DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

CONVEX HULLS

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg] and [Mount]

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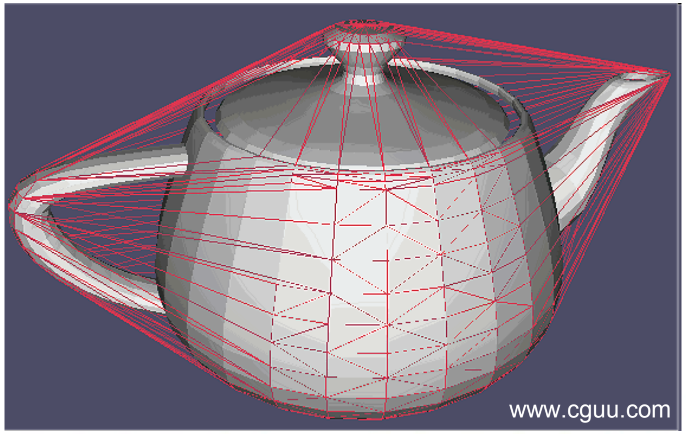
Talk overview

- Motivation and Definitions
- Graham’s scan – incremental algorithm
- Divide & Conquer
- Quick hull
- Jarvis’s March – selection by gift wrapping
- Chan’s algorithm – assymptotic optimal algorithm



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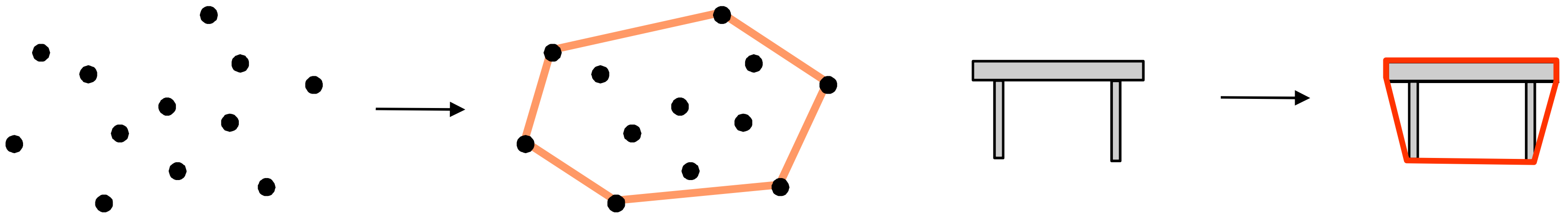
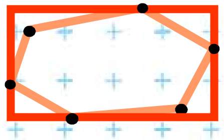
Convex hull (CH) – why to deal with it?

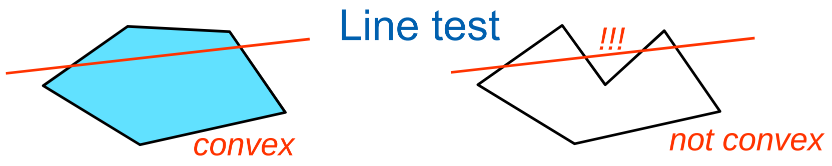


- *Shape approximation* of a point set or complex shapes (other common approximations include: minimal area enclosing rectangle, circle, and ellipse,…) – e.g., for collision detection
- *Initial stage* of many algorithms to filter out irrelevant points, e.g.:
  - diameter of a point set

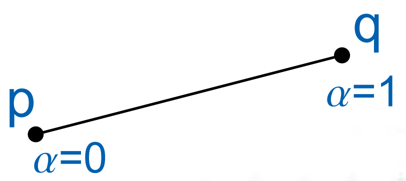
  - minimum enclosing convex shapes (such as rectangle, circle, and ellipse) depend only on points on CH



Convexity

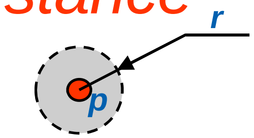


- A set *S* is *convex*
  - if for any points p,q $\in S$ the line segment $\overline{pq} \subseteq S$, or
  - if any convex combination of *p* and *q* is in *S*
- Convex combination of points *p, q* is any point that can be expressed as $(1-\alpha)\,p + \alpha q$, where $0 \le \alpha \le 1$

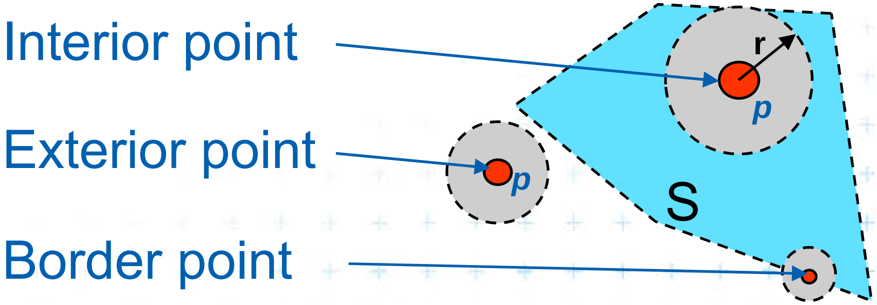


- *Convex hull* CH(S) of set *S* – is (similar definitions)
  - the smallest set that contains *S (convex)*
  - or: intersection of all convex sets that contain *S*
  - Or in 2D for points: the smallest convex polygon containing all given points

Definitions from topology in metric spaces



- *Metric space* – each two of points have defined a *distance*
- *r-neighborhood* of a point *p* and radius *r* > *0*
  = set of points whose distance to *p* is strictly less than *r*
  (open ball of diameter *r* centered about *p*)
- Given set S, point *p* is
  - Interior point of S – if $\exists r, r > 0$, (r-neighborhood about *p*) ⊂ S
  - Exterior point – if it lies in interior of the complement of S
  - Border point – is neither interior neither exterior

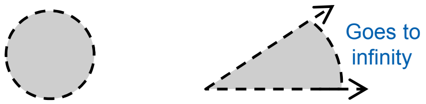


Definitions from topology in metric spaces

Are border points $p \in S$?

