VIR2: MLE for classification

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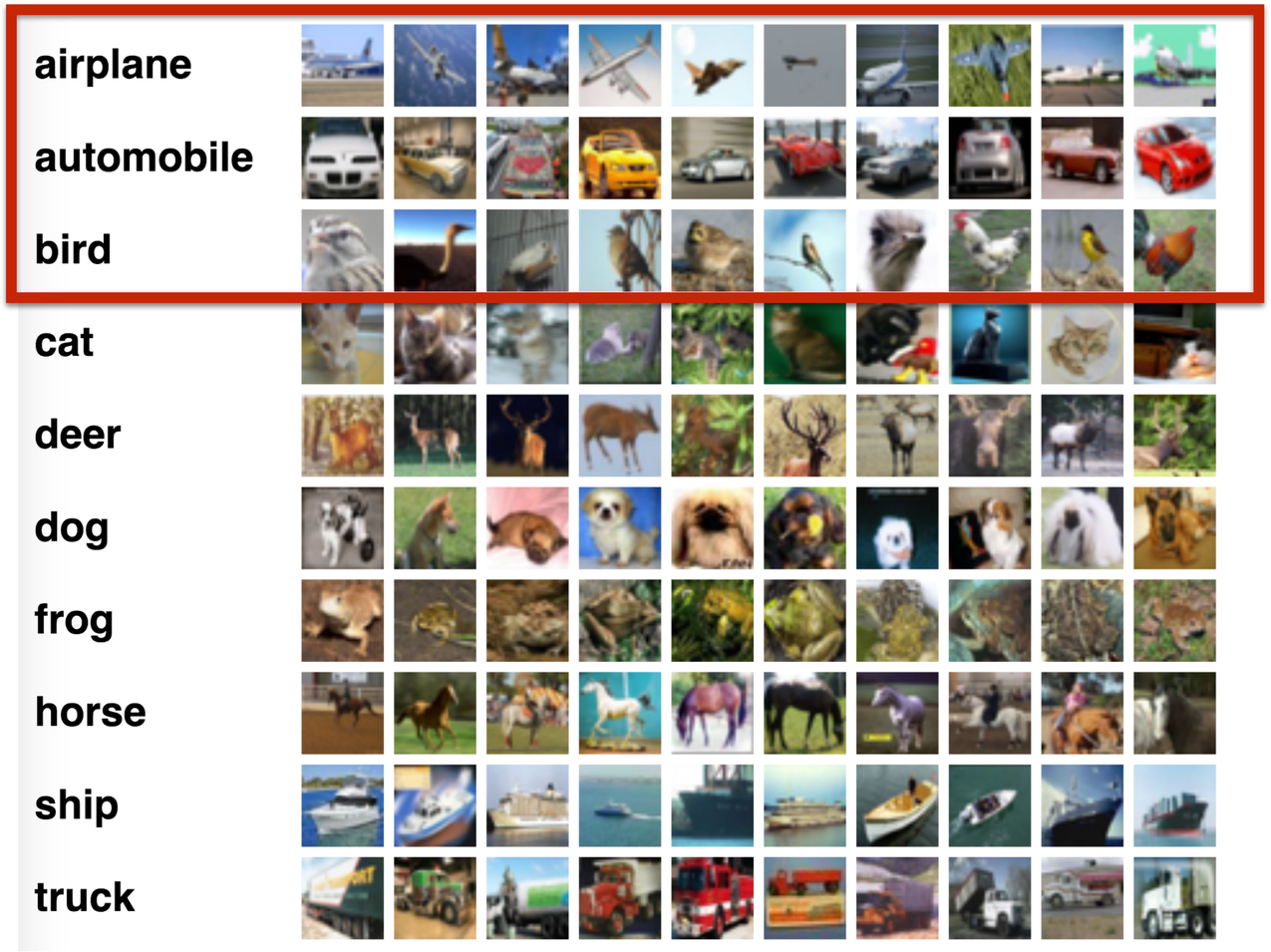
Outline

- Pre-requisites: linear algebra, Bayes rule
- MAP/ML estimation, prior and overfitting
- Linear regression
- Linear classification

Recognition problem



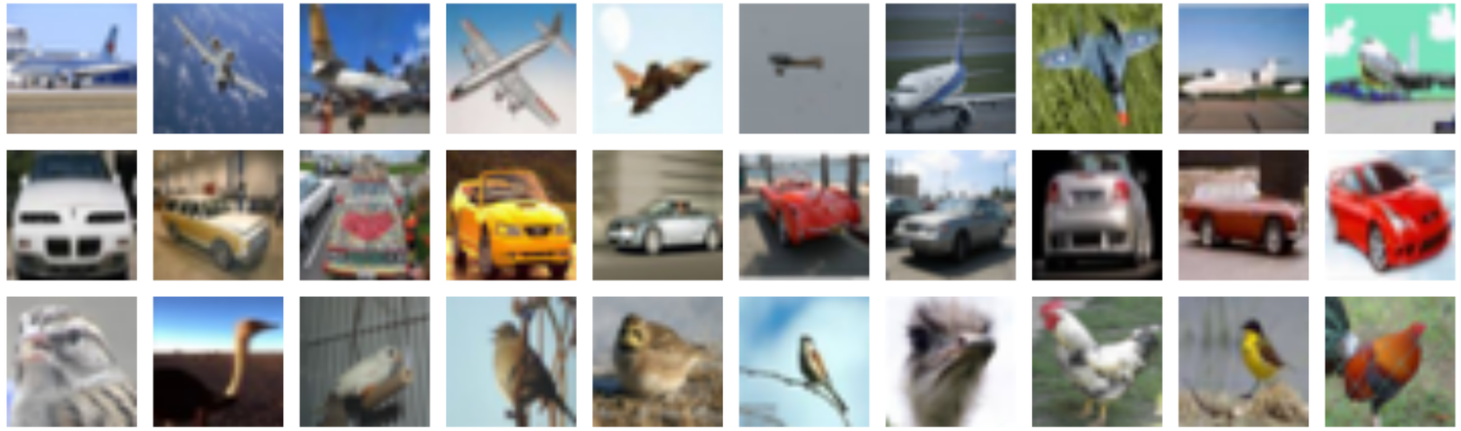
Why is it hard?

CIFAR-10: classify 32x32 RGB images into 10 categories

https://www.cs.toronto.edu/~kriz/cifar.html

Labels (y_i)

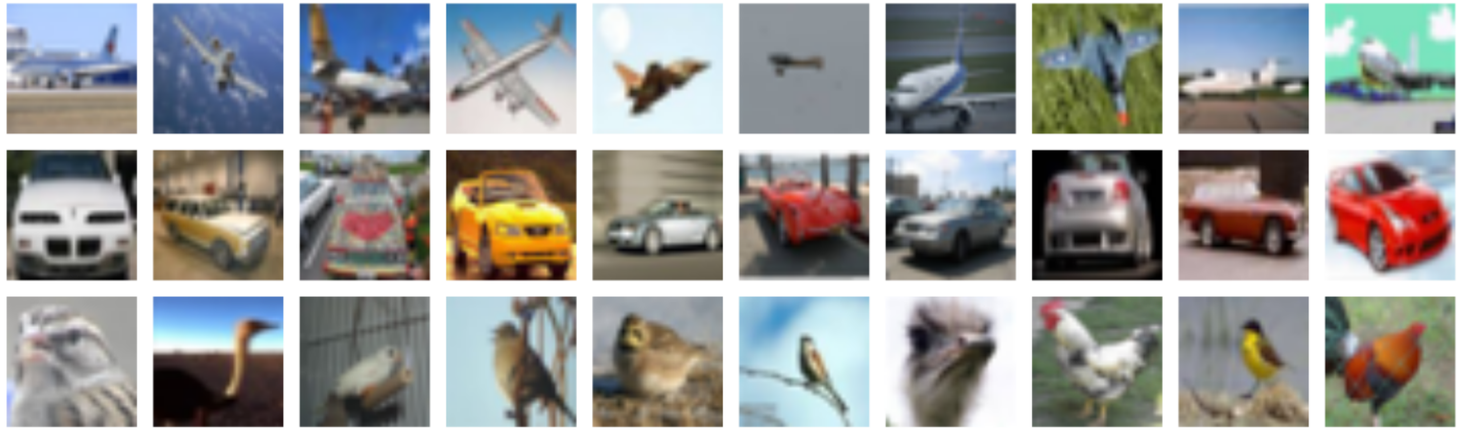
RGB images ($\mathbf{x}_i$)



Three-class recognition problem:

Labels (y_i)

RGB images ($\mathbf{x}_i$)



Three-class recognition problem:

def classify():

???

return $\mathbf{p}$

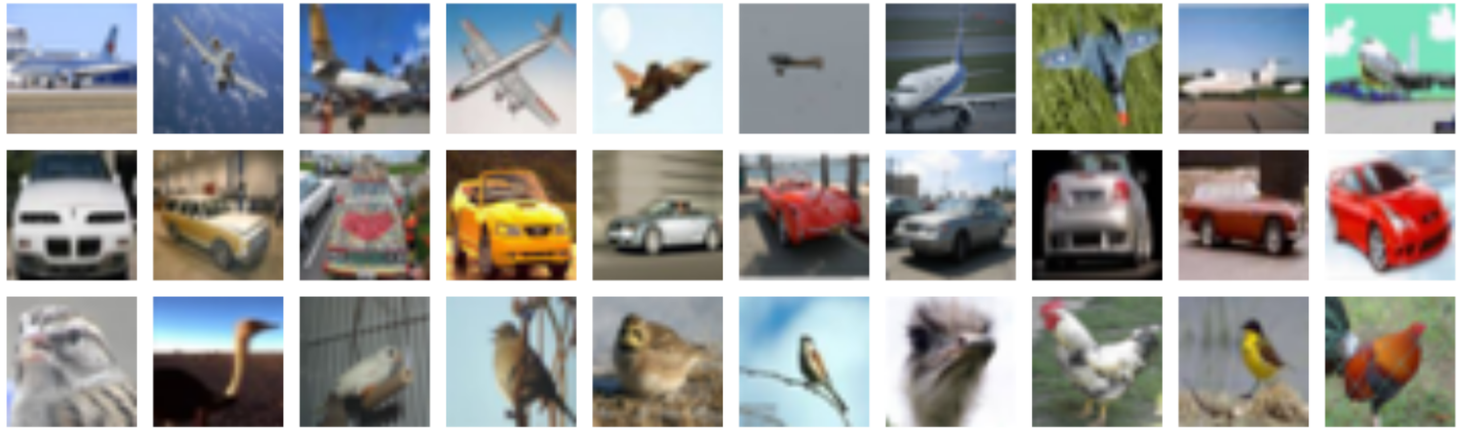
Labels (y_i)

RGB images ($\mathbf{x}_i$)

1

2

3



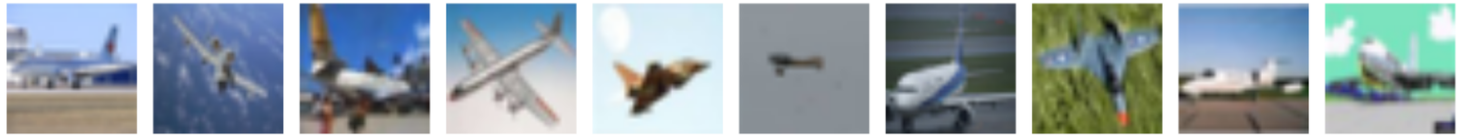
Model probability distribution over classes by softmax function

$$p(y|\mathbf{x},\mathtt{W}) = \begin{bmatrix} \exp(f(\mathbf{x},\mathbf{w}_1)) \\ \exp(f(\mathbf{x},\mathbf{w}_2)) \\ \exp(f(\mathbf{x},\mathbf{w}_3)) \end{bmatrix} / \sum_k \exp(f(\mathbf{x},\mathbf{w}_k) = \mathbf{s}(\mathbf{f}(\mathbf{x},\mathtt{W}))$$

Labels (y_i)

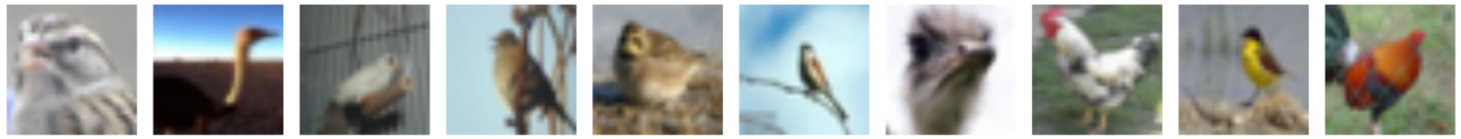
RGB images ($\mathbf{x}_i$)

1



2

3



Example: Linear classifier:

def classify():

 # Linear classifier

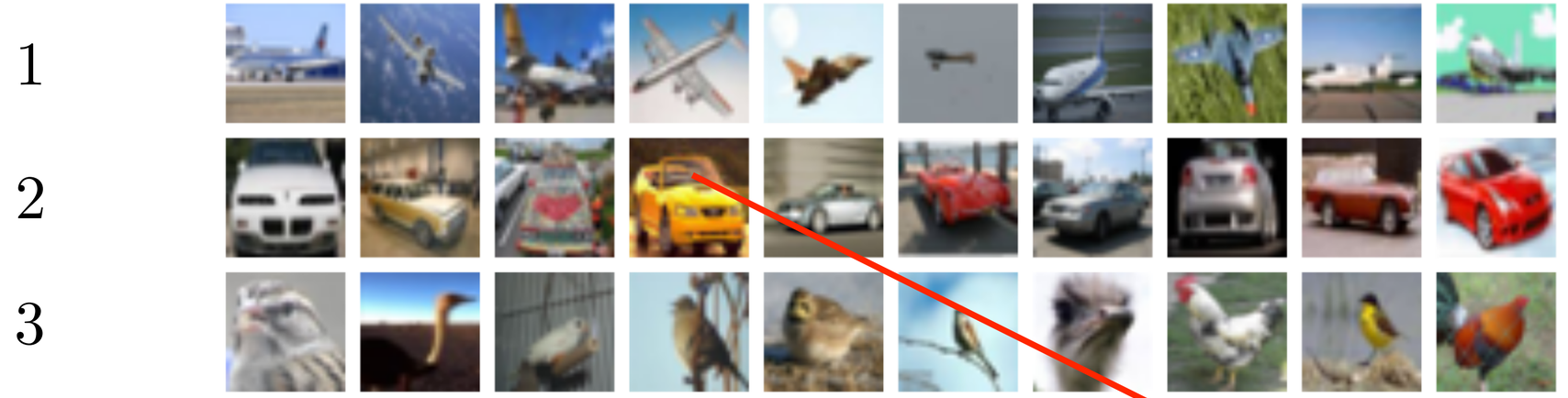
$\mathbf{x} = \text{vec}($  $)$

$\mathbf{p} = \mathbf{s}(\mathtt{W}\,\overline{\mathbf{x}})$

return $\mathbf{p}$

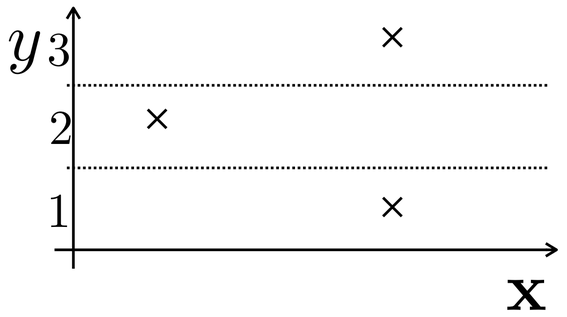
$$\boxed{\mathtt{W}}\ \boxed{\overline{\mathbf{x}}} = \begin{bmatrix} -2 \\ +1 \\ 0 \end{bmatrix}$$

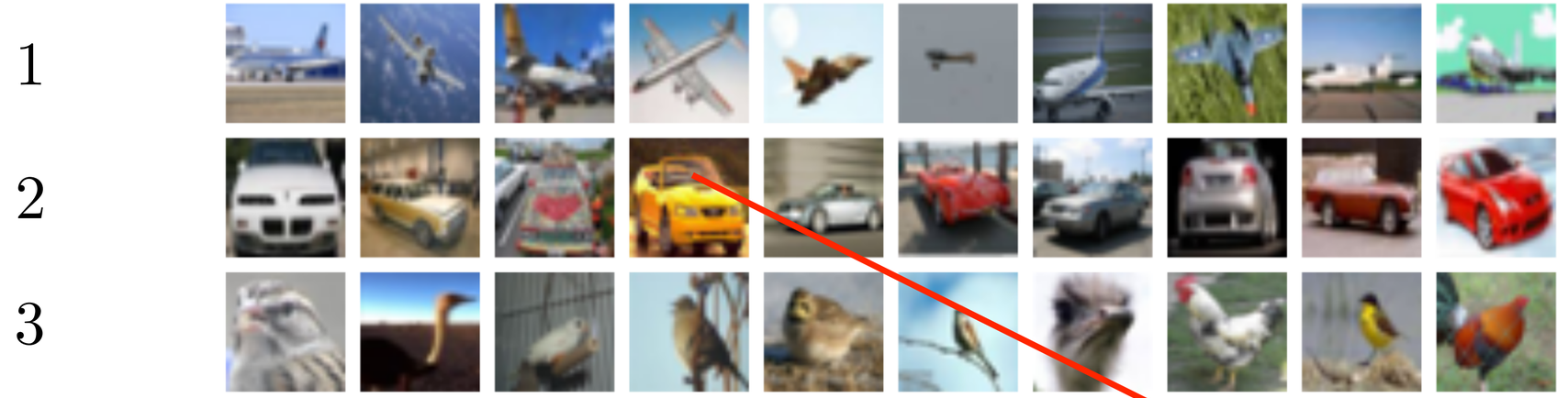
$$\mathbf{s}\left(\begin{bmatrix} -2 \\ +1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0.03 \\ 0.71 \\ 0.26 \end{bmatrix}$$



