Description Logics – Reasoning

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Outline

What can we conclude from description logics?

Which clinical findings can occur on a head?

How: Get subclasses of $\mathit{Finding} \sqcap \exists \mathit{FindingSite} \cdot \mathit{Head}$

e.g. *Heavyhead*, resulting from

$\mathit{Headache} \equiv \mathit{Pain} \sqcap \exists \mathit{FindingSite} \cdot \mathit{Head}$
$\mathit{Pain} \sqsubseteq \mathit{Finding}$
$\mathit{HeavyHead} \sqsubseteq \mathit{Headache}$

Which properties do I have to fill in when recording an allergic head?

How: For each property p check $AllergicHead \sqsubseteq \exists p \cdot T$

e.g. $FindingSite$**, resulting from**

$Pain \sqsubseteq \exists FindingSite \cdot T$
$ImmuneFunctionDisorder \sqsubseteq \exists PathologicalProcess \cdot T$
$AllergicHead \sqsubseteq Pain$
$AllergicHead \sqsubseteq ImmuneFunctionDisorder$

Is a Headache occurring in a Leg correct?

How: Check satisfiability of the concept $Headache \sqcap \exists FindingSite \cdot Leg$

No, because the concept is unsatisfiable, resulting from

$Headache \sqsubseteq Pain \sqcap \exists FindingSite \cdot Head$
$Pain \sqsubseteq \leq 1 FindingSite \cdot T$
$Leg \sqsubseteq \neg Head$

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- S is consistent, if S has at least one model

Inference problems

Inference Problems in TBOX

We have introduced syntax and semantics of the language $\mathcal{ALC}$. Now, let's look on automated reasoning. Having a $\mathcal{ALC}$ theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. For TBOX $\mathcal{T}$ and concepts $C_{(i)}$, we want to decide whether

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All these tasks can be reduced to unsatisfiability checking of a single concept ...

Reducting Subsumption to Unsatisfiability

Example

These reductions are straighforward – let's show, how to reduce subsumption checking to unsatisfiability checking. Reduction of other inference problems to unsatisfiability is analogous.

$$
\begin{aligned}
(\mathcal{T} \models C_1 \sqsubseteq C_2) & & \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad \mathcal{I} \models C_1 \sqsubseteq C_2) & \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}) & \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad C_1^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus C_2^{\mathcal{I}}) \subseteq \emptyset & \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \Longrightarrow & \quad \mathcal{I} \models C_1 \sqcap \neg C_2 \sqsubseteq \bot & \text{iff} \\
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\end{aligned}
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All these tasks, as well as concept unsatisfiability checking, can be reduced to consistency checking. Under which condition and how ?

Reduction of concept unsatisfiability to theory consistency

Example

Consider an $\mathcal{ALC}$ theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a concept C and a fresh individual a_f not occuring in $\mathcal{K}$:

$$
\begin{aligned}
(\mathcal{T} \models C \sqsubseteq \bot) & \qquad \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies \mathcal{I} \models C \sqsubseteq \bot) & \qquad \text{iff} \\
(\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies C^{\mathcal{I}} \subseteq \emptyset) & \qquad \text{iff} \\
\neg \left[(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge C^{\mathcal{I}} \nsubseteq \emptyset) \right] & \qquad \text{iff} \\
\neg \left[(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge a_f^{\mathcal{I}} \in C^{\mathcal{I}}) \right] & \qquad \text{iff} \\
(\mathcal{T}, \{C(a_f)\}) \quad \text{is inconsistent} &
\end{aligned}
$$

Note that for more expressive description logics than $\mathcal{ALC}$, the ABOX has to be taken into account as well due to its interaction with TBOX.

Inference Algorithms

Inference Algorithms in Description Logics

Structural Comparison is polynomial, but complete just for some simple DLs *without full negation*, e.g. $\mathcal{ALN}$, see [**dlh2003**].

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other ... – e.g. resolution-based, transformation to finite automata, etc.

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We will introduce tableau algorithms.

Tableaux Algorithms

(TAs are not new in DL – they were known in predicate logics as well.)