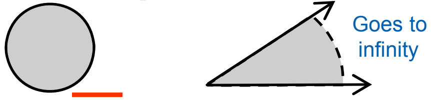
Goes to infinity?

- Set S is Open (otevřená)



  - $\forall p \in S \ \exists$ (r-neighborhood about p of radius r) $\subseteq$ S
  - it contains only interior points, none of its border points

- Closed (uzavřená)



  - If it is equal to its closure $\overline{S}$ (uzávěr = smallest closed set containing S in topol. space)
    $\forall$(r-neighborhood about p of radius r) $\cap$ S $\neq \emptyset$)

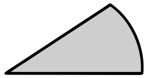
- Clopen (otevřená i uzavřená) – Ex.: empty set $\emptyset$, or finite set of disjoint components
  - if it is both closed and open

    (S= all positive rational numbers whose square is bigger than 2)

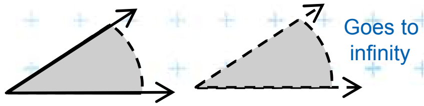
    space Q = rational numbers

    S = ($\sqrt{2}$, ∞) in Q, $\sqrt{2} \notin$ Q, S = $\overline{S}$

- Bounded (ohraničená)

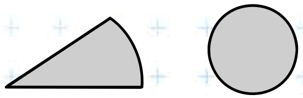


Unbounded



  - if it can be enclosed in a ball of finite radius

- Compact (kompaktní)



  - if it is both closed and bounded

Clopen (otevřená i uzavřená) example

If it is both closed and open => clopen

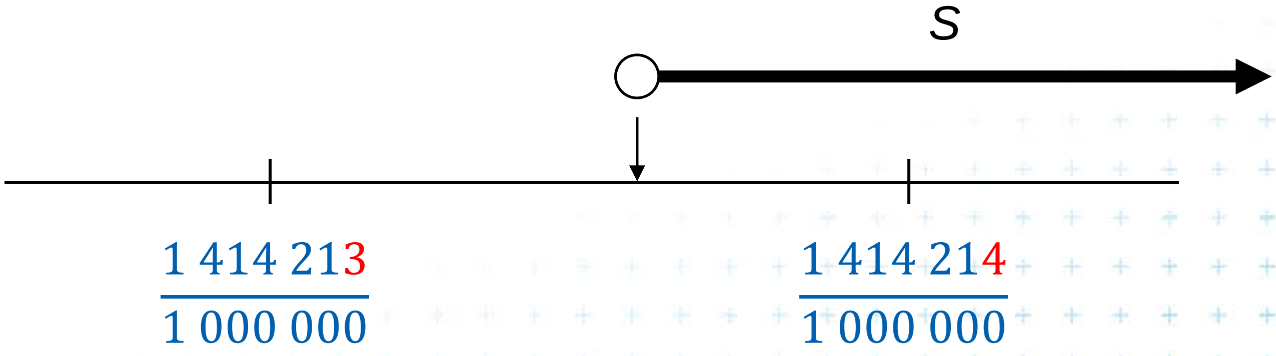
Space Q: rational numbers

Set S: all positive rational numbers whose square is bigger than 2

$S = (\sqrt{2}, \infty)$ in Q

$$\left.\begin{array}{l} \sqrt{2} \notin Q \Rightarrow \text{open (does not contain the border)} \\ S = \bar{S} \Rightarrow \text{closed (equal to its closure } \bar{S}) \end{array}\right\} \Rightarrow \text{clopen}$$

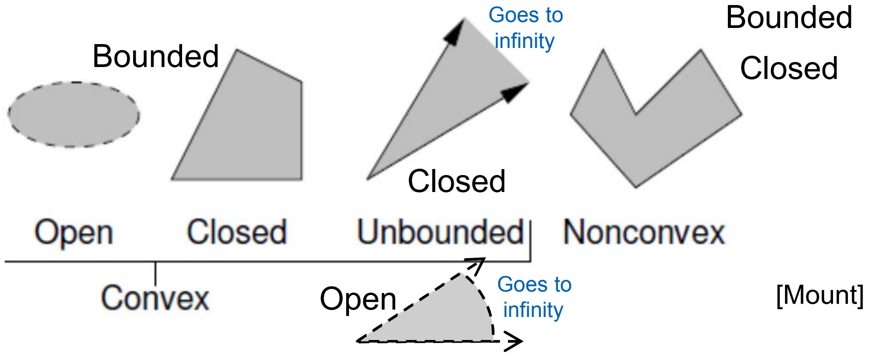
$\sqrt{2} = 1.414213562$



$\frac{1\,414\,213}{1\,000\,000}$ $\frac{1\,414\,214}{1\,000\,000}$

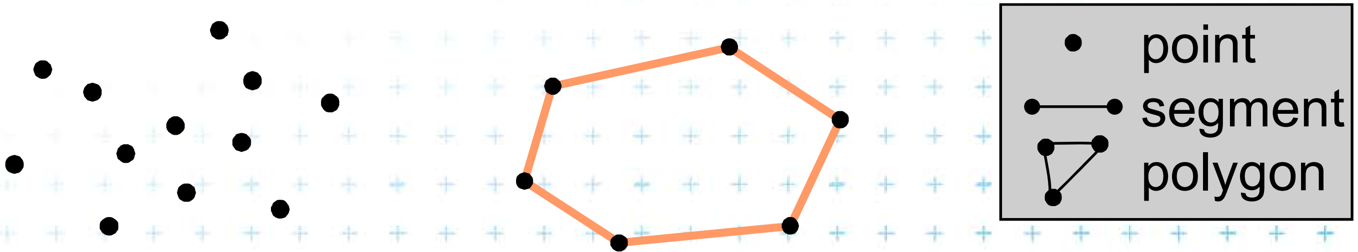
Definitions from topology in metric spaces