- **Classification** (probability modeled by soft-max function):

$$p(y|\mathbf{x}, \mathtt{W}) = \begin{bmatrix} \exp(f(\mathbf{x}, \mathbf{w}_1)) \\ \exp(f(\mathbf{x}, \mathbf{w}_2)) \\ \exp(f(\mathbf{x}, \mathbf{w}_3)) \end{bmatrix} / \sum_k \exp(f(\mathbf{x}, \mathbf{w}_k) = \mathbf{s}(\mathbf{f}(\mathbf{x}, \mathtt{W}))$$





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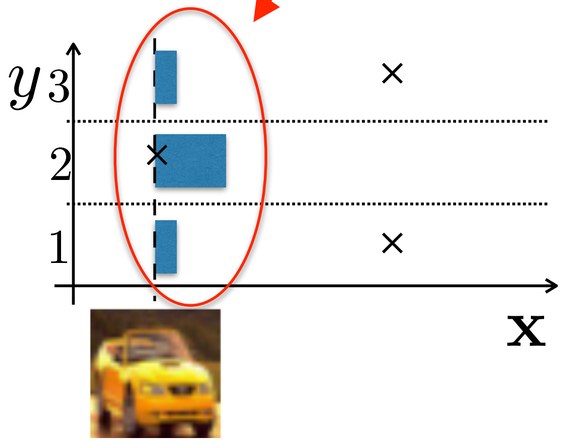

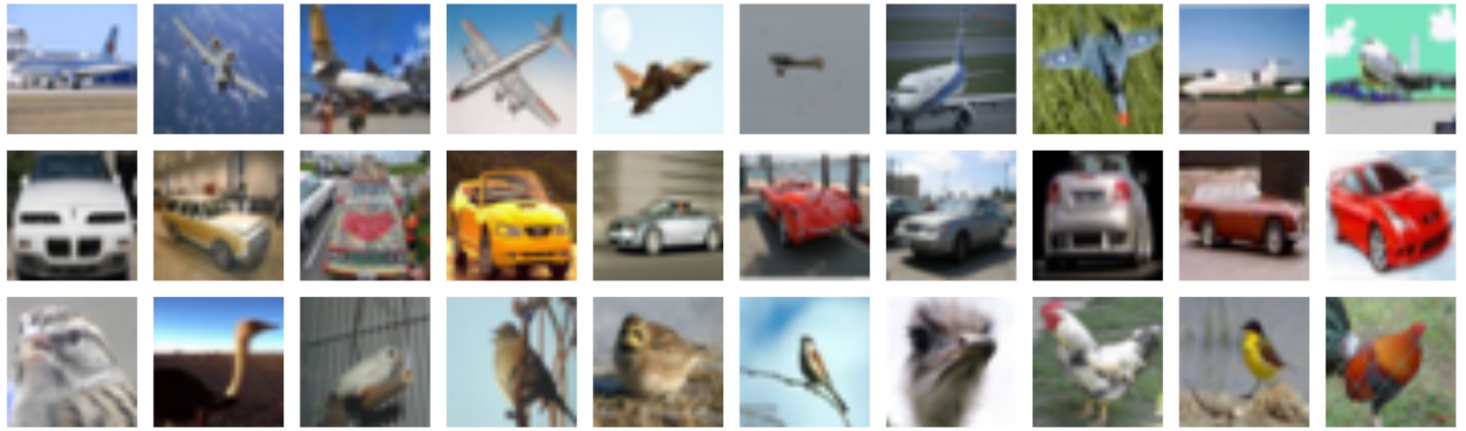

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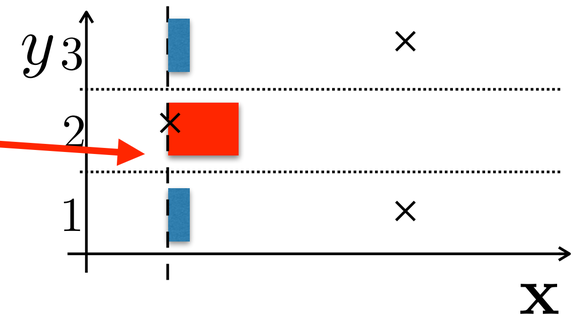
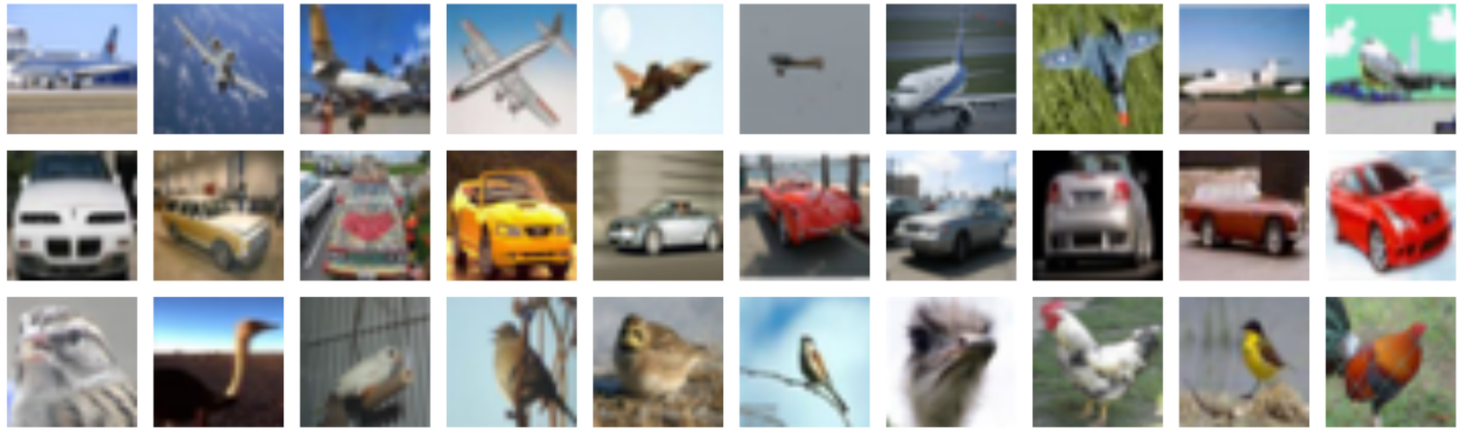
- Probability of observing y_i when measuring $\mathbf{x}_i$ is

$$p(y_i|\mathbf{x}_i, \mathtt{W}) = \mathbf{s}_{y_i}(\mathbf{f}(\mathbf{x}_i, \mathtt{W}))$$





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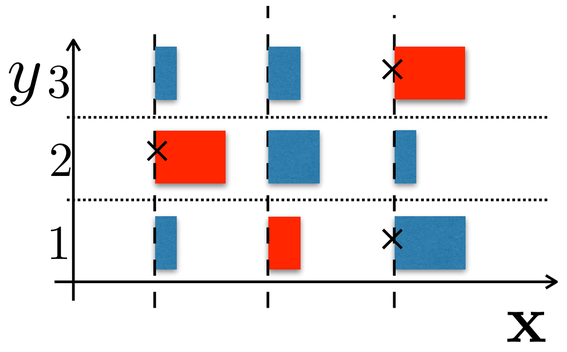
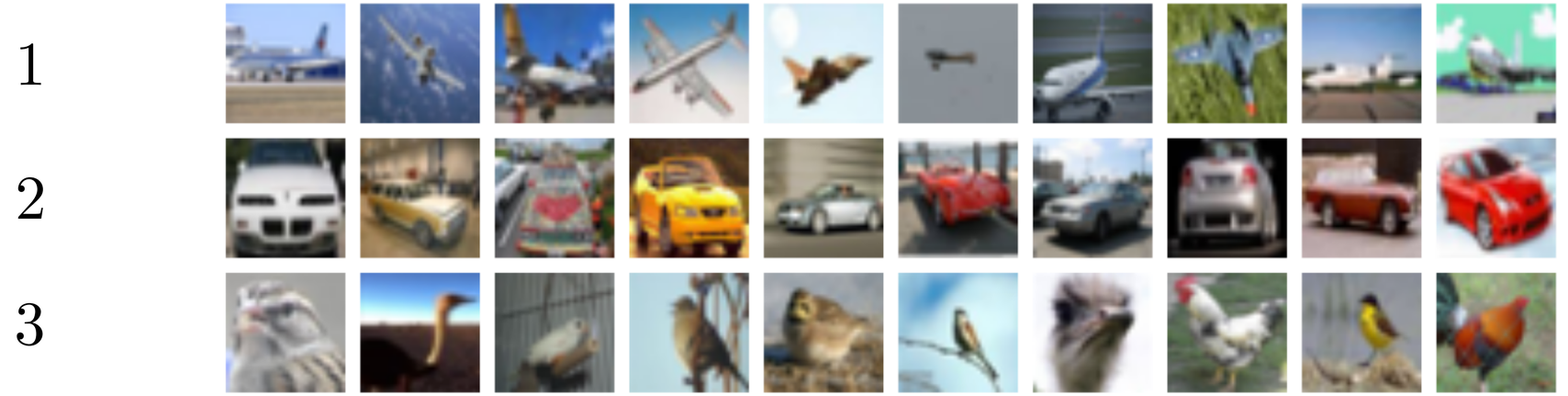


- **Training:** MLE estimate of W

$$\mathtt{W}^* = \arg\min_{\mathtt{W}} \sum_i -\log \mathbf{s}_{y_i}(\mathbf{f}(\mathbf{x}_i, \mathtt{W}))$$



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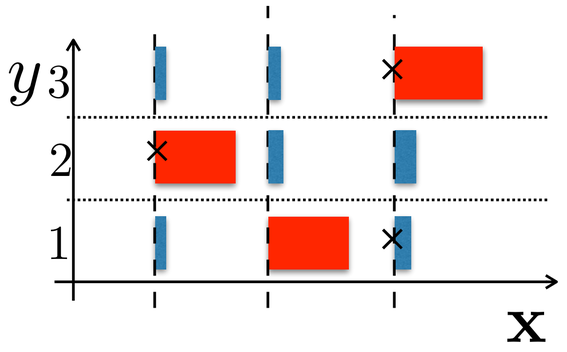

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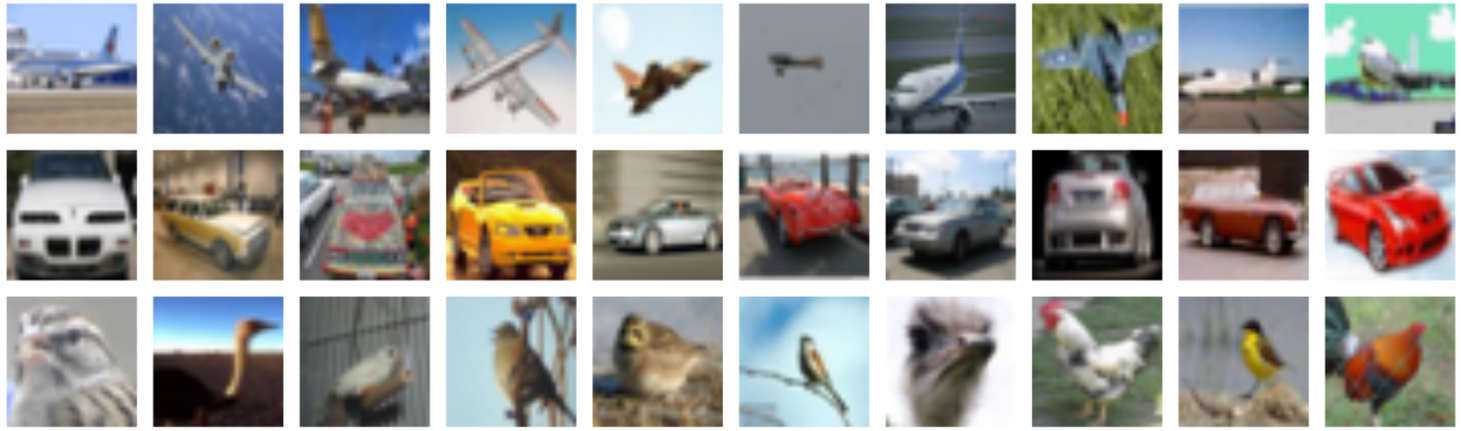

def train():

1 1 1 2 2 2 3 3 3

$\mathbf{x}_i = \text{vec}($  $)$

$$\mathtt{W}^* = \arg\min_{\mathtt{W}} \sum_i -\log \mathbf{s}_{y_i}(\mathtt{W}\,\overline{\mathbf{x}}_i))$$

return $\mathtt{W}^*$

label
$y_i = 2$

$$\mathbf{s}(\mathtt{W}\,\overline{\mathbf{x}}_i) = \begin{bmatrix} 0.03 \\ 0.71 \\ 0.26 \end{bmatrix}$$

Car classified as car yields small loss and gradient

$$\Rightarrow \quad -\log \mathbf{s}_{y_i}(\mathtt{W}\,\overline{\mathbf{x}}_i) = -\log(0.71) = 0.15$$

def train():

1 1 1 2 2 2 3 3 3

$$\mathbf{x}_i = \text{vec}(\ \ )$$

$$\mathtt{W}^* = \arg\min_{\mathtt{W}} \sum_i -\log \mathbf{s}_{y_i}(\mathtt{W}\,\overline{\mathbf{x}}_i))$$

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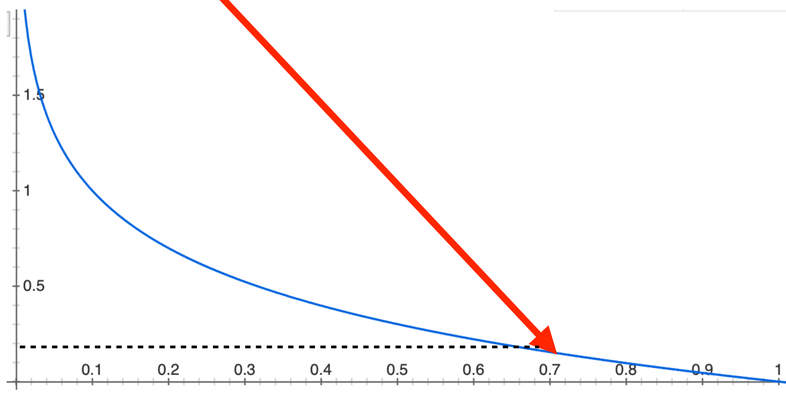

label
$y_i = 1$

$$\mathbf{s}(\mathtt{W}\,\overline{\mathbf{x}}_i) = \begin{bmatrix} 0.03 \\ 0.57 \\ 0.40 \end{bmatrix}$$

Plane classified as car yields huge loss and gradient

$$\Rightarrow \quad -\log \mathbf{s}_{y_i}(\mathtt{W}\,\overline{\mathbf{x}}_i) = -\log(0.03) = 1.52$$

def train():

1 1 1 2 2 2 3 3 3

$$\mathbf{x}_i = \text{vec}(\ \text{  }\ )$$

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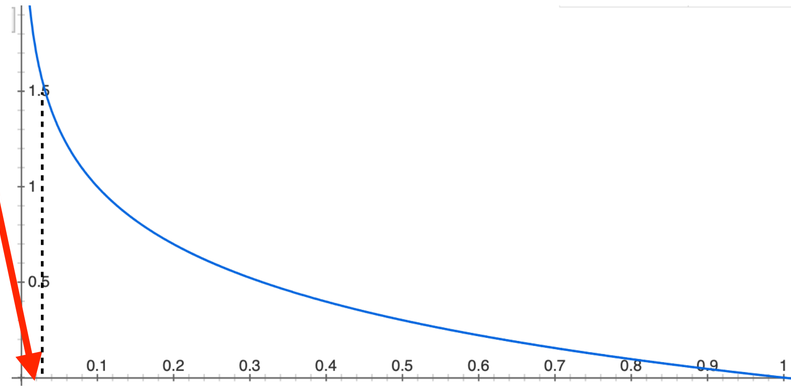
return $\mathtt{W}^*$



def train():

