among investors fairly
- Evaluate power of voters in collective decision-making

Game theory in AI and applications

- Checkers (1994)
- Chess (1998)
- AlphaGo (2015)
- DeepStack (2017)
- AlphaStar (2019)

- security games
- cybersecurity
- auctions
- voting
- social choice
- generative AI (GANs)
- explainable ML
- robotics

Game theory and other disciplines

- *Economics*
  - Rationality assumption
  - The concept of equilibrium
- *Optimization*
  - From unilateral optimization to fixed point computation
- *RL*
  - From MDPs to multiagent RL
- *Computer science*
  - PPAD completeness
- *Optimal control*
  - Pursuit-evasion games
- *Mathematics*
  - Fixed point theory

Plan of the course

1. Normal-form (strategic) games
2. Extensive-form games with imperfect information
3. Bayesian games and auctions
4. Cooperative games

Game theory in other FEL courses

- *Řešení problémů a hry* (RPH)
  - Prisoner's dilemma and rock-paper-scissors
- *Introduction to Artificial Intelligence* (ZUI)
  - Two-player zero-sum extensive-form games with perfect information (chess, go)
  - Backward induction and MCTS
- *AI in robotics* (UIR)
  - Solving very large matrix games
  - Solving two-player zero-sum stochastic games

Classification of games

- *Game forms*
  - normal
  - extensive
  - cooperative
- *Dynamics*
  - static
  - sequential
- *Strategy sets*
  - finite
  - infinite
- *Utility functions*
  - general-sum
  - zero-sum
- *Information*
  - complete
  - incomplete

Normal-form game

1. *Player set* $N = \{1, \ldots, n\}$
2. *Strategy set* S_i for each $i \in N$, let $\mathbf{S} = S_1 \times \cdots \times S_n$
3. *Utility function* $u_i\colon \mathbf{S} \to \mathbb{R}$ for each player $i \in N$

This captures a one-shot strategic situation:

- Each player i selects $s_i \in S_i$, let $\mathbf{s} = (s_1, \ldots, s_n)$.
- Each player i gets utility $u_i(\mathbf{s})$.

Two-player zero-sum games

1. *Player set* $N = \{1, 2\}$
2. *Strategy sets* S_1 and S_2
3. *Utility functions*: $u_1 \colon S_1 \times S_2 \to \mathbb{R}$ and $u_2 = -u_1$

- $u_1 + u_2 = 0$
- We often simply write $u := u_1$
- *Constant-sum games* ($u_1 + u_2 = c$) are not more general: define $u_2' := u_2 - c$ and observe that $u_1 + u_2' = 0$

Pure and mixed strategies

- A strategy $s_i \in S_i$ is called *pure*

Assumption: every S_i is finite

- *Mixed strategy* is a probability distribution p_i over S_i
- The set of all mixed strategies is denoted by Δ_i

Every pure strategy $s_i \in S_i$ is mixed:

$$\delta_{s_i}(t_i) = \begin{cases} 1 & t_i = s_i \\ 0 & \text{otherwise} \end{cases}$$

Expected utility

The *expected utility* of player i is $U_i\colon \Delta_1 \times \cdots \times \Delta_n \to \mathbb{R}$,

$$U_i(p_1, \ldots, p_n) = \sum_{\mathbf{s} \in \mathbf{S}} u_i(\mathbf{s}) \prod_{j \in N} p_j(s_j).$$

It is an extension of utility function since

$$U_i(\delta_{s_1}, \ldots, \delta_{s_n}) = u_i(s_1, \ldots, s_n)$$

for every $(s_1, \ldots, s_n) \in \mathbf{S}$.

Examples of normal-form games (1)

Rock paper scissors: $S_1 = S_2 = \{r, p, s\}$ with the payoff matrix

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Prisoner's dilemma: $S_1 = S_2 = \{c, d\}$ with the bimatrix

$$\begin{bmatrix} -1, -1 & -4, 0 \\ 0, -4 & -3, -3 \end{bmatrix}$$

Examples of normal-form games (2)

Matching pennies: $S_1 = S_2 = \{h, t\}$ with the payoff matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