- *Convex set S may be bounded or unbounded*



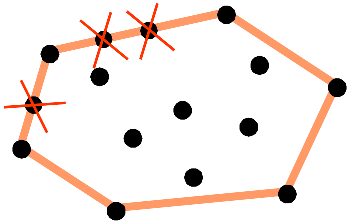
- *Convex hull* CH(S) of a finite set *S* of points in the plane

  = Bounded, closed, (= compact) convex polygon



Convex hull representation

- CCW enumeration of vertices
- Contains only the extreme points ("endpoints" of collinear points)



- Simplification for the whole semester:
  Assume the input points are in general position,
  - no two points have the same *x*-coordinates and
  - no three points are collinear

-> We avoid problem with non-extreme points on *x*
(solution may be simple – e.g. lexicographic ordering)

Online x offline algorithms

- Incremental algorithm
  - Proceeds one element at a time (step-by-step)
- Online algorithm (must be incremental)
  - is started on a partial (or empty) input and
  - continues its processing as additional input data becomes available (comes online, thus the name).
  - Ex.: insertion sort
- Offline algorithm (may be incremental)
  - requires the entire input data from the beginning
  - than it can start
  - Ex.: selection sort (any algorithm using sort)

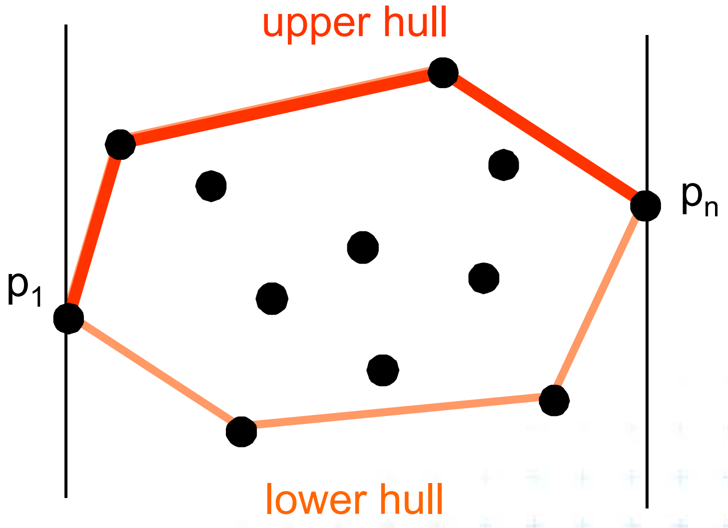
Graham's scan

- Incremental O(*n* log *n*) algorithm
- Objects (points) are added one at a time
- Order of insertion is important
  1. Random insertion $n\,O(n) = O(n^2)$
     –> we need to test: *is-point-inside-the-hull(p)* 
  2. Ordered insertion
     Find the point p with the smallest *y* coordinate first
     a) Sort points p_i according to *increasing angles* around the point p (angle of pp_i and x axis)
     b) Andrew's modification: sort points p_i according to *x* and add them left to right (construct upper & lower hull)

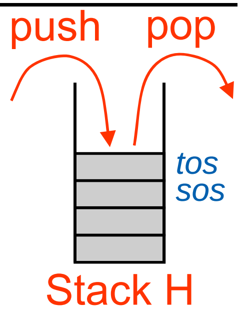
  Sorting *x-coordinates* is simpler to implement than sorting of angles

Graham's scan – b) modification by Andrew

- O(*n* log *n*) for unsorted points, O(*n*) for sorted pts.
- Upper hull, then lower hull. Merge.
- Minimum and maximum on *x* belong to CH



Graham’s scan – incremental algorithm



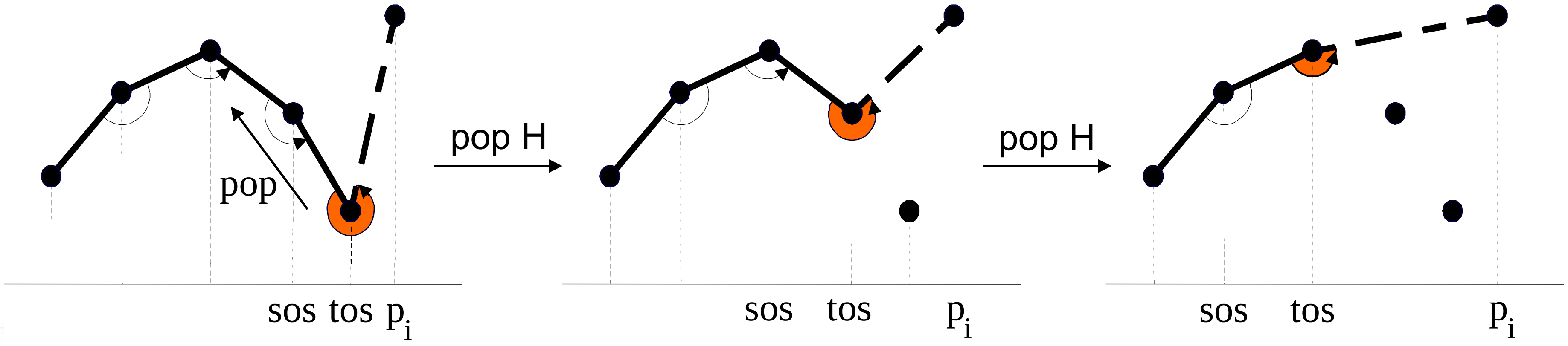
GrahamsScan(points p)

Input: points p

Output: CCW points on the convex hull

upper hull

```
sort points according to increasing x-coord -> {$p_1, p_2, \dots, p_n$}
push( $p_1$, H), push( $p_2$, H )
for $i = 3$ to $n$ do
    while( size(H) $\geq$ 2 and orient( sos, tos, $p_i$ ) $\geq$ 0 )  // skip left turns
        pop H                                                  // (back-tracking)
    push( $p_i$, H )                                           // store right turn
store H to the output (in reverse order)   // upper hull
Symmetrically the lower hull
```