1 1 1 2 2 2 3 3 3

$\mathbf{x}_i = \text{vec}($  $)$

$$\mathtt{W}^* = \arg\min_{\mathtt{W}} \underbrace{\sum_i -\log \mathbf{s}_{y_i}(\mathtt{W}\,\overline{\mathbf{x}}_i)}_{\mathcal{L}(\mathbf{w})} = \sum_i \ell(\mathtt{W}, \overline{\mathbf{x}}_\mathtt{i}, y_i)$$

return $\mathtt{W}^*$

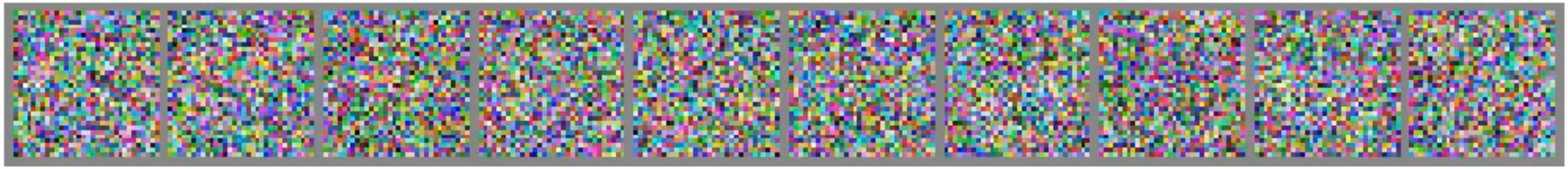
def train():

1 1 1 2 2 2 3 3 3

$\mathbf{x}_i = \mathrm{vec}($  $)$

$\mathtt{W}^* = \arg\min_{\mathtt{W}} \mathcal{L}(\mathtt{W})$ … gradient optimization

return $\mathtt{W}^*$



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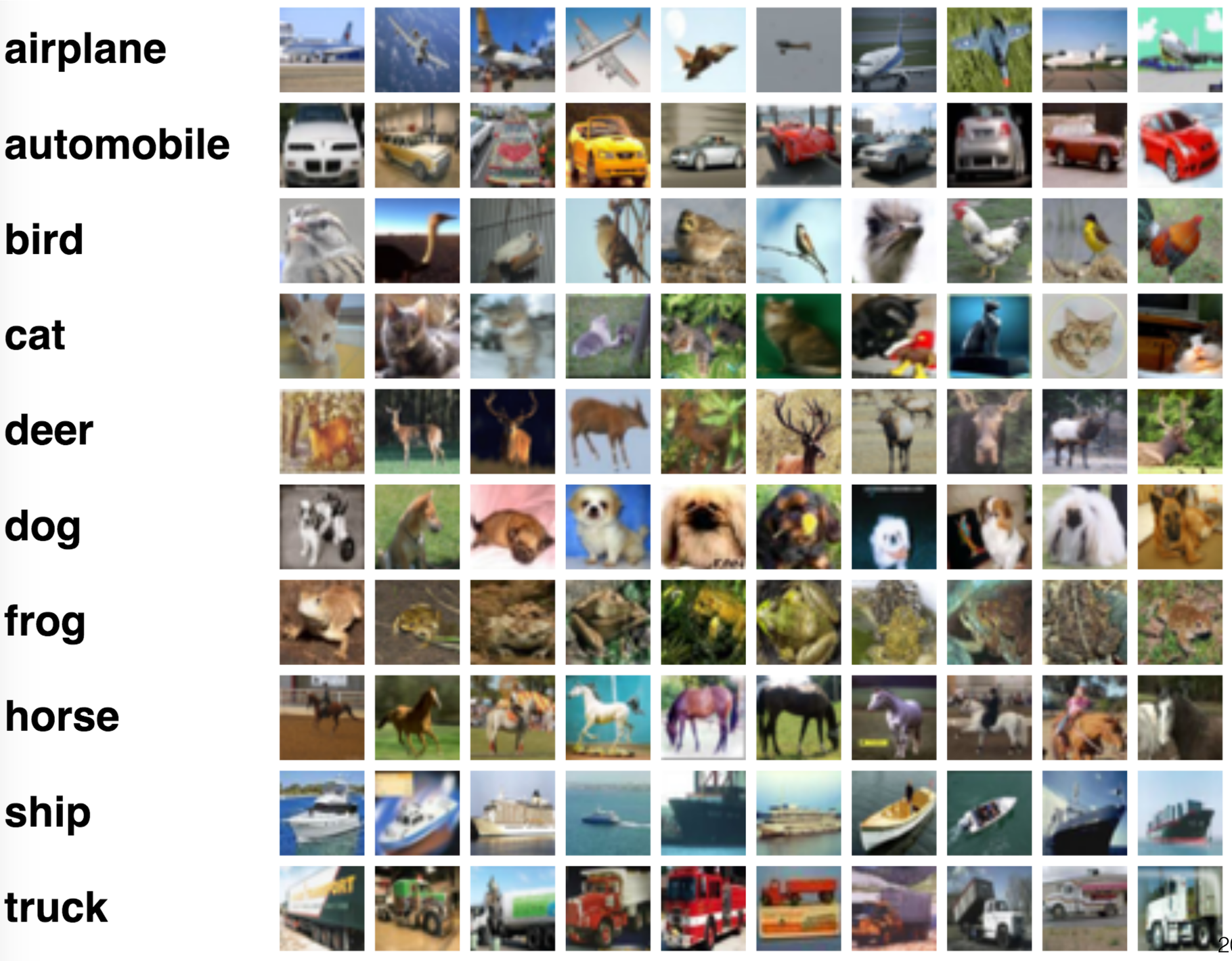
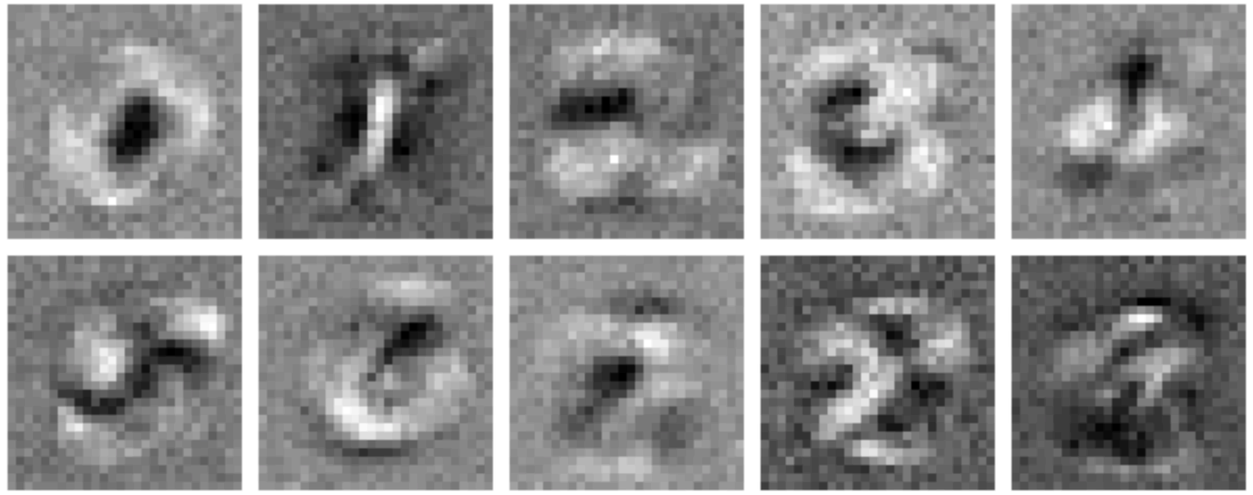
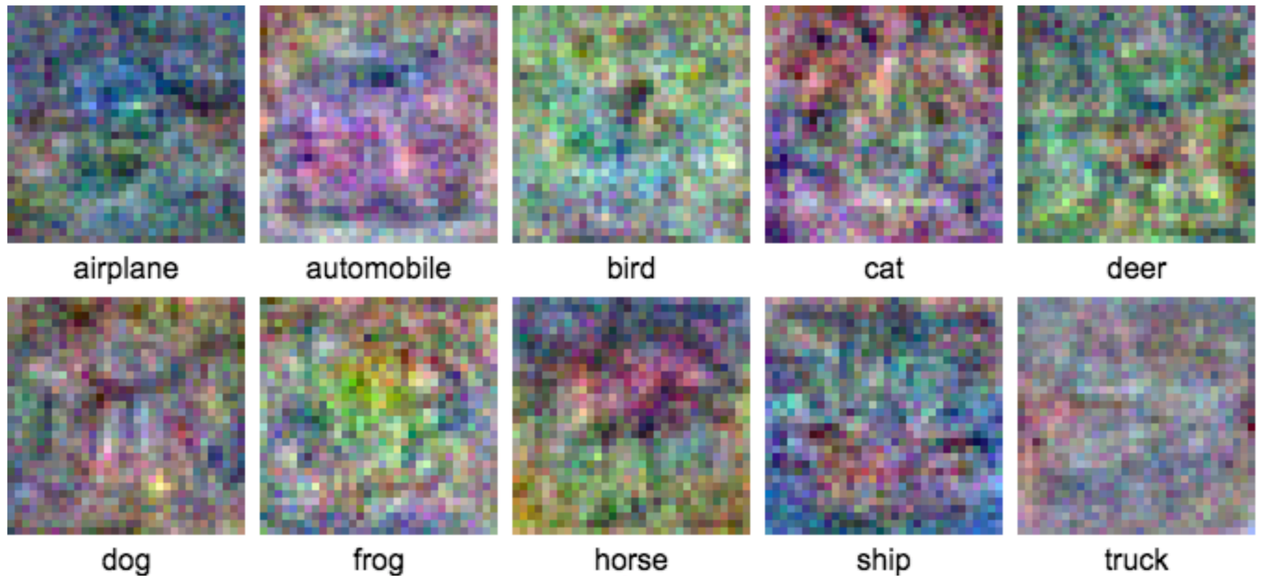
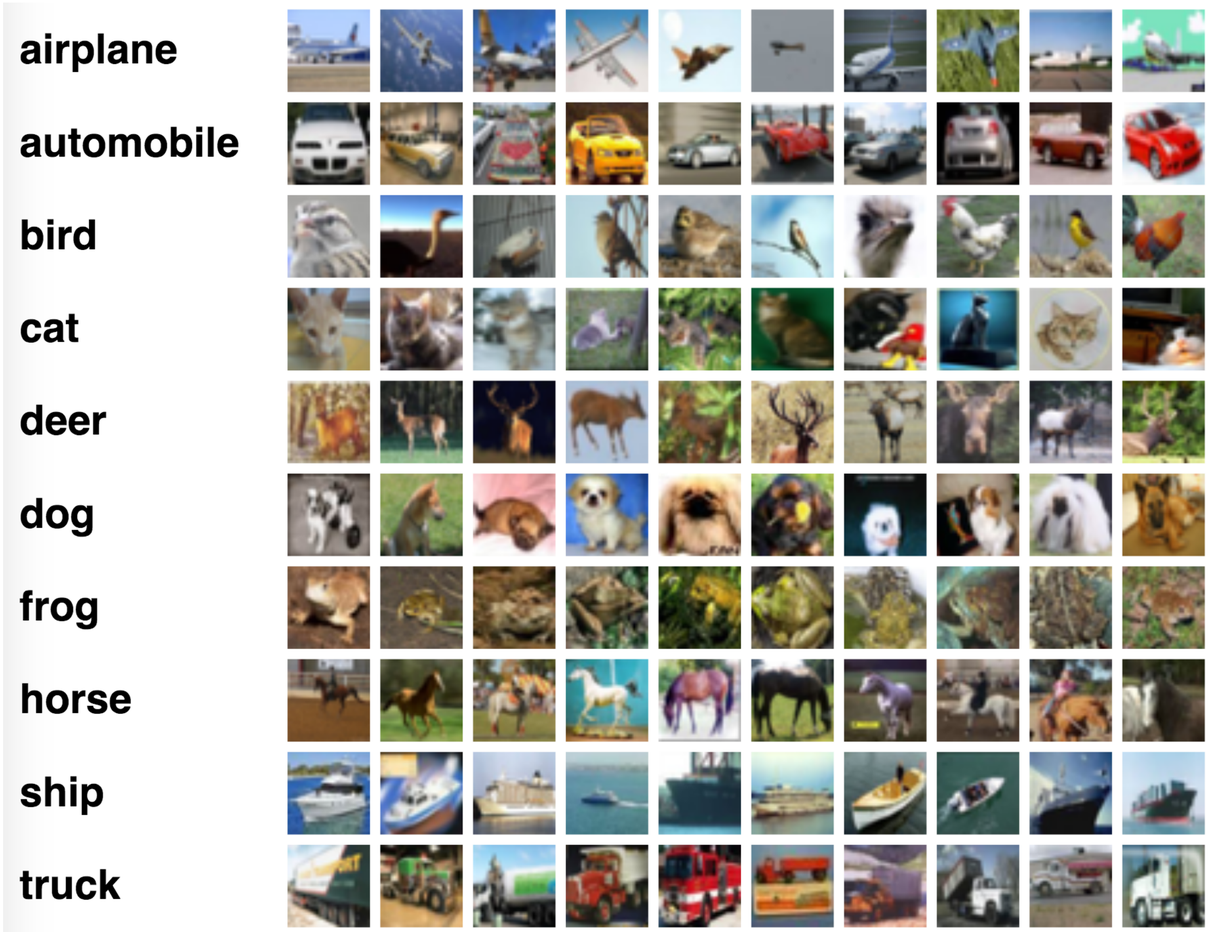

airplane
automobile
bird
cat
deer
dog
frog
horse
ship
truck

Dataset	Learned weights of linear classifier	Error
MNIST		8%
CIFAR-10		63%

https://benchmarks.ai

Recognition problem



Why is it hard?

CIFAR-10: classify 32x32 RGB images into 10 categories
https://www.cs.toronto.edu/~kriz/cifar.html

Recognition problem

Huge within-class variability!

Why it is hard?

- Viewpoint
- Occlusion
- Illumination
- Pose
- Type
- Context

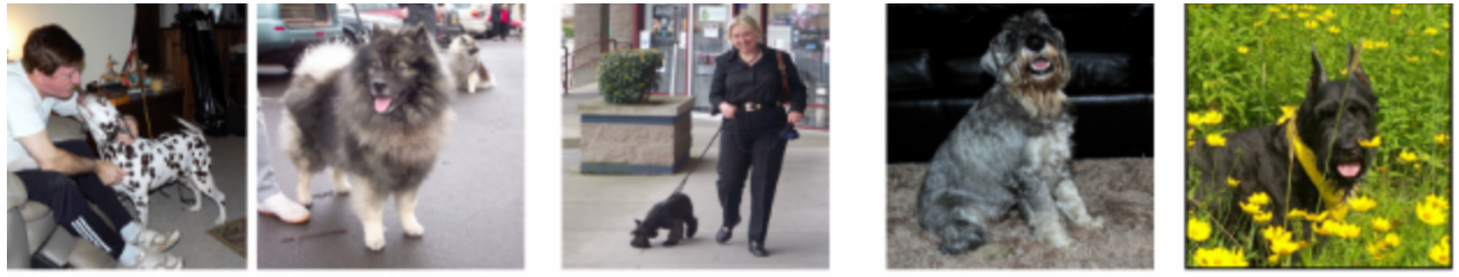
Why is it hard?

Huge (i) within-class variability

bird

cat

dog



Why is it hard?

Huge (i) within-class variability (ii) among class similarity!