Main idea

"ABOX $\mathcal{A}$ is consistent w.r.t. TBOX $\mathcal{T}$ if we find a model of $\mathcal{T} \cup \mathcal{A}$." (similarly for theory $\mathcal{K}$ as a whole)

Each TA can be seen as a *production system* :

state ($\sim$ data base) containing a set of *completion graphs* (see next slides),

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strategy for picking the most suitable rule for application

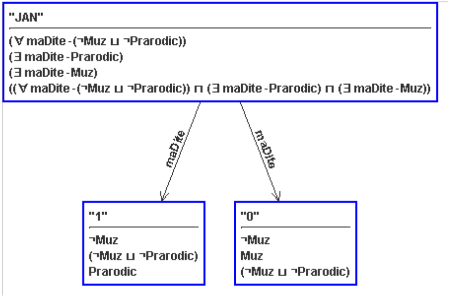
Completion Graphs

(Do not mix with *complete graphs* from the graph theory.)

Completion graph

is a labeled oriented graph $G = (V_G, E_G, L_G)$, where each

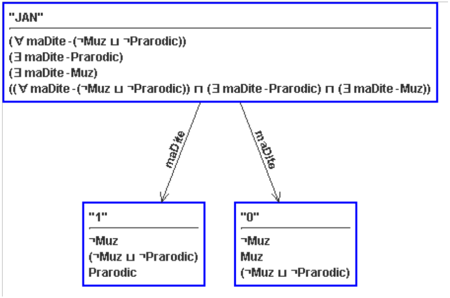
- node $x \in V_G$ is labeled with a set $L_G(x)$ of concepts and
- each edge $\langle x, y \rangle \in E_G$ is labeled with a set of edges $L_G(\langle x, y \rangle)$ (or shortly $L_G(x, y)$)



Completion Graphs

Direct Clash

occurs in a completion graph $G = (V_G, E_G, L_G))$, if $\{A, \neg A\} \subseteq L_G(x)$, or $\perp \in L_G(x)$ for some atomic concept A and a node $x \in V_G$



Completion Graphs

Complete Completion Graph

is a completion graph $G = (V_G, E_G, L_G))$, to which no inference rule can be applied (any more).

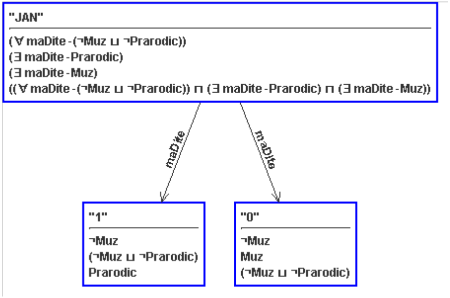


Tableau Algorithm for $\mathcal{ALC}$

Tableau Algorithm for $\mathcal{ALC}$ when $\mathcal{T} = \emptyset$

Let's have $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T} = \emptyset$ for now.

0 (Preprocessing) Transform all concepts appearing in $\mathcal{K}$ to the "negational normal form" (NNF), "shifting" negation $\neg$ to the atomic concepts (using equivalent operations known from propositional and predicate logics).

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- Sets $V_{G_0}, E_{G_0}, L_{G_0}$ are smallest possible with these properties.

Tableau algorithm for $\mathcal{ALC}$ without TBOX (2)

...

2. Current algorithm state is S. If each $G \in S$ contains a direct clash, terminate as "INCONSISTENT".

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3 Let's choose one $G \in S$ that doesn't contain a direct clash. If G is *complete* w.r.t. rules shown next, terminate as "CONSISTENT"

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3. Let's choose one $G \in S$ that doesn't contain a direct clash. If G is *complete* w.r.t. rules shown next, terminate as "CONSISTENT"
4. Find a rule that is applicable to G and apply it. As a result, we obtain from the state S a new state S'. Jump to step 2.

TA for $\mathcal{ALC}$ without TBOX – Inference Rules

$\rightarrow_{\sqcap}$ rule

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$\rightarrow_{\sqcap}$ rule

if $(C_1 \sqcap C_2) \in L_G(a)$ and $\{C_1, C_2\} \nsubseteq L_G(a)$ for some $a \in V_G$.