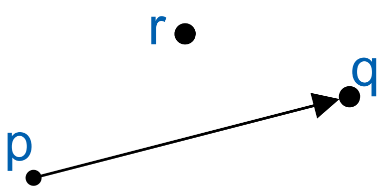


Position of point in relation to segment

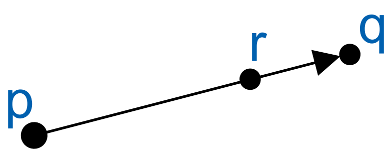
$$\text{orient}(\,p,\ q,\ r\,)\begin{cases} > 0 & r \text{ is left from } pq\text{, CCW orient} \\ = 0 & \text{if } (\,p,\ q,\ r\,) \text{ are collinear} \\ < 0 & r \text{ is right from } pq\text{, CW orient} \end{cases}$$

Point *r* is:

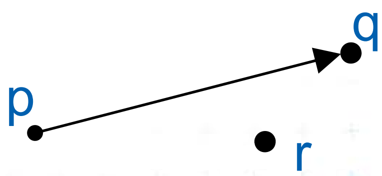
left from *pq*



on segment *pq*



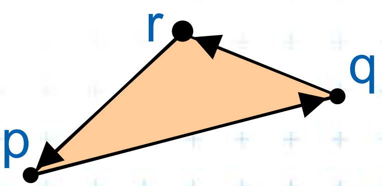
right from *pq*



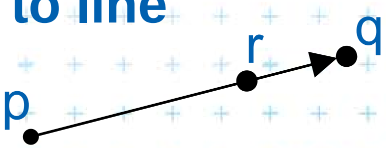
Convex polygon with edges pq and qr or

Triangle *pqr*:

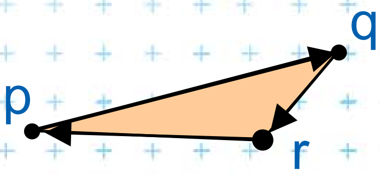
is CCW oriented



degenerated to line



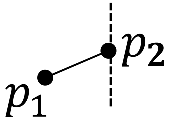
is CW oriented



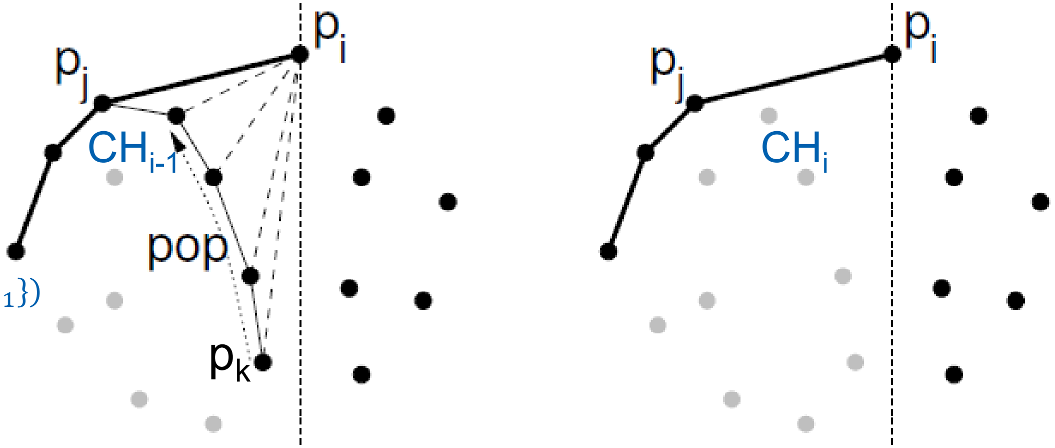
Is Graham’s scan correct?

Stack H at any stage contains upper hull of the points $\{p_1, \dots, p_j, p_i\}$, processed so far

- For induction basis $H = \{p_1, p_2\}$ … true



- p_i = last added point to CH, p_j = its predecessor on CH
- Each point p_k that lies between p_j and p_i lies below $p_j p_i$ and should not be part of UH after addition of p_i => is removed before push p_i. [orient$(p_j, p_k, p_i) > 0$, p_k is right from $p_j p_i \Rightarrow p_k$ is removed from UH]
- Stop, if 2 points in the stack or after construction of the upper hull



Points on stack H = CH $(\{p_1, p_2, \dots, p_{i-1}\})$

last added point

[Mount]

Complexity of Graham’s scan

- Sorting according x – O(n log n)
- Each point pushed once – O(n)
- Some ($d_i \leq$ n) points deleted while processing p_i

  – O(n)
- The same for lower hull – O(n)

- Total O(n log n) for unsorted points

  O(n) for sorted points

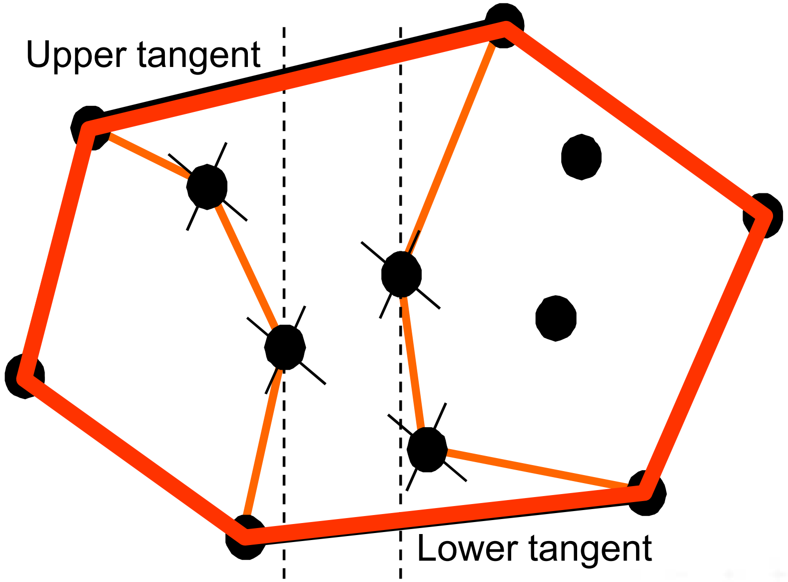
Divide & Conquer

- $\Theta(n \log(n))$ algorithm
- Extension of mergesort
- Principle
  - Sort points according to x-coordinate,
  - recursively partition the points and solve CH.