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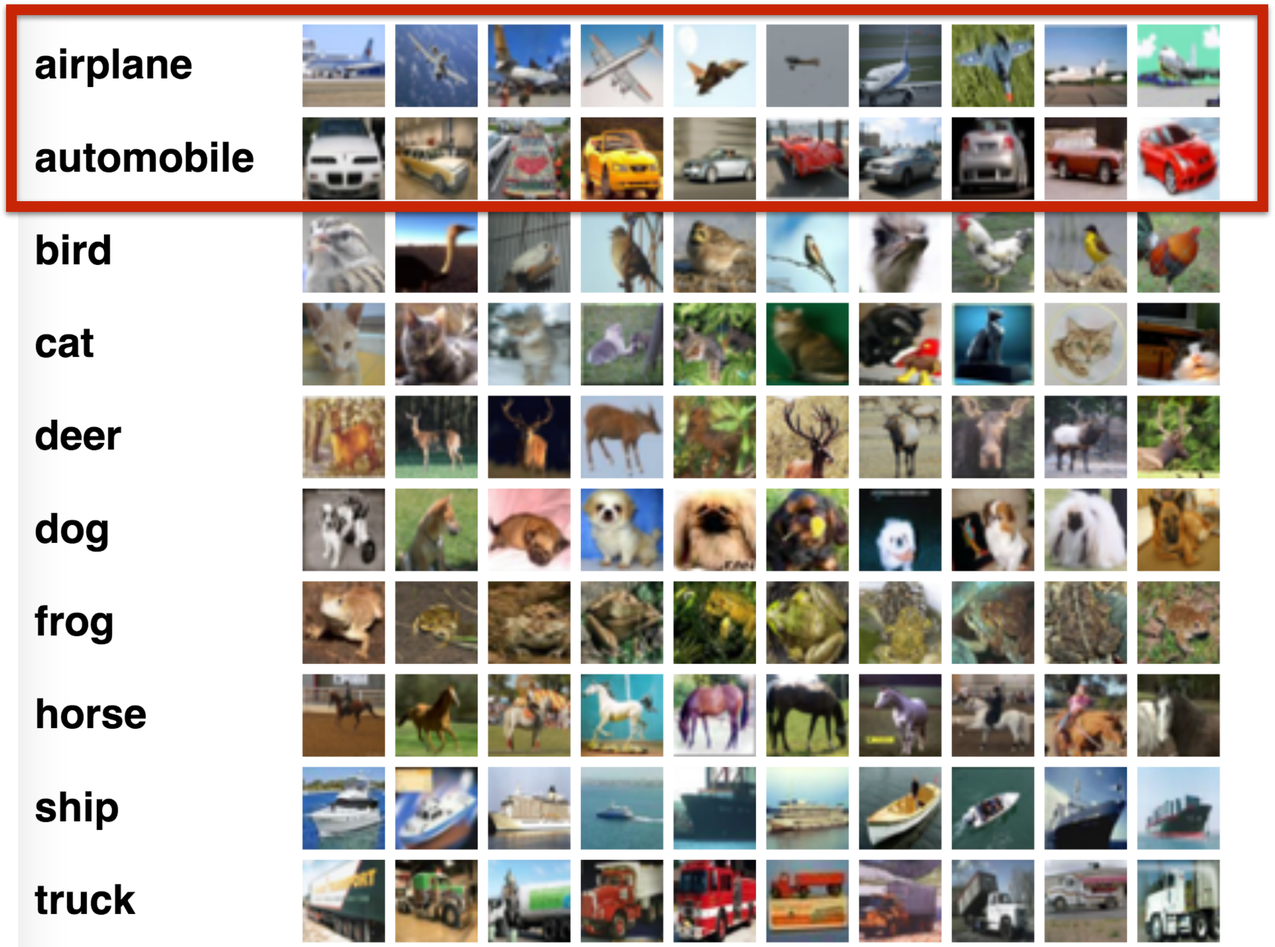

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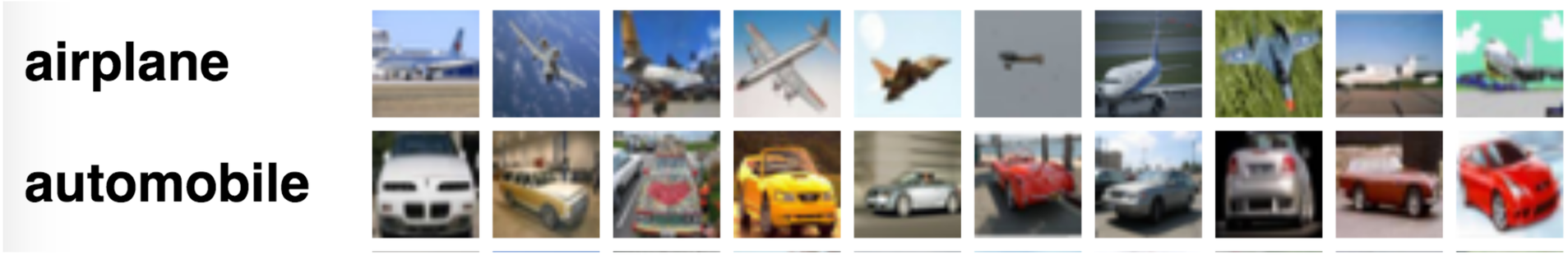
RGB images ($\mathbf{x}_i$)



Two-class recognition problem: classify airplane/automobile

def classify():

???

return p

Probability of image being from the class airplane

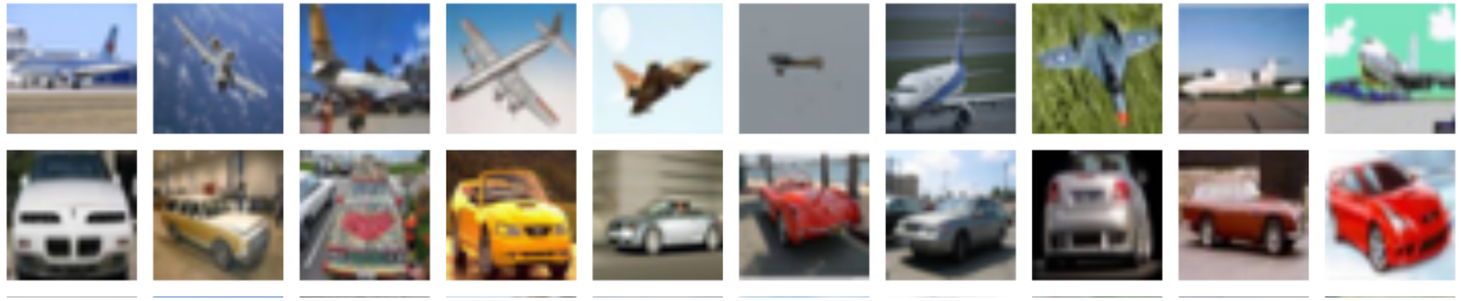
How to model it?

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Classification

We model probability of image $\mathbf{x}$ being label +1 or -1 as

$$p(y|\mathbf{x},\mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x},\mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x},\mathbf{w})) & y = -1 \end{cases}$$

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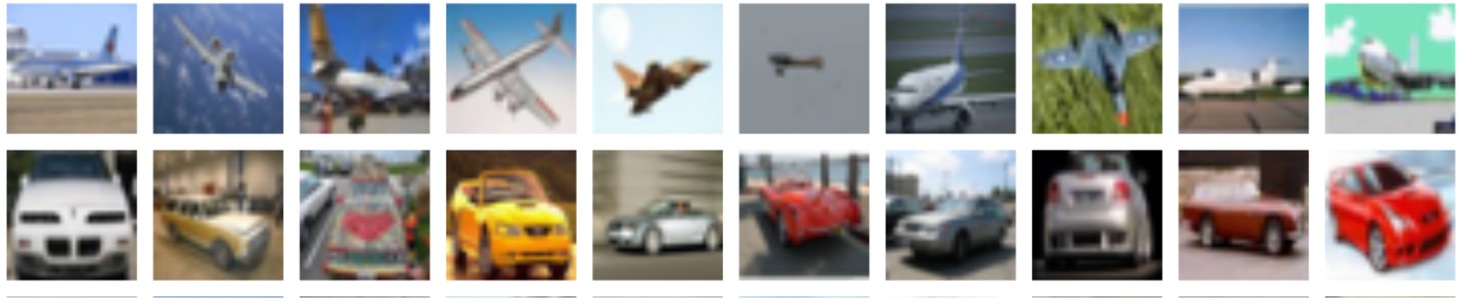
$+1$

-1



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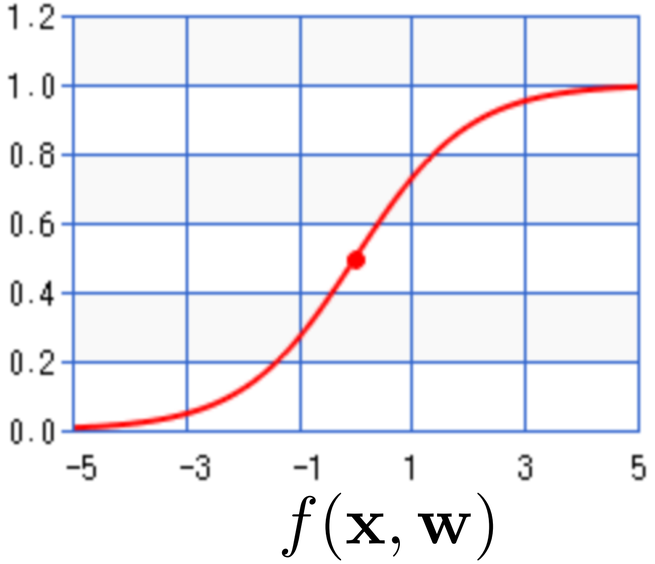
where

$$\sigma(f(\mathbf{x},\mathbf{w})) = \frac{1}{1+\exp(-f(\mathbf{x},\mathbf{w}))}$$

is sigmoid function.



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$+1$

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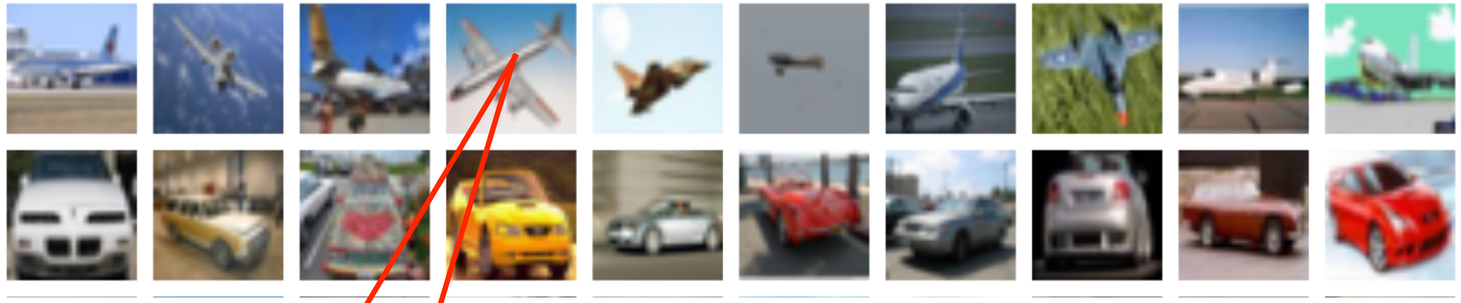

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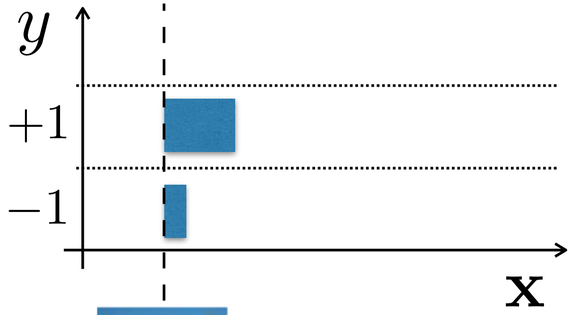

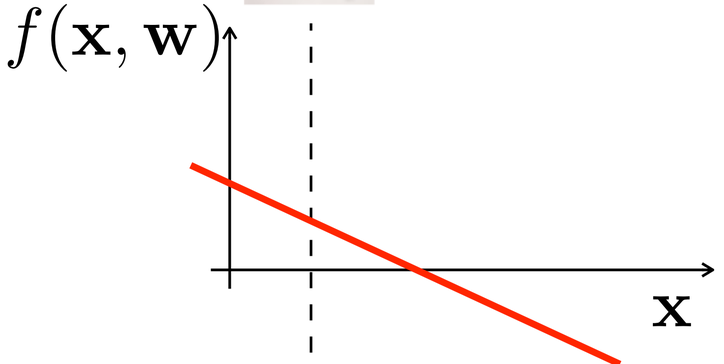

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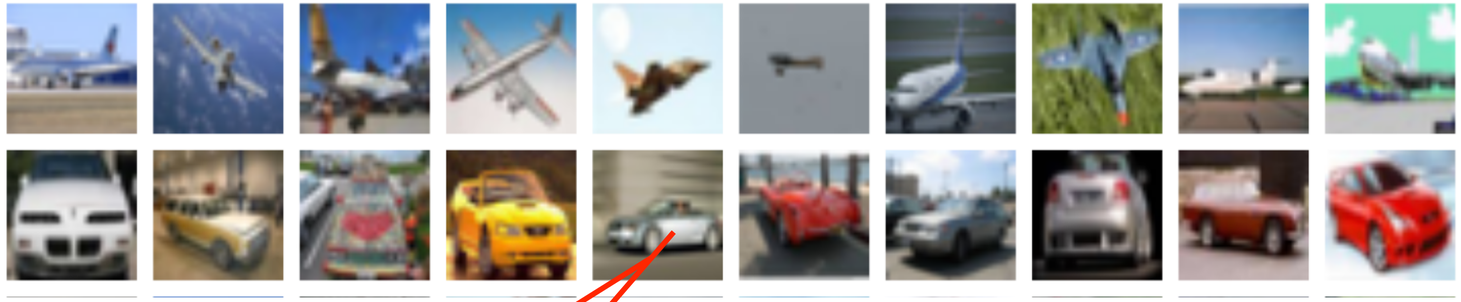

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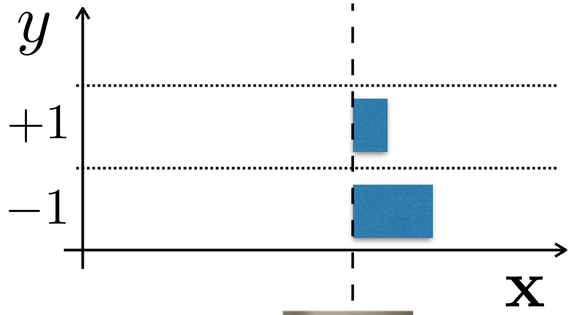

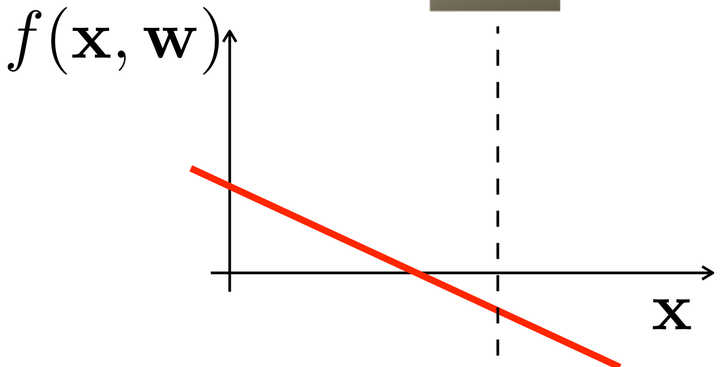

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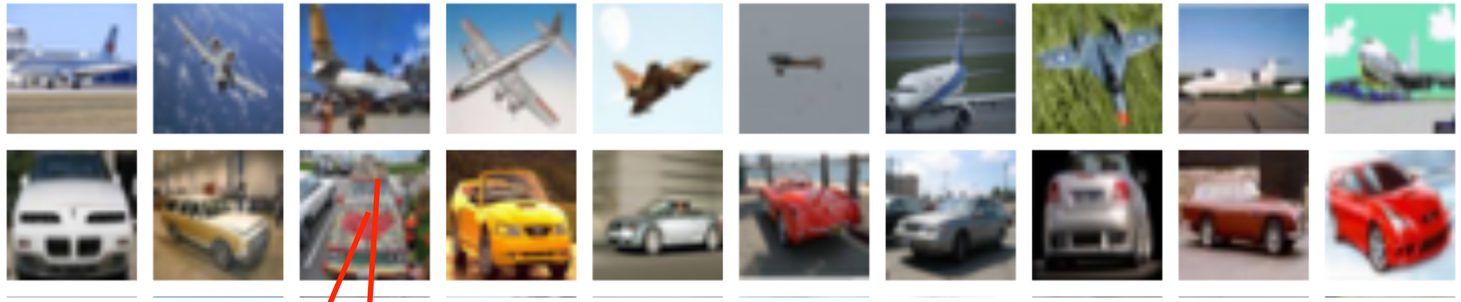

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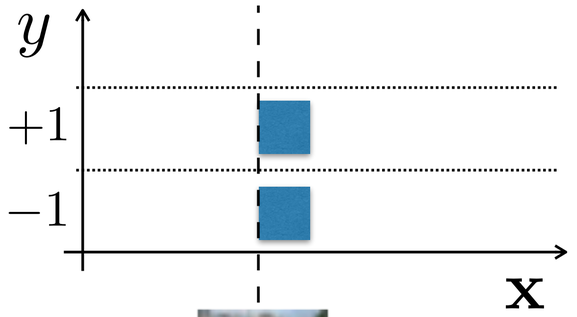
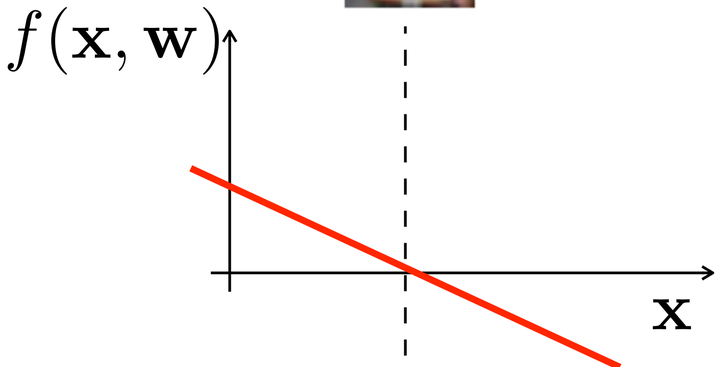
$$p(y|\mathbf{x},\mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x},\mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x},\mathbf{w})) & y = -1 \end{cases}$$