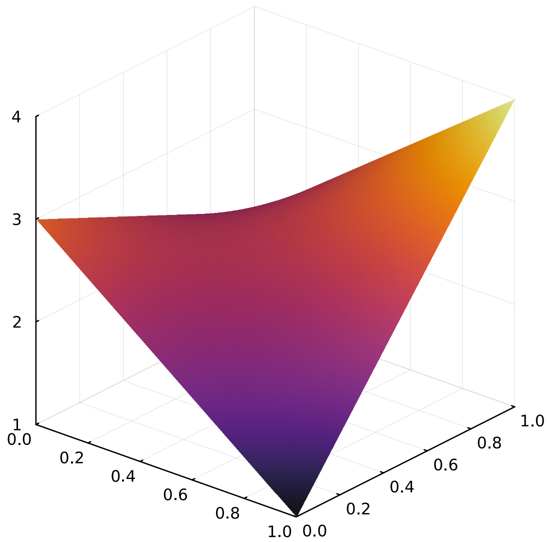
Expected utility of player 1 is

$$U(p, q) = 4pq - 2p - 2q + 1,$$

where $p := p_1(h)$ and $q := p_2(h)$

Examples of normal-form games (3)

Two-player zero-sum *continuous game*: $S_1 = S_2 = [0, 1]$ and $u(x, y) = 4xy - 2x - y + 3$.



Nash equilibrium in pure strategies

A strategy profile $\mathbf{s}^* = (s_1^*, \ldots, s_n^*) \in \mathbf{S}$ is a *Nash equilibrium* if, for each $i \in N$ and every $s_i \in S_i$,

$$u_i(s_i, \mathbf{s}_{-i}^*) \leq u_i(\mathbf{s}^*).$$

Equivalently: for each $i \in N$,

$$s_i^* \in \mathbf{BR}_i(\mathbf{s}_{-i}^*),$$

where

$$\mathbf{BR}_i(\mathbf{s}_{-i}^*) = \{s_i \in S_i \mid u_i(s_i, \mathbf{s}_{-i}^*) = \max_{t_i \in S_i} u_i(t_i, \mathbf{s}_{-i}^*)\}.$$

NE in pure strategies - Examples

- *Matching pennies* and *rock paper scissors* have no pure NE
- *Prisoner's dilemma* has a pure NE with utilities -3

$$\begin{bmatrix} -1,-1 & -4,0 \\ 0,-4 & -3,-3 \end{bmatrix}$$

- The continuous game $u(x,y) = 4xy - 2x - y + 3$ on the unit square has the unique NE $(\frac{1}{4}, \frac{1}{2})$ with value $\frac{5}{2}$

NE in mixed strategies for finite games

A profile of mixed strategies $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$ is a *Nash equilibrium* if, for each $i \in N$ and every $p_i \in \Delta_i$,

$$U_i(p_i, \mathbf{p}^*_{-i}) \leq U_i(\mathbf{p}^*).$$

Equivalently: for each $i \in N$,

$$p_i^* \in \mathbf{BR}_i(\mathbf{p}^*_{-i}),$$

where
$\mathbf{BR}_i(\mathbf{p}^*_{-i}) = \{p_i \in \Delta_i \mid U_i(p_i, \mathbf{p}^*_{-i}) = \max_{q_i \in \Delta_i} U_i(q_i, \mathbf{p}^*_{-i})\}.$

NE in mixed strategies - Examples

- The unique NE in *Matching pennies* and *rock paper scissors* are uniform probability distributions
- *Battle of the sexes* game

$$\begin{bmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{bmatrix}$$

has two pure NE (with payoffs 2 and 1) and one mixed NE

$$((\tfrac{1}{3}, \tfrac{2}{3}), (\tfrac{2}{3}, \tfrac{1}{3}))$$

Existence of Nash equilibria

Nash's theorem (1950)

Every n-player strategic game with finite strategy spaces has at least NE in mixed strategies.

A minimax theorem

Every two-player zero-sum strategic game with compact convex strategy sets and continuous concave-convex utility function u has a pure NE.

Dominated strategies

- A strategy s_i *strongly dominates* s'_i if, for every $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$, $u_i(s_i, \mathbf{s}_{-i}) > u_i(s'_i, \mathbf{s}_{-i})$.
- A strategy s_i *weakly dominates* s'_i if, for every $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$, $u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s'_i, \mathbf{s}_{-i})$ and there exists some $\mathbf{s}_{-i} \in \mathbf{S}_{-i}$ such that $u_i(s_i, \mathbf{s}_{-i}) > u_i(s'_i, \mathbf{s}_{-i})$.

A rational player doesn't adopt strongly dominated strategy.

Removal of dominated strategies

- An iterative procedure yields a smaller game and does not depend on the order of elimination if we remove only *strictly* dominated strategies
- It preserves all the existing pure NE
- Example: Iterated removal applied to

$$\begin{bmatrix} 1,0 & 1,2 & 0,1 \\ 0,3 & 0,1 & 2,0 \end{bmatrix}$$

  yields a unique NE with payoffs $(1,2)$