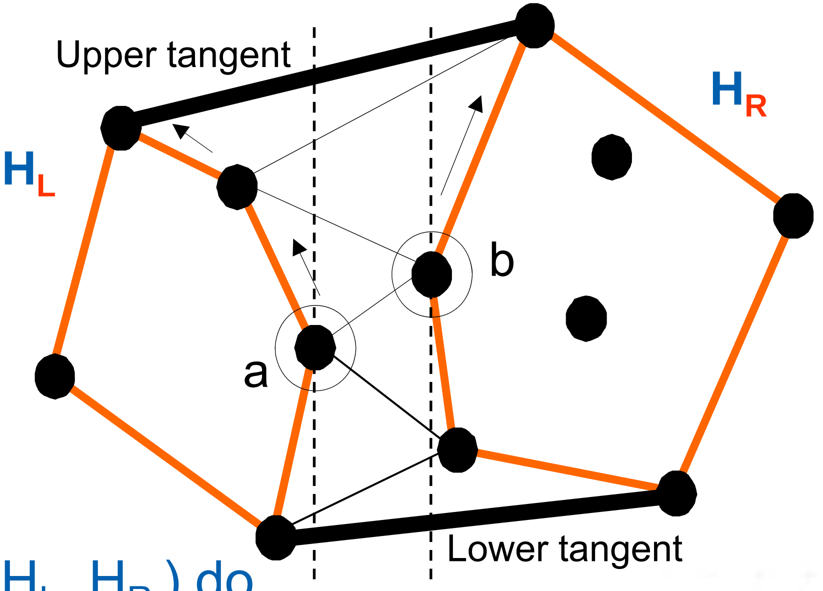
Convex hull by D&C

```
ConvexHullD&C( points P )
Input:    points p
Output:   CCW points on the convex hull
Sort points P according to $x$
return hull( P )

hull( points P )
    if $|P| \leq 3$ then
            compute CH by brute force,
            return
    Partition P into two sets L and R (with lower & higher coords $x$)
    Recursively compute $H_L$ = hull(L), $H_R$ = hull(R)
    H = Merge hulls($H_L$, $H_R$) by computing
        Upper_tangent( $H_L$, $H_R$) // find nearest points, $H_L$ CCW, $H_R$ CW
        Lower_tangent( $H_L$, $H_R$) // ($H_L$ CW, $H_R$ CCW)
        discard points between these two tangents
    return H
```



Search for upper tangent (lower is symmetrical)



Upper_tangent(H_L, H_R)
Input: two non-overlapping CH's
Output: upper tangent *ab*

```
1. a = rightmost $H_L$
2. b = leftmost $H_R$

3. while( ab is not the upper tangent for $H_L$, $H_R$ ) do
4.     while( ab is not the upper tangent for $H_L$)  a = a.succ    // move CCW
5.     while( ab is not the upper tangent for $H_R$)  b = b.pred    // move  CW
6. Return ab
```

Where: (ab is not the upper tangent for H_L) => orient*(a, b, a.succ)* ≥ 0
which means *a.succ* is left from line *ab*

$$m = |H_L| + |H_R| \leq |L| + |R| \quad \Rightarrow \text{Upper Tangent: } O(m) = O(n)$$

Convex hull by D&C complexity

- Initial sort O(*n* log(*n*))
- Function hull()
  - Upper and lower tangent O(*n*)
  - Merge hulls O(1)
  - Discard points between tangents O(*n*)

$$\left.\begin{array}{ll} \text{Upper and lower tangent} & O(n) \\ \text{Merge hulls} & O(1) \\ \text{Discard points between tangents} & O(n) \end{array}\right\} O(n)$$

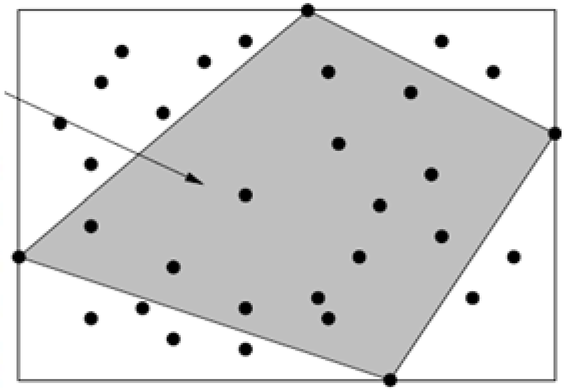
- Overall complexity
  - Recursion

$$T(n) = \begin{cases} 1 & \ldots \text{ if } n \leq 3 \\ 2T(n/2) + O(n) & \ldots \text{ otherwise} \end{cases}$$

  - Overall complexity of CH by D&C: => O(*n* log(*n*))