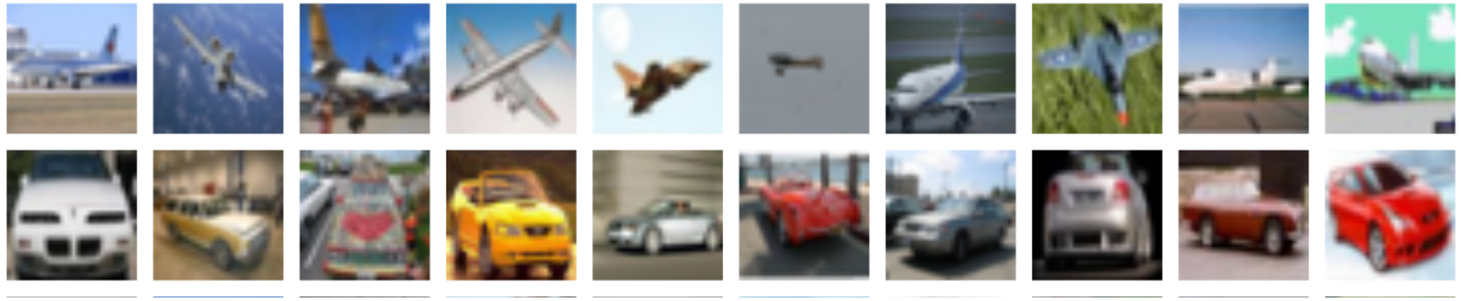


Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Classification

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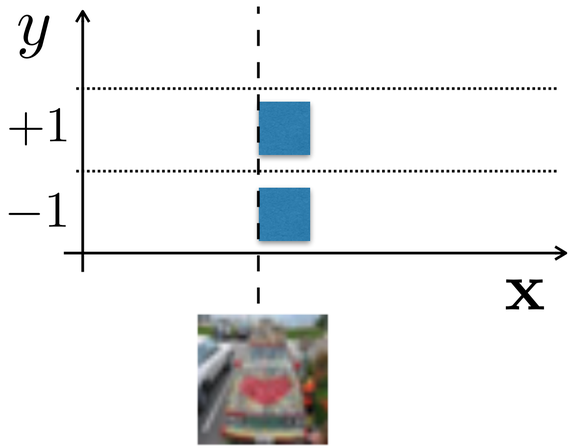


Linear classifier model probability of being from class +1 as $p = \sigma\left(\mathbf{w}^\top \overline{\mathbf{x}}\right)$

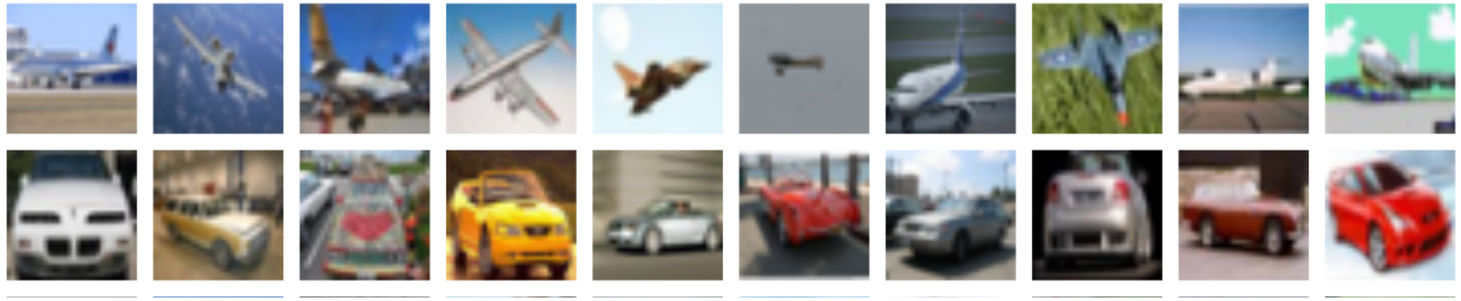
What is dimensionality of $\mathbf{x}$ and $\mathbf{w}$?

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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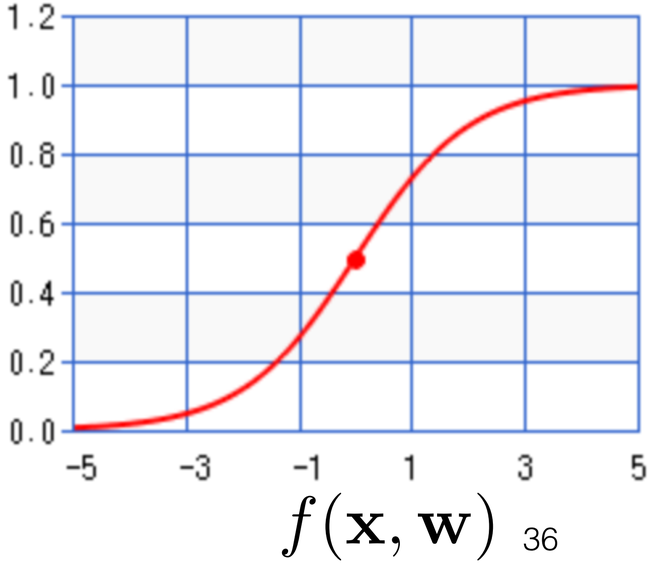
where

$$\sigma(f(\mathbf{x},\mathbf{w})) = \frac{1}{1+\exp(-f(\mathbf{x},\mathbf{w}))}$$

is sigmoid function.



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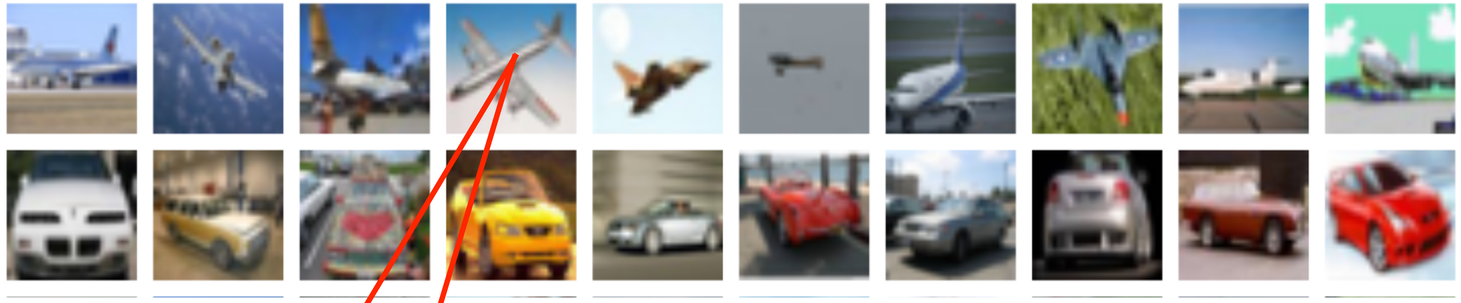
$+1$

-1



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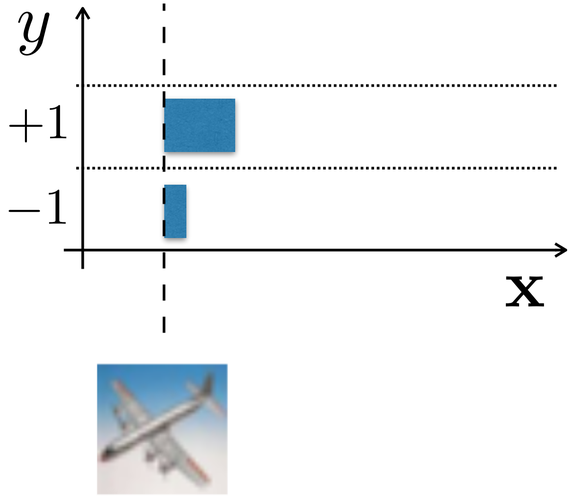

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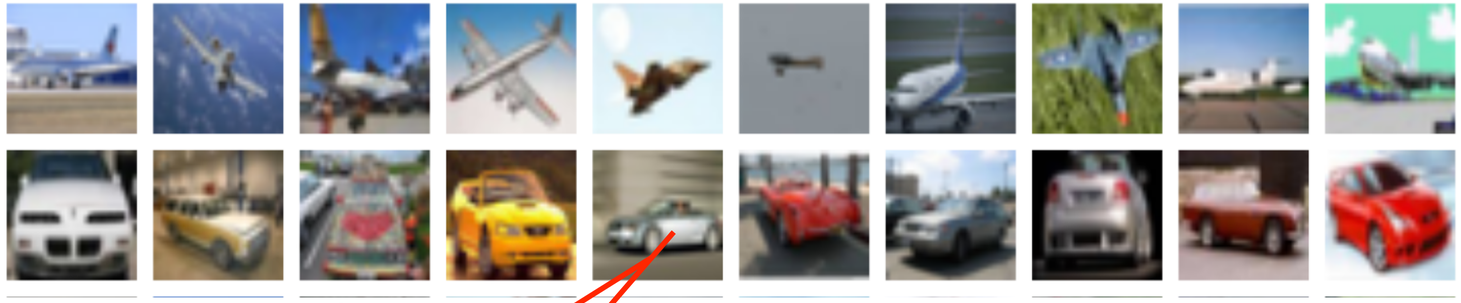
RGB images ($\mathbf{x}_i$)

$+1$

-1



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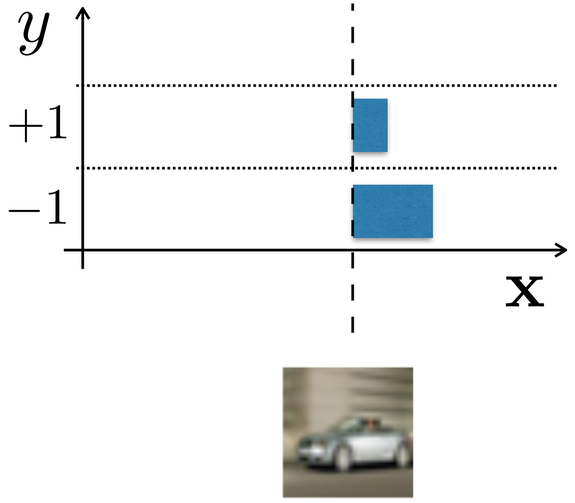

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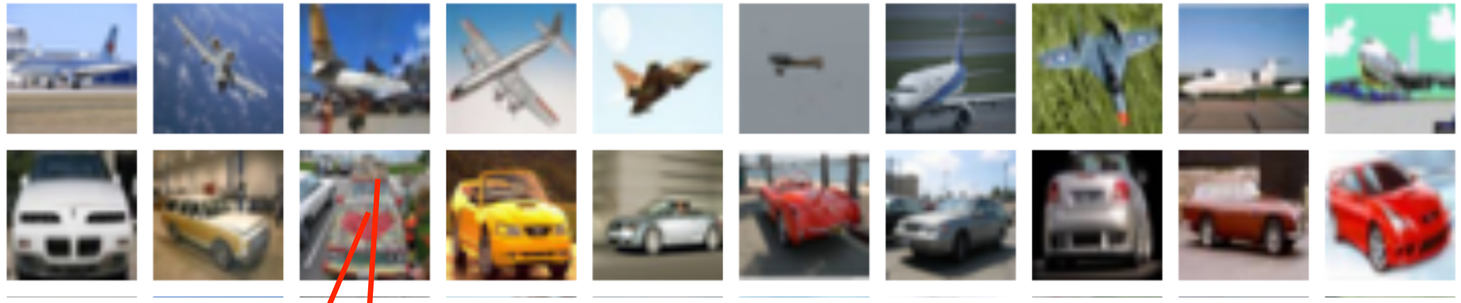
$+1$

-1



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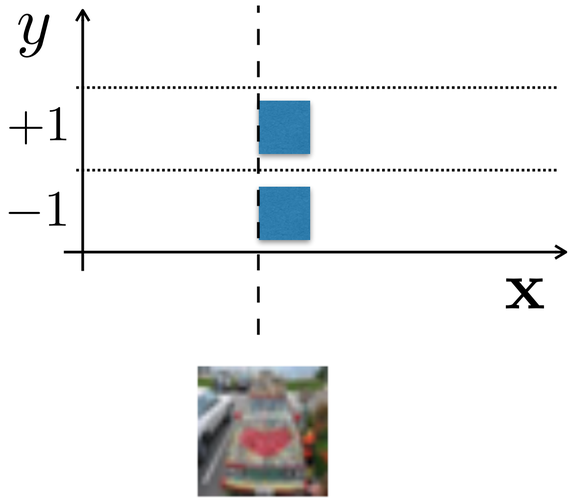

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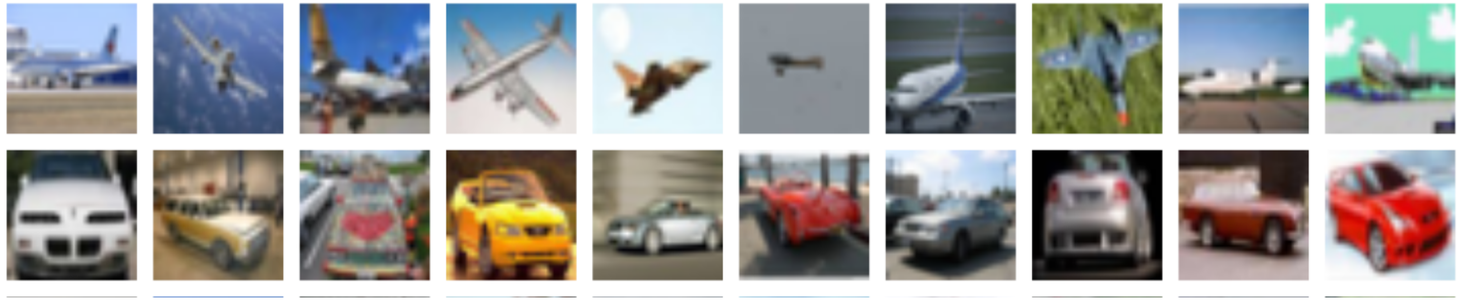
$+1$

-1



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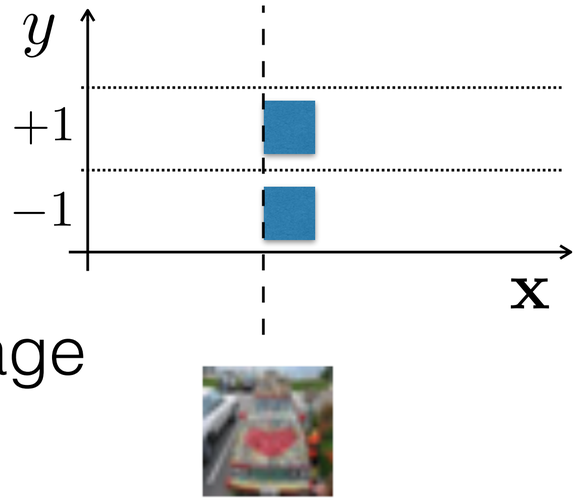

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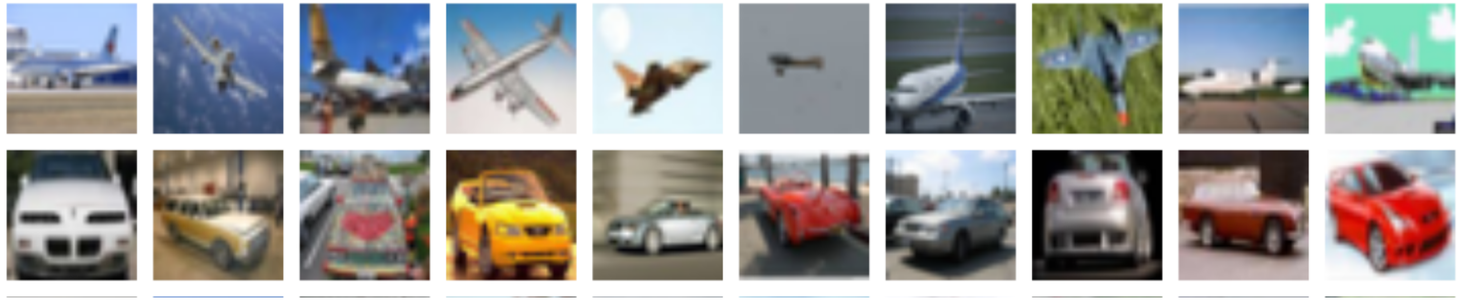
-1