TA for $\mathcal{ALC}$ without TBOX – Inference Rules

$\rightarrow_\sqcap$ rule

if $(C_1 \sqcap C_2) \in L_G(a)$ and $\{C_1, C_2\} \not\subseteq L_G(a)$ for some $a \in V_G$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, and $L_{G'}(a) = L_G(a) \cup \{C_1, C_2\}$ and otherwise is the same as L_G.

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TA for $\mathcal{ALC}$ without TBOX – Inference Rules

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$\rightarrow_\exists$ rule

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TA for $\mathcal{ALC}$ without TBOX – Inference Rules

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TA Run Example

Example – Consistency Checking

$\mathcal{K}_2 = (\emptyset, \mathcal{A}_2)$, where $\mathcal{A}_2 = \{(\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \neg\exists maDite \cdot (Muz \sqcap Prarodic))(JAN)\})$.

- Let's transform the concept into NNF: $\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \forall maDite \cdot (\neg Muz \sqcup \neg Prarodic)$

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TA Run Example (2)

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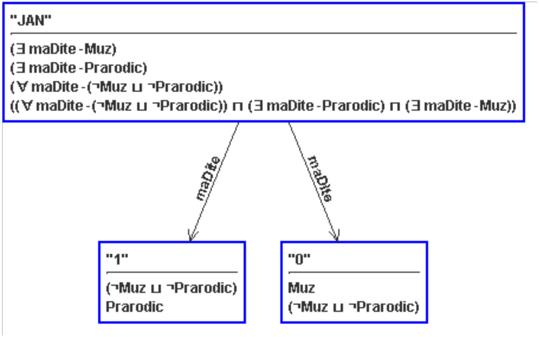
- $\{G_0\} \xrightarrow{\sqcap\text{-rule}} \{G_1\} \xrightarrow{\exists\text{-rule}} \{G_2\} \xrightarrow{\exists\text{-rule}} \{G_3\} \xrightarrow{\forall\text{-rule}} \{G_4\}$, where G_4 is



TA Run Example (3)

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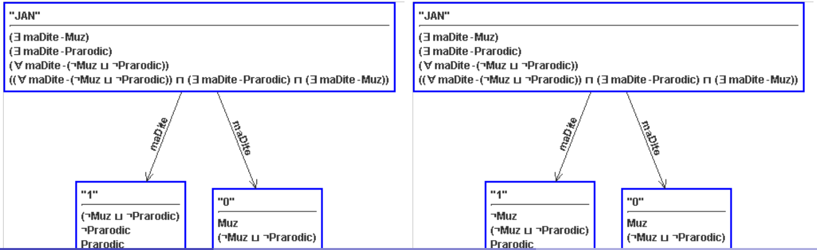
- Now, we have to apply the $\sqcup$-rule to the concept $\neg Muz \sqcup \neg Rodic$ either in the label of node "0", or in the label of node "1". Its application e.g. to node "1" we obtain the state $\{G_5, G_6\}$ (G_5 left, G_6 right)



TA Run Example (4)

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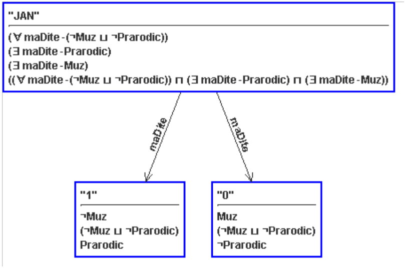
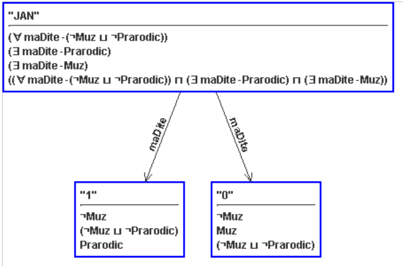
- We see that G_5 contains a direct clash in node "1". The only other option is to go through the graph G_6. By application of $\sqcup$-rule we obtain the state $\{G_5, G_7, G_8\}$, where G_7 (left), G_8 (right) are derived from G_6 :