Quick hull

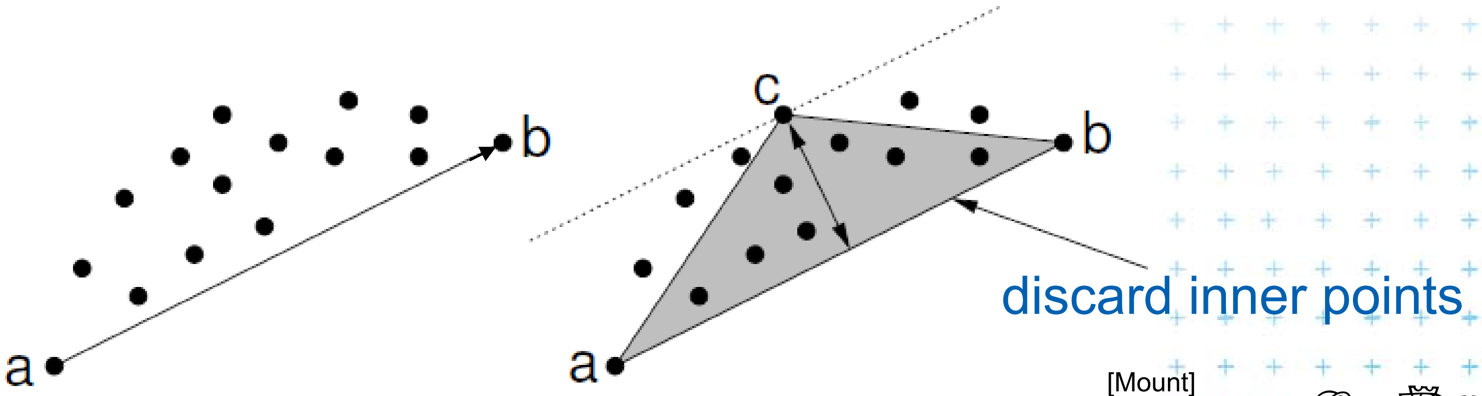
- A variant of Quick Sort
- O(n log n) expected time, max O(n^2)
- Principle
  - in praxis, most of the points lie in the interior of CH
  - E.g., for uniformly distributed points in unit square, we expect only O(log n) points on CH
- Find extreme points (parts of CH) quadrilateral, discard inner points
  - Add 4 edges to temp hull T
  - Process points outside 4 edges



[Mount]

Process each of four groups of points outside

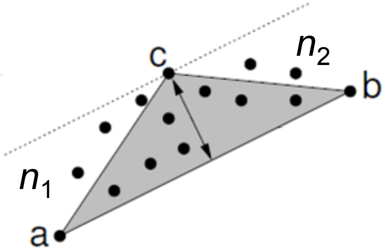
- For points outside *ab* (left from *ab* for clockwise CH)
  - Find point *c* on the hull – max. perpend. distance to *ab*
  - Discard points inside triangle *abc* (right from the edges)
  - Split points into two subsets
    - outside *ac* (left from *ac*) and outside *cb* (left from *cb*)
  - Replace edge *ab* in *T* by edges *ac* and *cb*
  - Process points outside *ac* and *cb* recursively



discard inner points

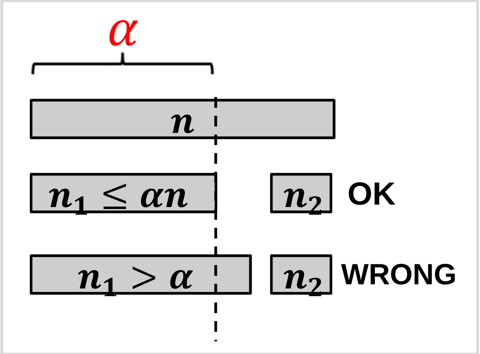
[Mount]

Quick hull complexity



- *n* points remain outside the hull
- *T*(*n*) = running time for such *n* points outside
  - O(*n*) - selection of splitting point *c*
  - O(*n*) - point classification to inside & (n_1+n_2) outside
  - $n_1+n_2 \leq n$
  - The running time is given by recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n_1) + T(n_2) & \text{where } n_1+n_2 \leq n \end{cases}$$

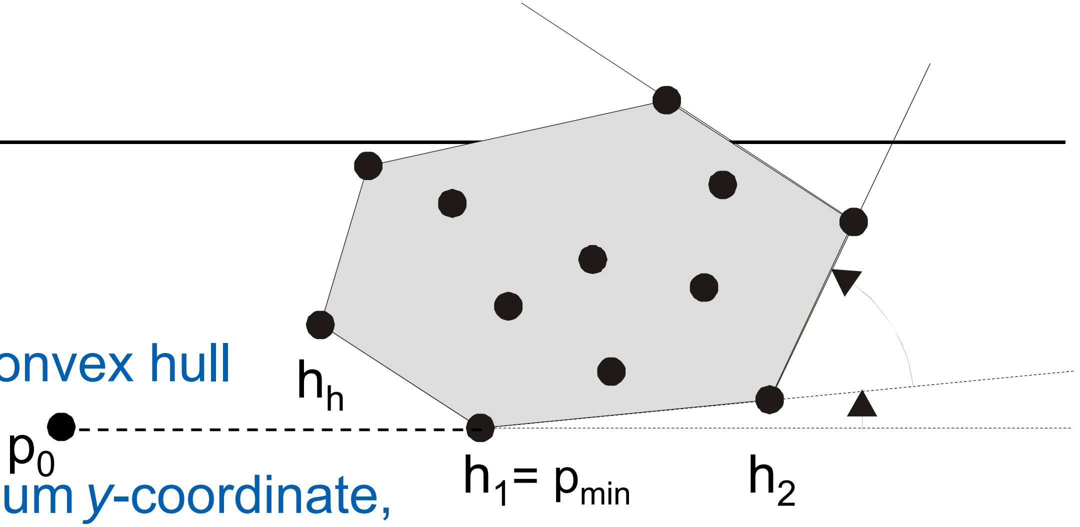


  - If evenly distributed that $\max(n_1, n_2) \leq \alpha n, 0 < \alpha < 1$ then solves as Quicksort to O(c*n* log *n*) where c=f(α) else O(n^2) for unbalanced splits
  - Output sensitive algorithm

Jarvis’s March – selection by gift wrapping

- Variant of $O(n^2)$ selection sort
- Output sensitive algorithm
- $O(nh)$ … h = *number of points on convex hull*

Jarvis’s March



JarvisCH(points P)
Input: points p
Output: CCW points on the convex hull

```
1. Take point $p_{min}$ with minimum $y$-coordinate,
   // $p_{min}$ will be the first point in the hull – append it to the hull as $h_1$
2. Take a horizontal line, i.e., create temporary point $p_0 = (-\infty, h_1.y)$
3. j = 1
4. repeat
5.     Rotate the line around $h_j$ until it bounces to the nearest point q = $p_q$
       // compute the smallest angle by the “smallest orient($h_{j-1}$, $h_j$, $q$)”   0..90°!
6.     j++
       append the bounced nearest point q to the hull as next $h_j$
7. until ($q \neq p_{min}$)
```