Classification

Example: Linear classifier

def classify():

 # Linear classifier

$\mathbf{x} = \text{vec}($  $)$

$p = \sigma\left(\mathbf{w}^\top \overline{\mathbf{x}}\right)$

return p

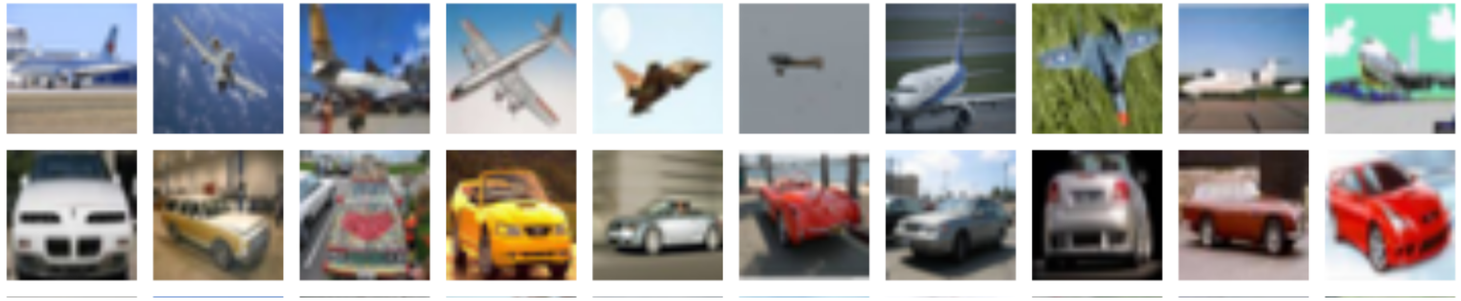
$\mathbf{w}^\top$ $\overline{\mathbf{x}}$ $= 2.5$ score

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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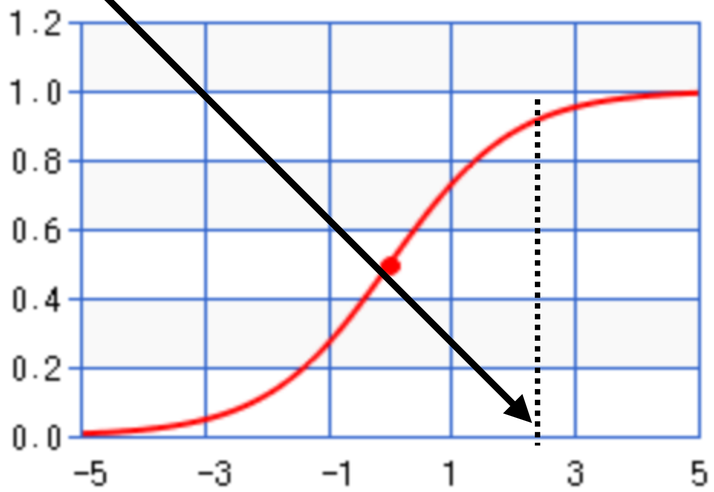
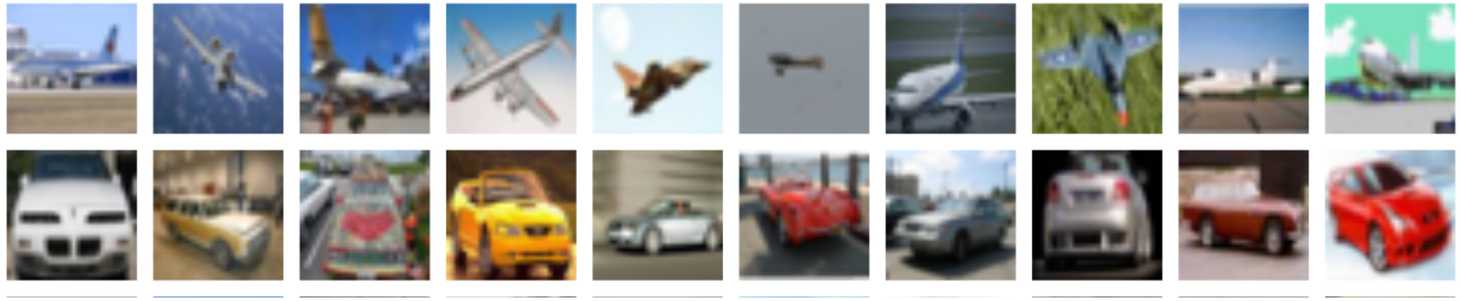


Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Classification

Example: Linear classifier

def classify():

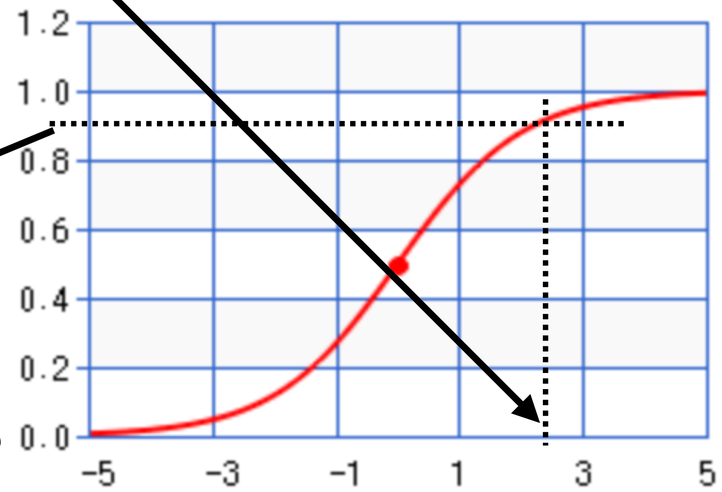
 # Linear classifier

$\mathbf{x} = \text{vec}($  $)$

$p = \sigma\left(\mathbf{w}^\top \overline{\mathbf{x}}\right)$

return $p = \sigma(2.5) = 0.92$

$\mathbf{w}^\top \overline{\mathbf{x}} = 2.5$ score



is it a good classifier?

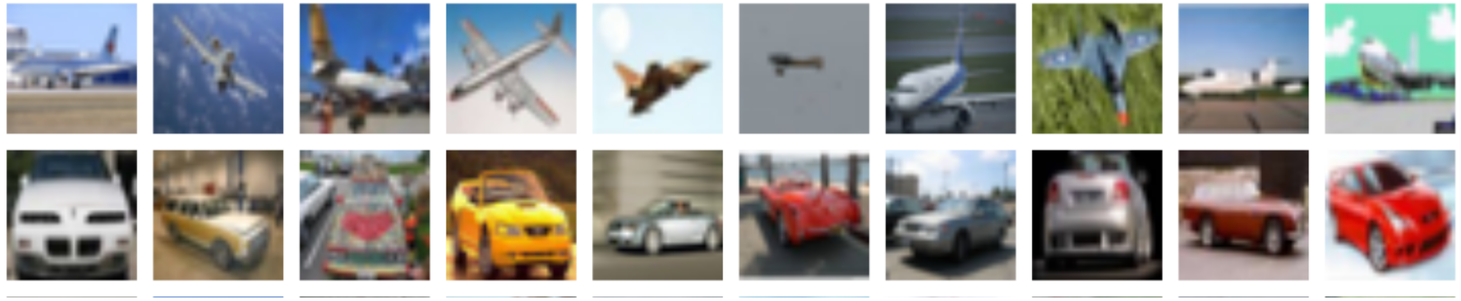
Show python code

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

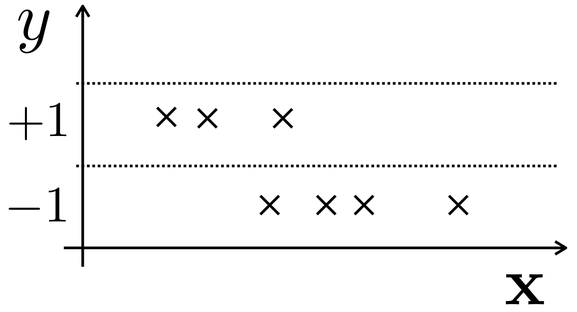
-1



Training

Training = search for unknown parameters $\mathbf{w}$
which fits a given data

Training data

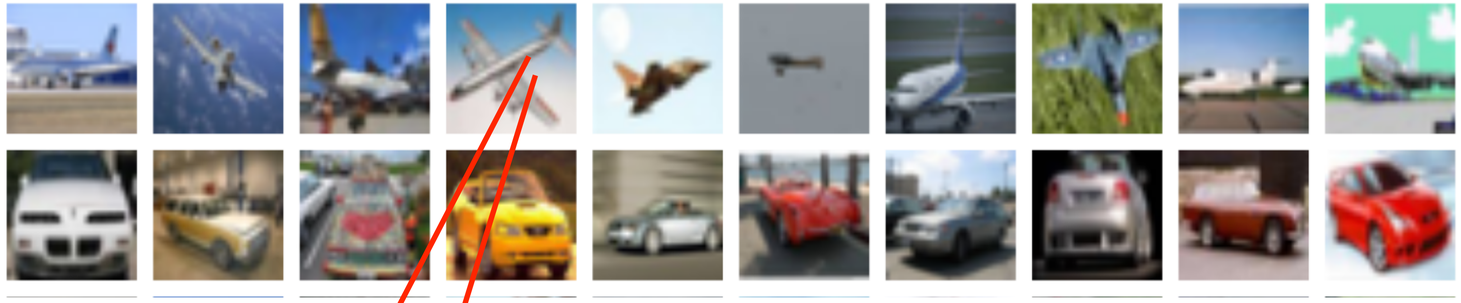


Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Training

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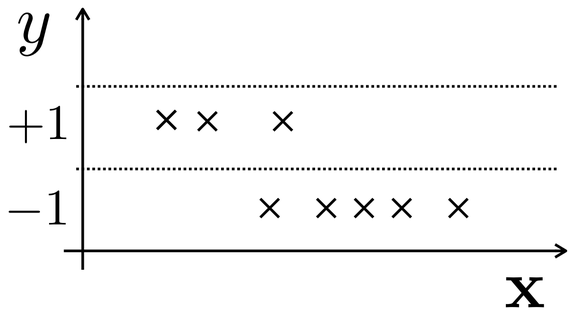
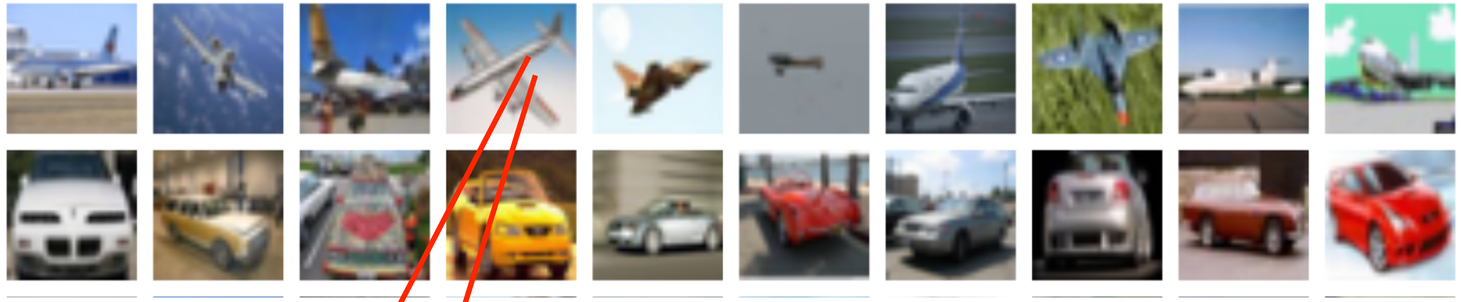


Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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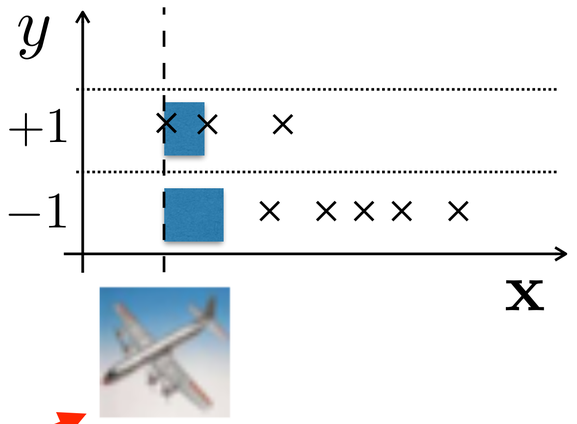
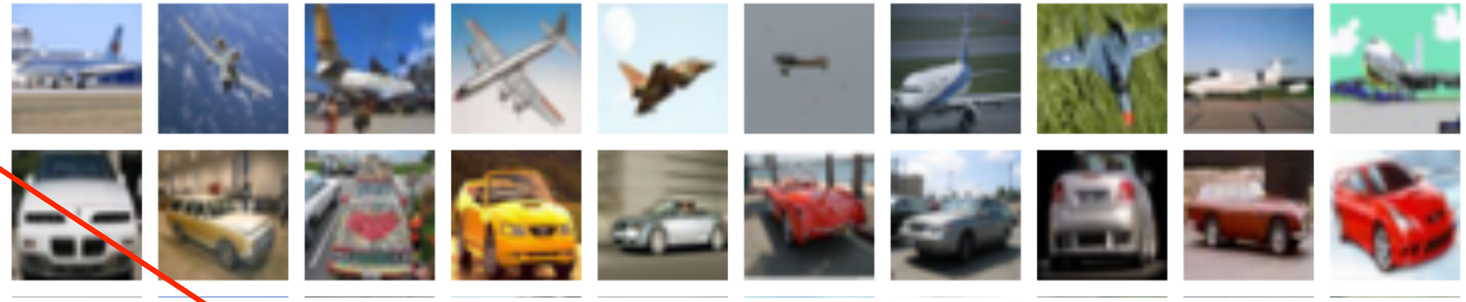


Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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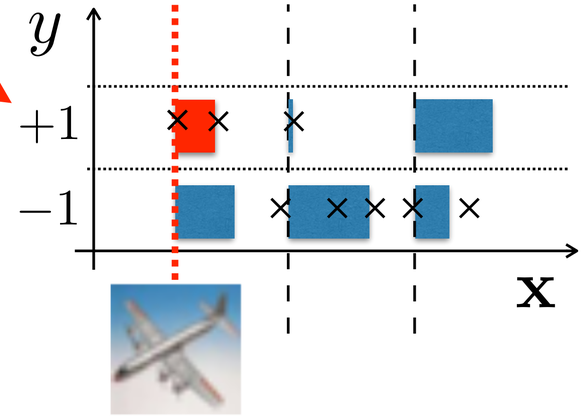


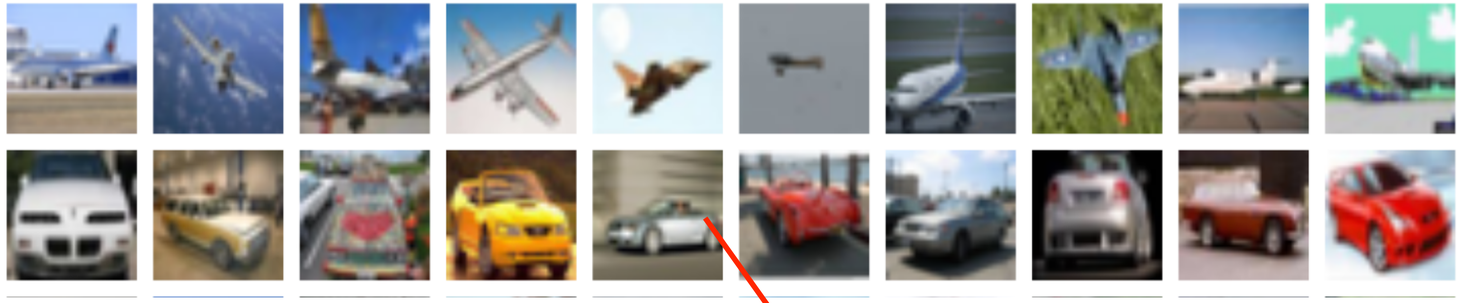
Probability of observing y_i when measuring $\mathbf{x}_i$ is

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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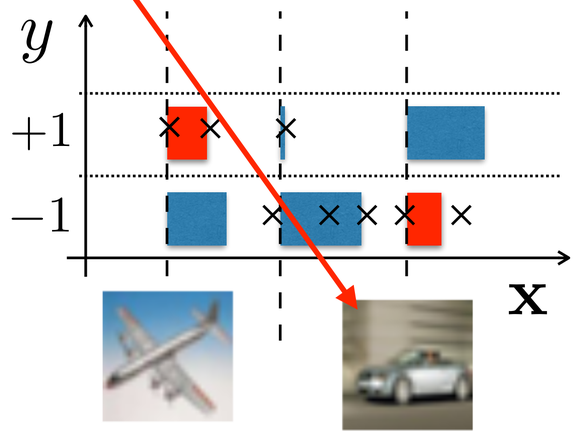