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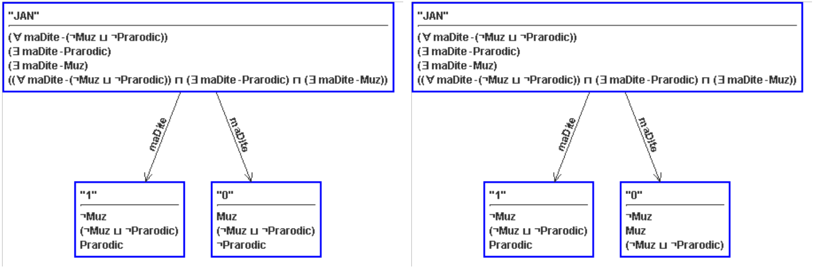

- G_7 is complete and without direct clash.

TA Run Example (5)

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- after application of any of the following rules $\rightarrow_{\sqcap}, \rightarrow_{\exists}, \rightarrow_{\forall}$ graph G is either enriched with a new node, new edge, or labeling of an existing node/edge is enriched. All these operations are finite.

Relation between ABOXes and Completion Graphs

We define also $\mathcal{I} \models G$ iff $\mathcal{I} \models \mathcal{A}_G$, where $\mathcal{A}_G$ is an ABOX constructed from G, as follows

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- For other rules, the soundness is shown in a similar way.

Completeness

- To prove completeness of the TA, it is necessary to construct a model for each complete completion graph G that doesn't contain a direct clash. Canonical model $\mathcal{I}$ can be constructed as follows:
  - the domain $\Delta^{\mathcal{I}}$ will consist of all nodes of G.
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- What about complexity of the algorithm ?
  - P-SPACE (between NP and EXP-TIME).

What if $\mathcal{T}$ is not empty?

- consider $\mathcal{T}$ containing axioms of the form $C_i \sqsubseteq D_i$ for $1 \leq i \leq n$. Such $\mathcal{T}$ can be transformed into a single axiom

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  where $\top_C$ denotes a concept $(\neg C_1 \sqcup D_1) \sqcap \ldots \sqcap (\neg C_n \sqcup D_n)$
- for each model $\mathcal{I}$ of the theory $\mathcal{K}$, each element of $\Delta^{\mathcal{I}}$ must belong to $\top_C^{\mathcal{I}}$. How to achieve this ?

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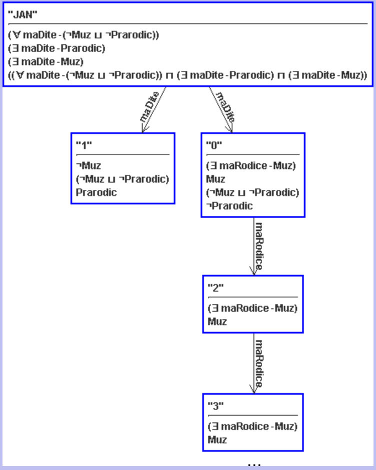
then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, a $L_{G'}(a) = L_G(a) \cup \{\top_C\}$ and otherwise is the same as L_G.

Example

Consider $\mathcal{K}_3 = (\{Muz \sqsubseteq \exists maRodice \cdot Muz\}, \mathcal{A}_2)$. Then $\top_C$ is $\neg Muz \sqcup \exists maRodice \cdot Muz$. Let's use the introduced TA enriched by $\rightarrow_{\sqsubseteq}$ rule. Repeating several times the application of rules $\rightarrow_{\sqsubseteq}$, $\rightarrow_{\sqcup}$, $\rightarrow_{\exists}$ to G_7 (that is not complete w.r.t. to $\rightarrow_{\sqsubseteq}$ rule) from the previous example we can get into an infinite loop

General Inclusions (3)

Example



. . . this algorithm doesn't necessarily terminate ☹.

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- $\exists-$ rule is only applicable if the node a_1 in its definition is not blocked by another node.

Blocking in TA (2)

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- **Introduced TA with subset blocking is sound, complete and finite decision procedure for $\mathcal{ALC}$.**

Let's play . . .

- http://kbss.felk.cvut.cz/tools/dl

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