Output sensitive algorithm

Complexity: $O(n) + O(n) * h \Rightarrow O(h*n)$

good for low number of points on convex hull

Output sensitive algorithm

- Worst case complexity analysis analyzes the worst case data
  - Presumes, that all (const. fraction of) points lie **on** the CH

    $\sim n$ points on CH
  - The points are ordered along CH

    => We need sorting => $\Omega(n \log n)$ of CH algorithm
- Such assumption is rare
  - usually only much less of points are on CH

    $\ll n$ points on CH
- Output sensitive algorithms
  - Depend on: input size n and the size of the output h
  - Are more efficient for small output sizes
- Reasonable time for CH is $O(n \log h)$

---

h = Number of points on the CH

Chan’s algorithm

Cleverly combines Graham’s scan and Jarvis’s march algorithms

Goal is $O(n \log h)$ running time

h points on CH

- We cannot afford sorting of all points - $\Omega(n \log n)$

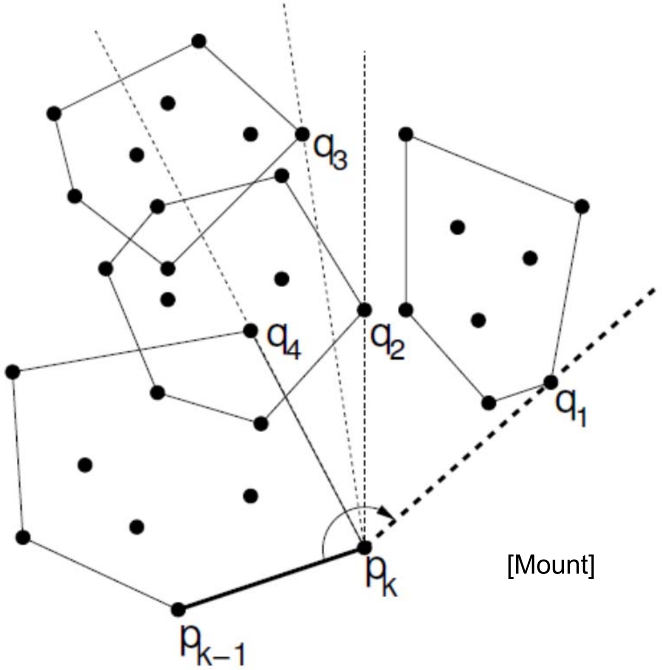
=> Idea: work on parts, limit the part sizes to polynomial h^c
the complexity does not change => $\log h^c = \log h$

- *h* is unknown – we get the estimation later
- Use estimation m, better not too high => $h \leq m \leq h^2$

1. Partition points *P* into r-groups of size m, $r = n/m$
2. Merge r-group CHs as “fat points”

Chan's algorithm

1. Partition points *P* into r-groups of size m, $r = n/m$
   - Each group take $O(m \log m)$ time - sort + Graham
   - r-groups take $O(r\, m \log m) = O(n \log m)$ – Jarvis



[Mount]

↓

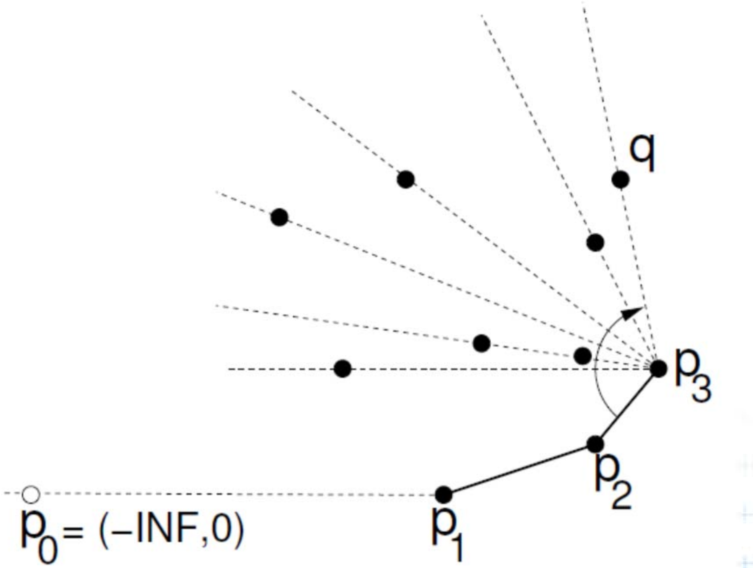
$$h \le m \le h^2$$

↓

goal $O(n \log h)$

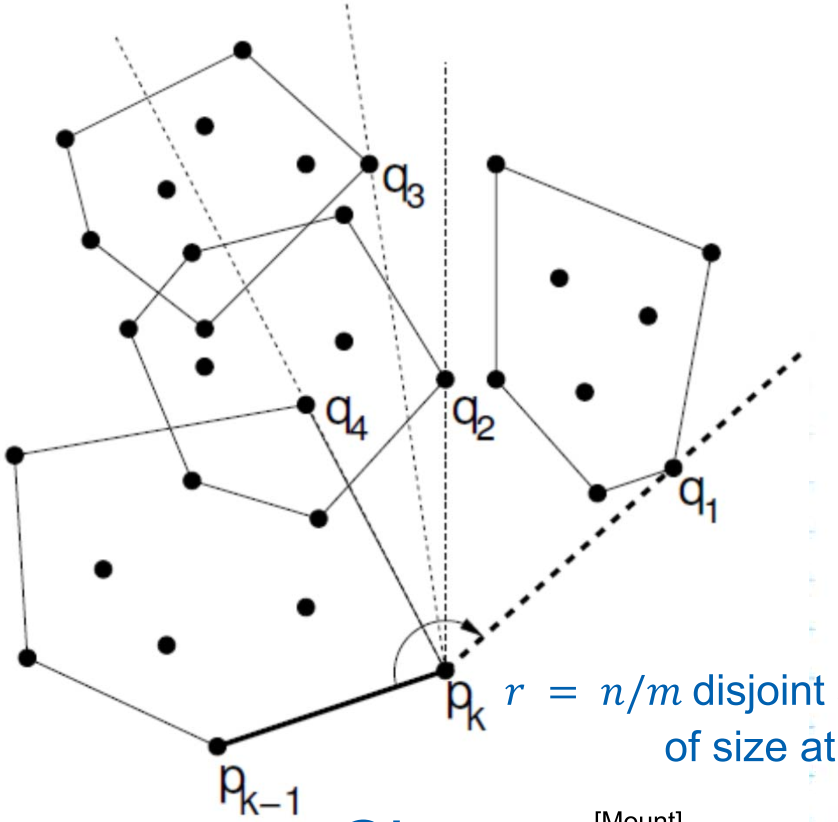
Merging of *m* parts in Chan's algorithm