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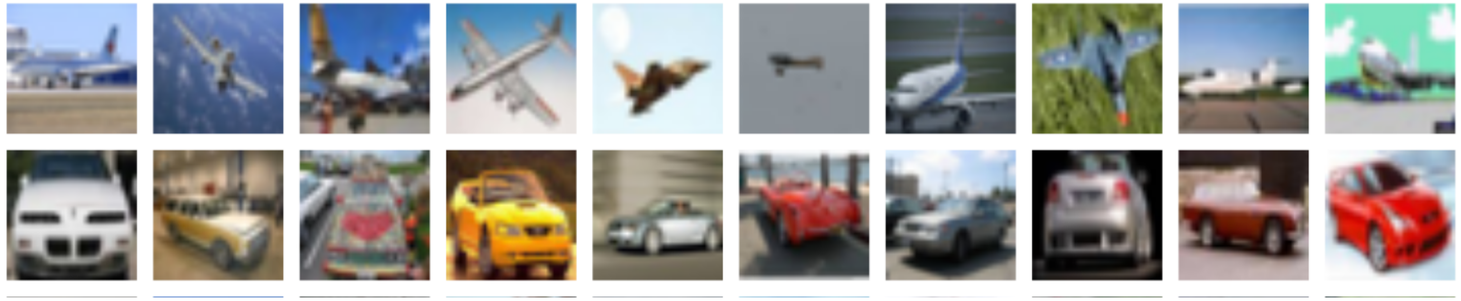
RGB images ($\mathbf{x}_i$)

$+1$

-1



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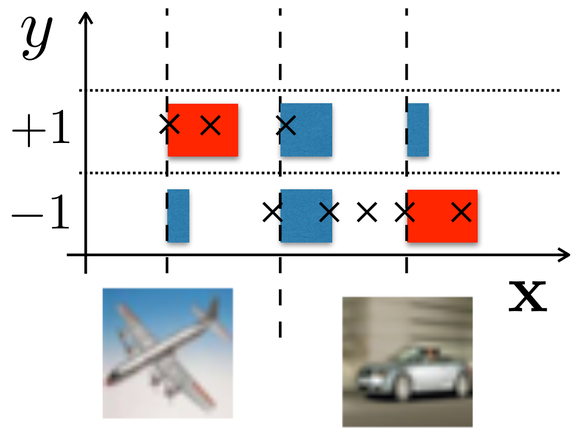

Probability of observing y_i when measuring $\mathbf{x}_i$ is

Two-class classification problem

$$p(y|\mathbf{x},\mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x},\mathbf{w})) & y=+1 \\ 1-\sigma(f(\mathbf{x},\mathbf{w})) & y=-1 \end{cases} = \sigma\Big(y \cdot f(\mathbf{x},\mathbf{w})\Big)$$

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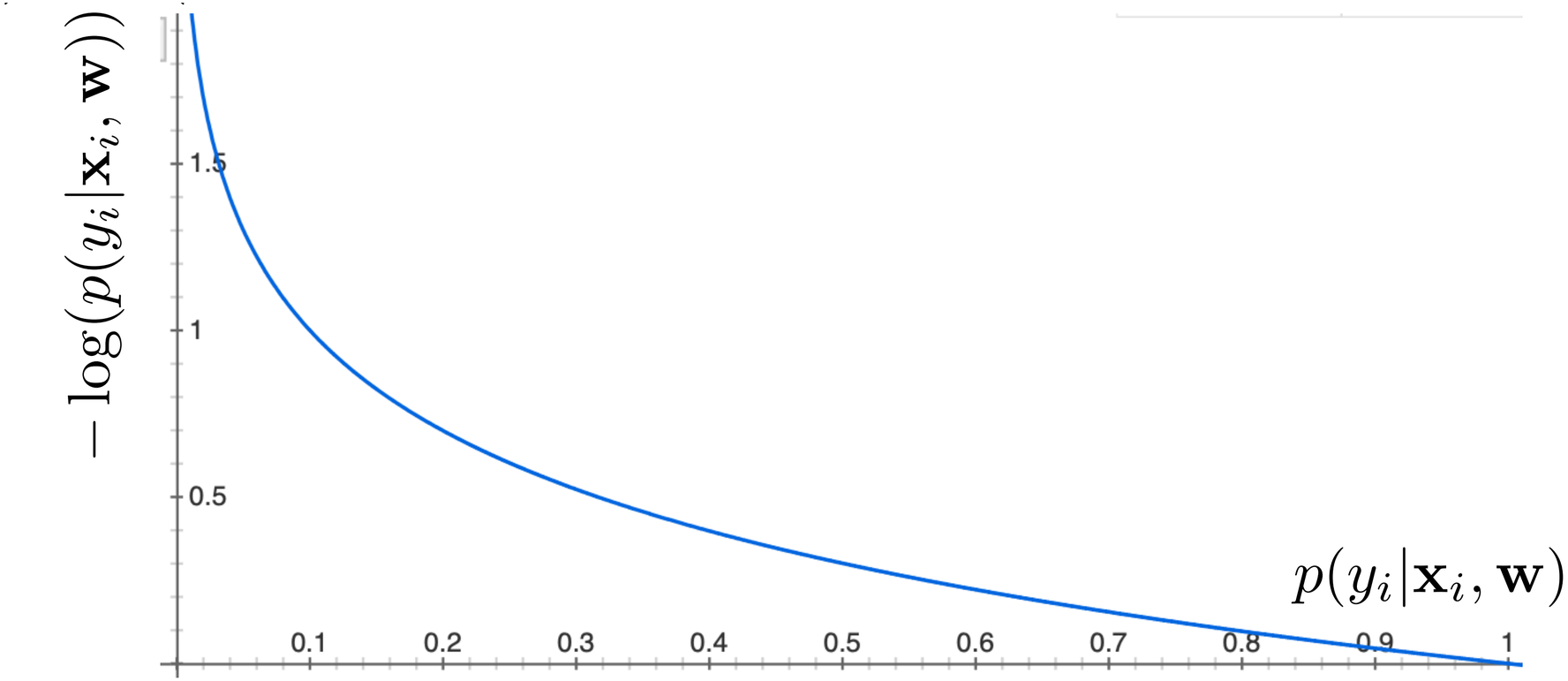
MLE: $$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i,\mathbf{w})) \right)$$



Equivalence of common binary classification losses

$$p(y|\mathbf{x},\mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x},\mathbf{w})) & y=+1 \\ 1-\sigma(f(\mathbf{x},\mathbf{w})) & y=-1 \end{cases} = \sigma\Big(y \cdot f(\mathbf{x},\mathbf{w})\Big)$$

MLE:

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i,\mathbf{w})) \right)$$

$$= -\log\left[\sigma\Big(y_i\, f(\mathbf{x}_i,\mathbf{w})\Big)\right]$$

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$\Leftrightarrow$

Logistic loss

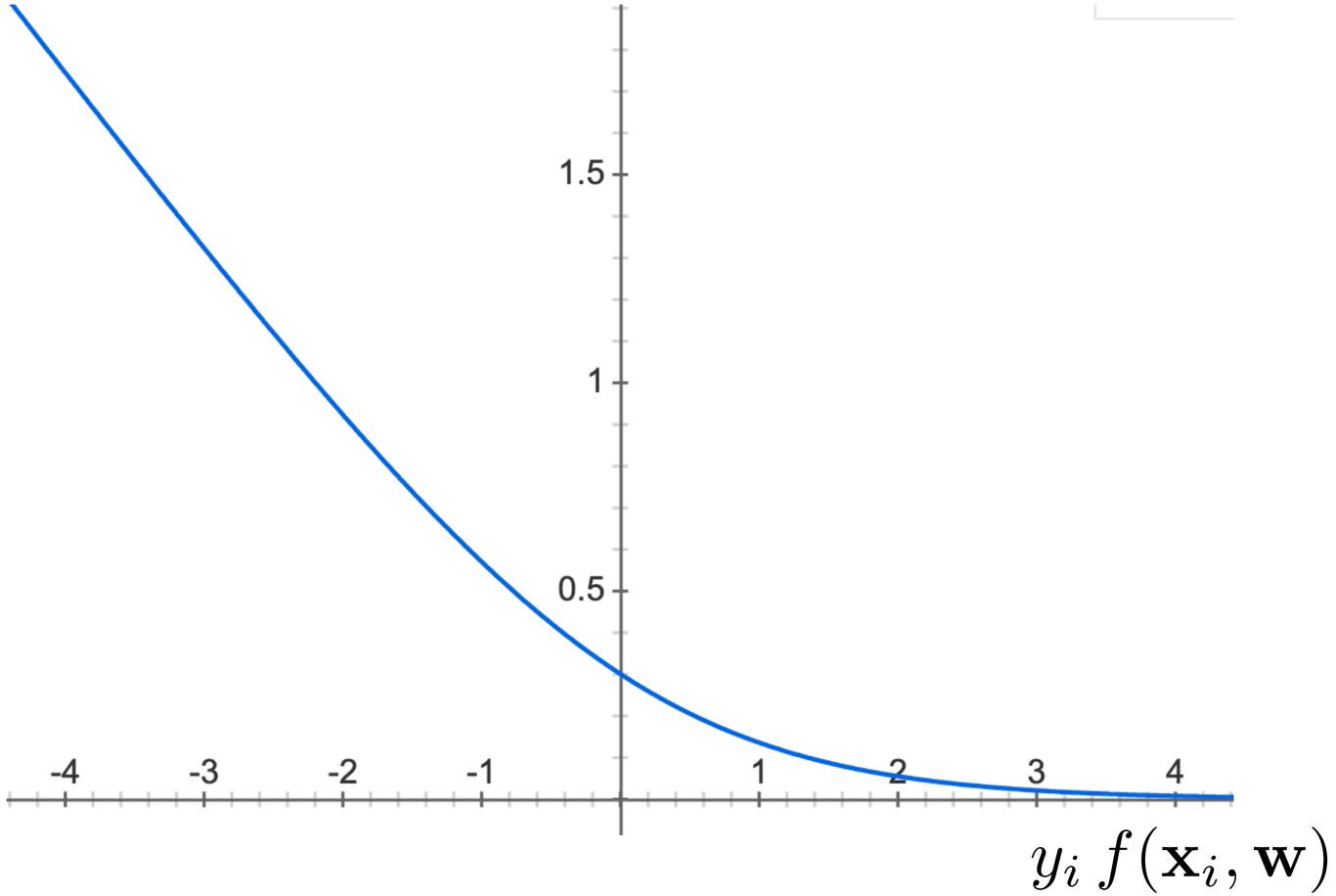
$$\log\left[1+\exp(-y_i\, f(\mathbf{x}_i,\mathbf{w}))\right]$$

Equivalence of common binary classification losses

$\Leftrightarrow$

Logistic loss

$$\log\left[1 + \exp(-y_i\, f(\mathbf{x}_i, \mathbf{w}))\right]$$



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$$\Leftrightarrow$$

Logistic loss

$$\log\left[1+\exp(-y_i\, f(\mathbf{x}_i,\mathbf{w}))\right]$$

$$\Leftrightarrow$$

Cross-entropy loss

$$-\left[\, \bar{y}_i \cdot \log\left(\sigma(\, f(\mathbf{x}_i,\mathbf{w}))\right) + (1-\bar{y}_i)\cdot \log\left(1-\sigma(\, f(\mathbf{x}_i,\mathbf{w}))\right) \,\right]$$

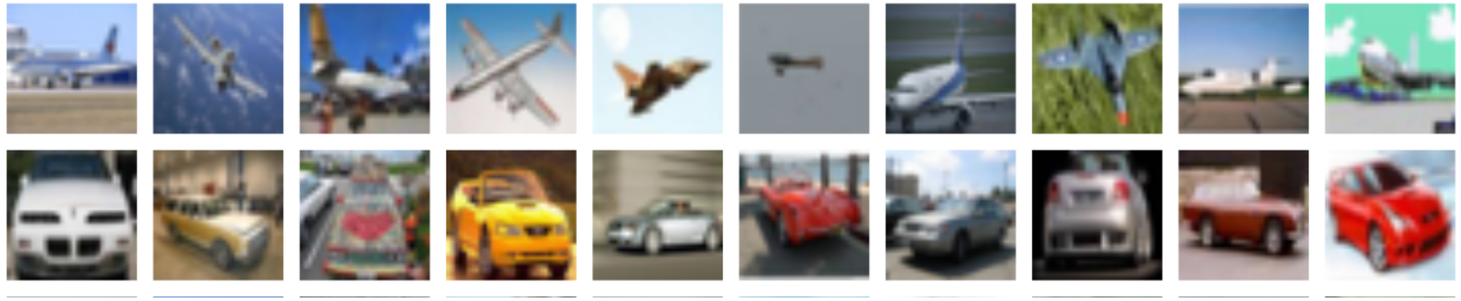
$$\bar{y}_i = \frac{y_i+1}{2} \in \{0,1\}$$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Training

Example: Training linear classifier

def train(

$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$\mathbf{x}_i = \text{vec}($  $) \quad \forall_i$

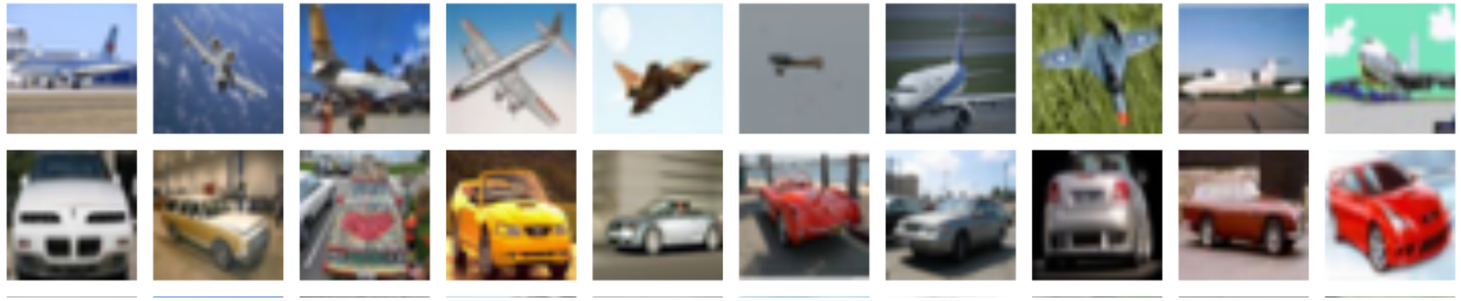
return $\mathbf{w}^*$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



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$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$$\mathbf{x}_i = \text{vec}(\quad) \quad \forall_i$$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i \, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

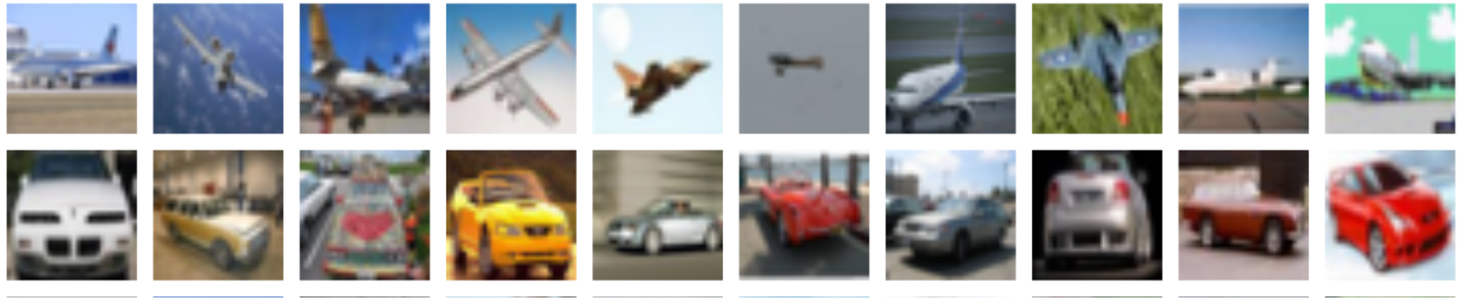
return $\mathbf{w}^*$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Training

Example: Training linear classifier

score

label

-2.5

$y_i = -1$

def train(

$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$\mathbf{x}_i = \text{vec}($  $) \quad \forall_i$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

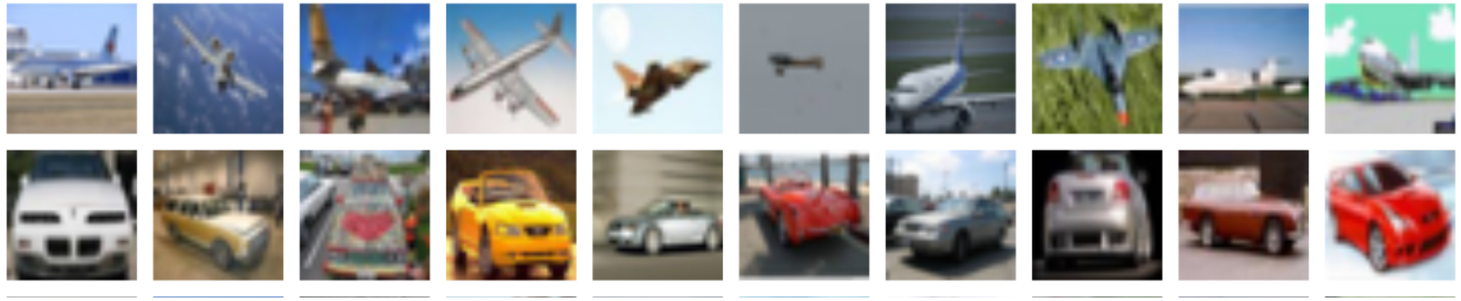
return $\mathbf{w}^*$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Training

Example: Training linear classifier

def train(

$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$$\mathbf{x}_i = \text{vec}(\ \text{  }\ ) \quad \forall_i$$

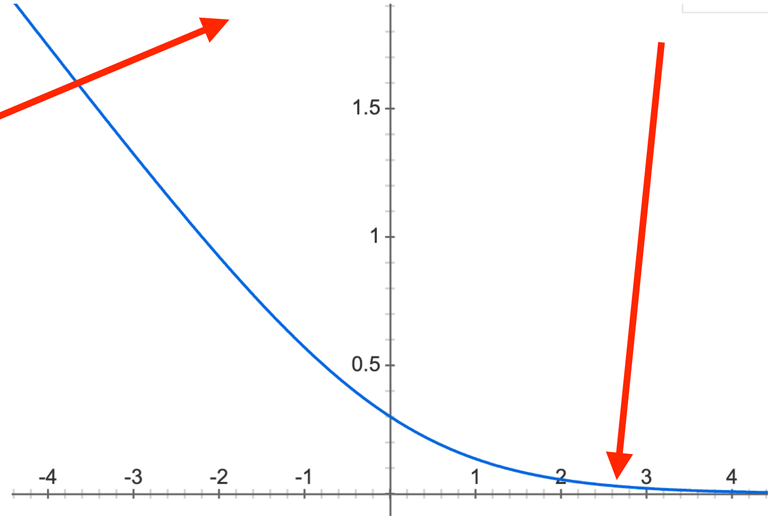
$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

return $\mathbf{w}^*$

score: -2.5

label: $y_i = -1$

(-1)x(-2.5) = 2.5

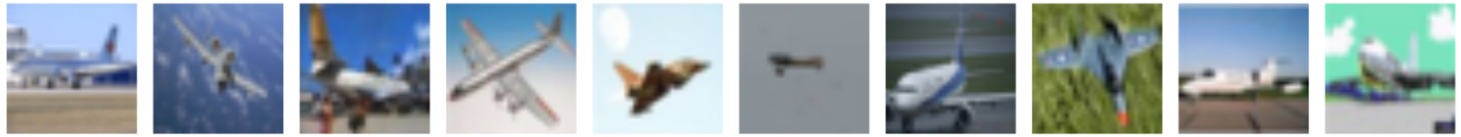


$y_i\, f(\mathbf{x}_i, \mathbf{w})$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$



-1

Training

Example: Training linear classifier

score

label

-2.5

$y_i = -1$

def train(

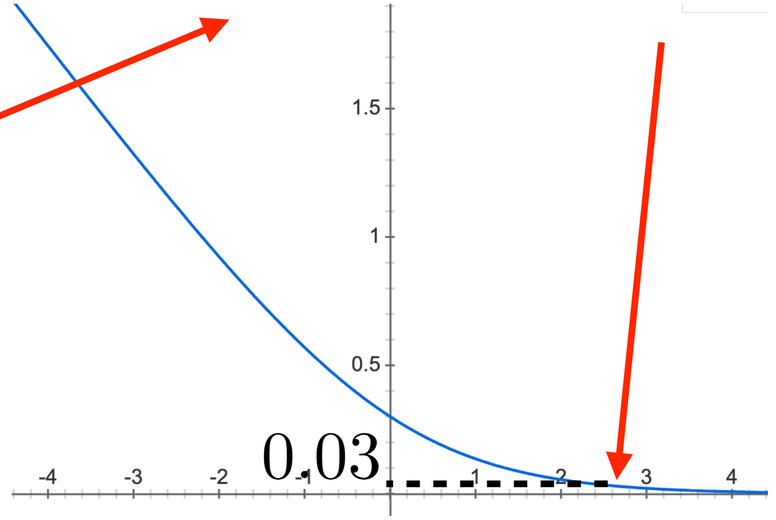
$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$\mathbf{x}_i = \text{vec}($  $) \quad \forall_i$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i \, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

return $\mathbf{w}^*$

(-1)x(-2.5) = 2.5



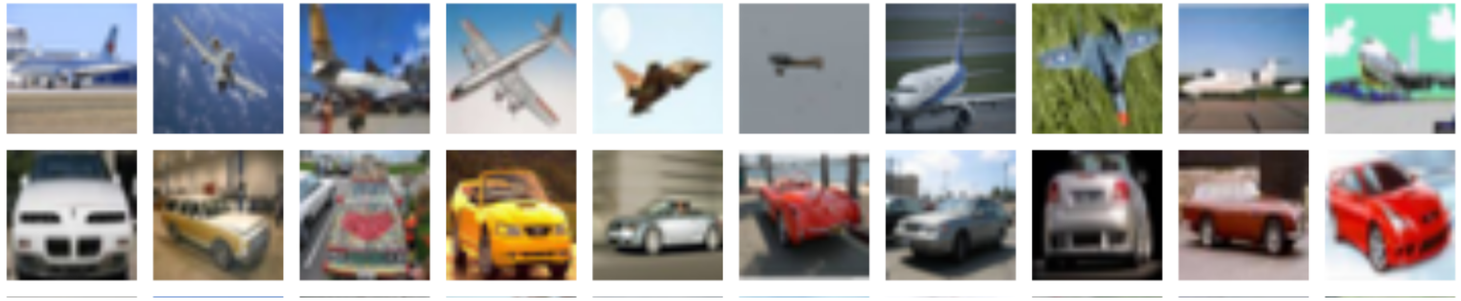
$y_i \, f(\mathbf{x}_i, \mathbf{w})$

Labels (y_i)

RGB images ($\mathbf{x}_i$)

$+1$

-1



Training

Example: Training linear classifier

def train(

$+1 \quad +1 \quad +1 \quad -1 \quad -1 \quad -1$):

$$\mathbf{x}_i = \text{vec}(\ \ ) \quad \forall_i$$

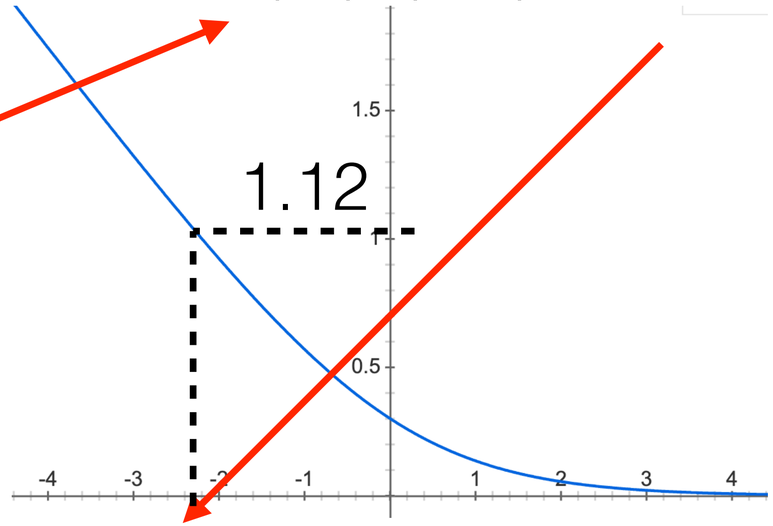
$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

return $\mathbf{w}^*$

score: 2.5

label: $y_i = -1$

(-1)x(2.5) = -2.5



$y_i\, f(\mathbf{x}_i, \mathbf{w})$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]$$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \underbrace{\sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]}_{\mathcal{L}(\mathbf{w})}$$

- There is no closed-form solution
- Gradient optimization

$$\mathbf{w} = \mathbf{w} - \alpha \left[\frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}}\right]^\top \text{where} \quad \frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}} = \quad \textbf{?}$$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \underbrace{\sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]}_{\mathcal{L}(\mathbf{w})}$$

- There is no closed-form solution
- Gradient optimization

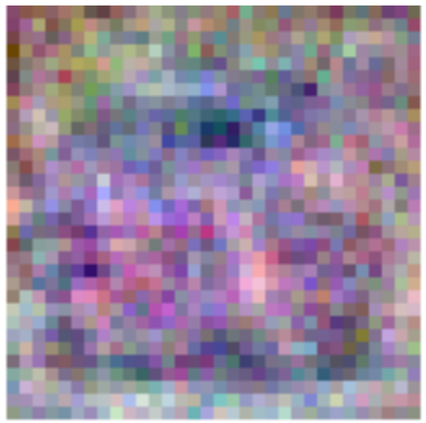
$$\mathbf{w} = \mathbf{w} - \alpha \left[\frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}}\right]^\top \text{where} \quad \frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}} = \sum_i \frac{-y_i \overline{\mathbf{x}}_i^\top}{1 + \exp(y_i \mathbf{w}^\top \overline{\mathbf{x}}_i)}$$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \underbrace{\sum_i \log\left[1 + \exp(-y_i\, \mathbf{w}^\top \overline{\mathbf{x}}_i)\right]}_{\mathcal{L}(\mathbf{w})}$$

- There is no closed-form solution
- Gradient optimization

$$\mathbf{w} = \mathbf{w} - \alpha \left[\frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}}\right]^\top \text{where } \frac{\partial \mathcal{L}(\mathbf{w})}{\partial \mathbf{w}} = \sum_i \frac{-y_i \overline{\mathbf{x}}_i^\top}{1 + \exp(y_i \mathbf{w}^\top \overline{\mathbf{x}}_i)}$$

Learned weights as a template:



automobile

Show python code

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

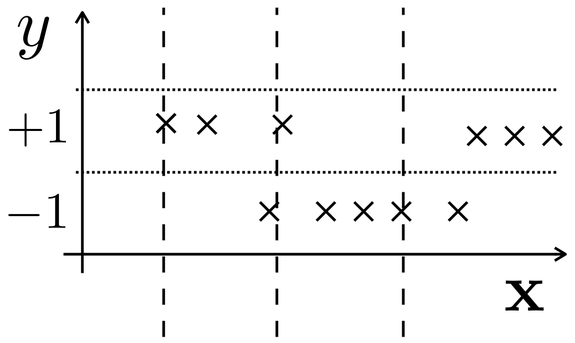
- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
- Choice of $f(\mathbf{x}, \mathbf{w})$ is crucial

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
- Linear $f(\mathbf{x}, \mathbf{w})$ cannot generate wild decision boundary

1D example:



$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
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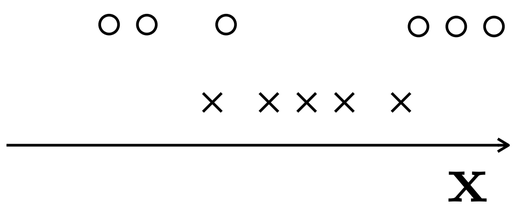
1D example:



$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
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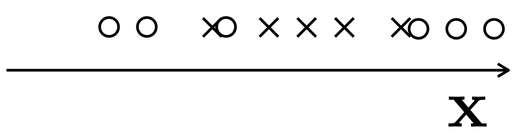
1D example:

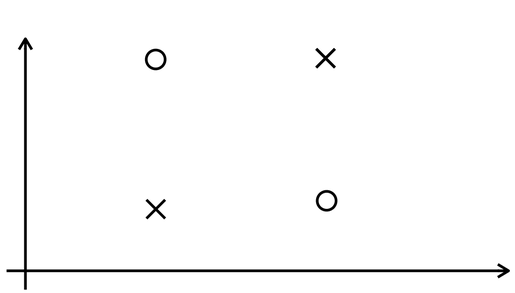


$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

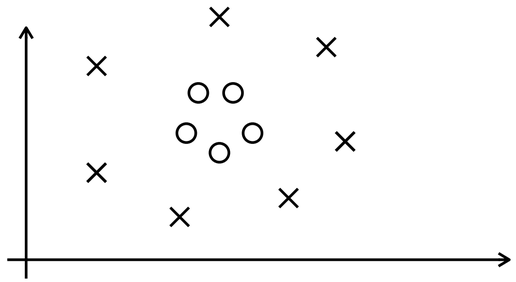
loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
- Linear $f(\mathbf{x}, \mathbf{w})$ cannot generate wild decision boundary

2D example:



XOR

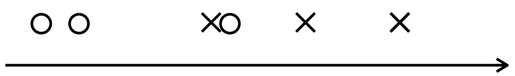


circle

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
- Wild $f(\mathbf{x}, \mathbf{w})$ with high-dimensional $\mathbf{w}$ suffers from the curse of dimensionality and overfitting



1D case

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
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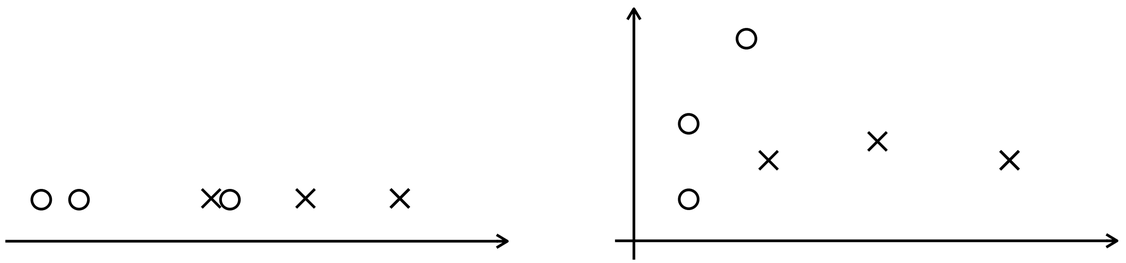

1D case 2D case

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
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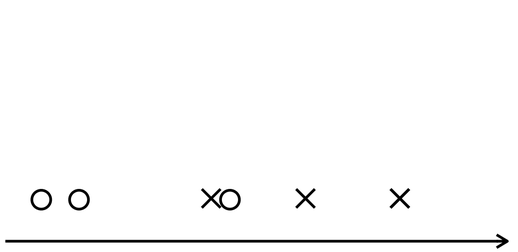
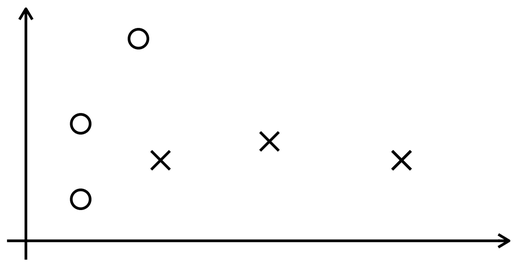


1D case



2D case

???

CIFAR case

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \left(\sum_i -\log(p(y_i|\mathbf{x}_i, \mathbf{w})) \right) + (-\log p(\mathbf{w}))$$

loss function prior/regulariser

- **Classification**: $p(y|\mathbf{x}, \mathbf{w}) = \begin{cases} \sigma(f(\mathbf{x}, \mathbf{w})) & y = +1 \\ 1 - \sigma(f(\mathbf{x}, \mathbf{w})) & y = -1 \end{cases}$
- Wild $f(\mathbf{x}, \mathbf{w})$ with high-dimensional $\mathbf{w}$ suffers from the curse of dimensionality and overfitting
- We exploit prior $p(\mathbf{w})$ to restrict the wildness of $f(\mathbf{x}, \mathbf{w})$
  - L2 regulariser $p(\mathbf{w}) = \mathcal{N}_{\mathbf{w}}(0, \sigma^2) \Rightarrow \|\mathbf{w}\|_2^2$
  - L1 regulariser, L1+L2 regulariser (elastic net)
  - prior on $f(\mathbf{x}, \mathbf{w})$ structure (e.g. consists of convolutions)
  - batch normalization

Conclusions

- Explained regression and linear classifier as MAP/ML estimator
- Discussed under/overfitting and regularisations
- Next lesson will go deeper

Competencies required for the test T1

- Derive MAP/ML estimate for two-class and K-class classification problem.
- Compute logistic-loss and cross-entropy-loss
- Understand when classifier score has high/low values.
- Understand when loss has high/low values.