2. Merge *r*-group CHs as "fat points"
   - Tangents to convex m-gon can be found in O(log *m*) by binary search



Jarvis

[Mount]



$r = n/m$ disjoint subsets of size at most m

Chan

[Mount]

Chan’s algorithm complexity

- *h* points on the final convex hull
  - => at most h steps in the Jarvis march algorithm
  - – each step computes r-tangents, $O(\log m)$ each
  - – merging together $O(hr \log m)$

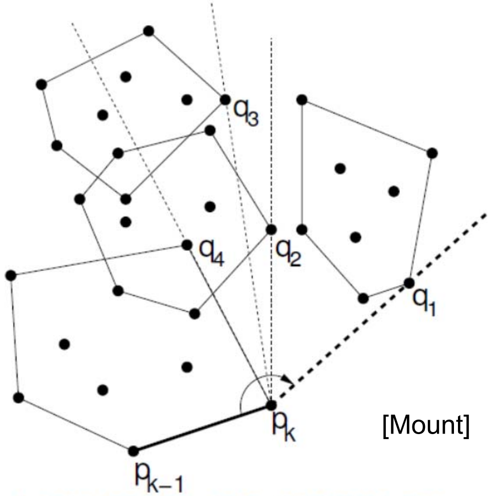
r-groups of size m, $r = n/m$

- Complete algorithm O(*n* log *h*)
  - – Graham’s scan on partitions $O(r\, m \log m) = O(n \log m)$
  - – Jarvis Merging: $O(hr \log m) = O(h\, n/m \log m)$, …4a)

    $h \leq m \leq h^2$ $= O(n \log m)$
  - – Altogether $\underline{O(n \log m)}$
  - – How to guess m? *Wait!*

1) use m as an estimation of h 2) if it fails, increase m

Chan's algorithm for known *m*



[Mount]

PartialHull(*P*, *m*)
Input: points P
Output: group of size *m*

```
1. Partition $P$ into $r = \lceil n/m \rceil$ disjoint subsets $\{p_1, p_2, \ldots, p_r\}$ of size at most $m$
2. for $i=1$ to $r$ do
     a) Convex hull by GrahamsScan($P_i$), store vertices in ordered array
3. let $p_1$ = the bottom most point of P and $p_0 = (-\infty, p_1.y)$
4. for $k = 1$ to $m$ do      // compute merged hull points
     a) for $i = 1$ to $r$ do  // angle to all $r$ subsets => points $q_i$
            Compute the point $q_i \in P$ that maximizes the angle $\angle\, p_{k-1}, p_k, q_i$
     b) let $p_{k+1}$ be the point $q \in \{q_1, q_2, \ldots, q_r\}$ that maximizes $\angle\, p_{k-1}, p_k, q$
        ($p_{k+1}$ is the new point in CH)
     c) if $p_{k+1} = p_1$ then return $\{p_1, p_2, \ldots, p_k\}$
5. return "Fail, $m$ was too small"
```

$O(\log m)$

Jarvis

Chan’s algorithm – estimation of *m*

```
ChansHull
Input:   points P
Output:  convex hull $p_1 \dots p_k$

1. for $t$ = 1, 2, … , $[\lg \lg h]$ do {
     a) let $m = \min(2^{2^{\wedge}t}, n)$          $m = \{2^2, 2^4, 2^8, \dots, n\}$
     b) $L$ = PartialHull( $P$, $m$)
     c) if $L \neq$ “Fail, $m$ was too small” then return $L$
   }
```

Sequence of choices of *m* are { 4, 16, 256,…, $2^{2^{\wedge}t}$,…, *n* } … squares

Example: for h = 23 points on convex hull of n = 57 points, the algorithm will try this sequence of choices of *m* { 4, 16, ~~256~~, 57 }

1. 4 and 16 will fail
2. 256 will be replaced by *n=57*

Complexity of Chan’s Convex Hull?

- The worst case: Compute all *t* iterations
- t^th iteration takes $O(n \log \overset{m}{2^{2^t}}) = O(n\, 2^t)$

one iteration

- Algorithm stops when $2^{2^t} \geq h \Rightarrow t = \lceil \lg \lg h \rceil$
- All $t = \lceil \lg \lg h \rceil$ iterations take:

Using the fact that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

t iterations

$$\sum_{t=1}^{\lg \lg h} n 2^t = n \sum_{t=1}^{\lg \lg h} 2^t \leq n 2^{\overset{1+k}{1+\lg \lg h}} = 2n \lg h = O(n \log h)$$

2x more work in the worst case

Conclusion in 2D

- Graham's scan: $O(n \log n), O(n)$ for sorted pts
- Divide & Conquer: $O(n \log n)$
- Quick hull: $O(n \log n), \max O(n^2)$ ~ distrib.
- Jarvis's march: $O(hn), \max O(n^2)$ ~ pts on CH
- Chan's alg.: $O(n \log h)$ ~ pts on CH

asymptotically optimal

but

constants are too high to be useful

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