Where the hell does the loss and the overfitting come from?

KL divergence, MAP, MLE view of regression, classification and overfitting

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Prerequisites: Bayes theorem

$$p(A, B) = p(A \mid B) \cdot p(B) = p(B \mid A) \cdot p(A)$$

$$p(A \mid B) = \frac{p(B \mid A) p(A)}{p(B)}$$

The same valid even if all probabilities conditioned by another event C

$$p(A, B \mid C) = p(A \mid B, C) \cdot p(B \mid C) = p(B \mid A, C) \cdot p(A \mid C)$$

$$p(A \mid B, C) = \frac{p(B \mid A, C) p(A \mid C)}{p(B \mid C)}$$

Prerequisites: Independence

Bayes theorem: $p(A, B) = p(A \mid B) \cdot p(B) = p(B \mid A) \cdot p(A)$

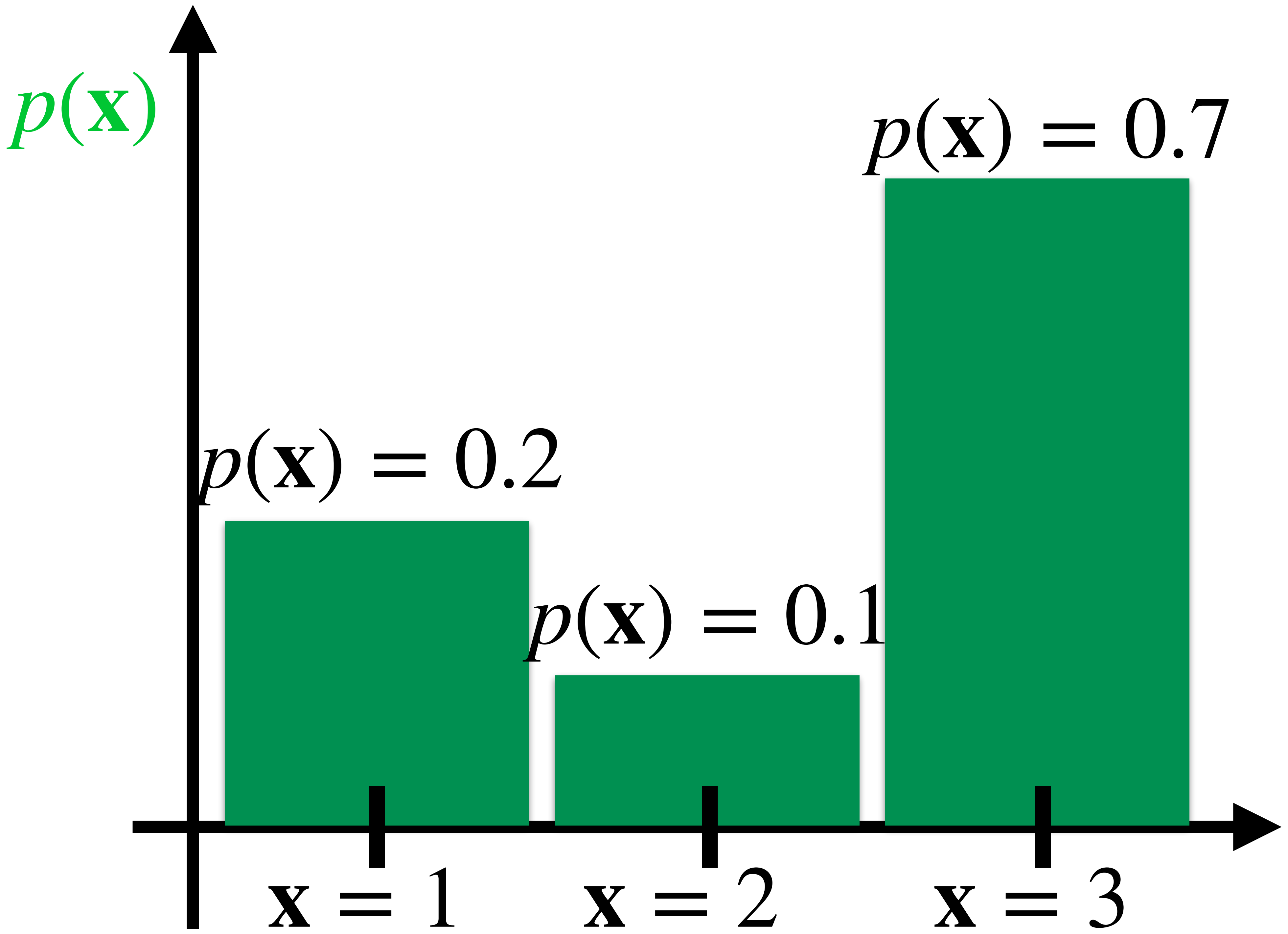
If A and B independent: $p(A, B) = p(A) \cdot p(B)$

Let’s put it together: $p(A, B) = p(A) \cdot p(B) = p(A \mid B) \cdot p(B)$

$$p(A) = p(A \mid B)$$

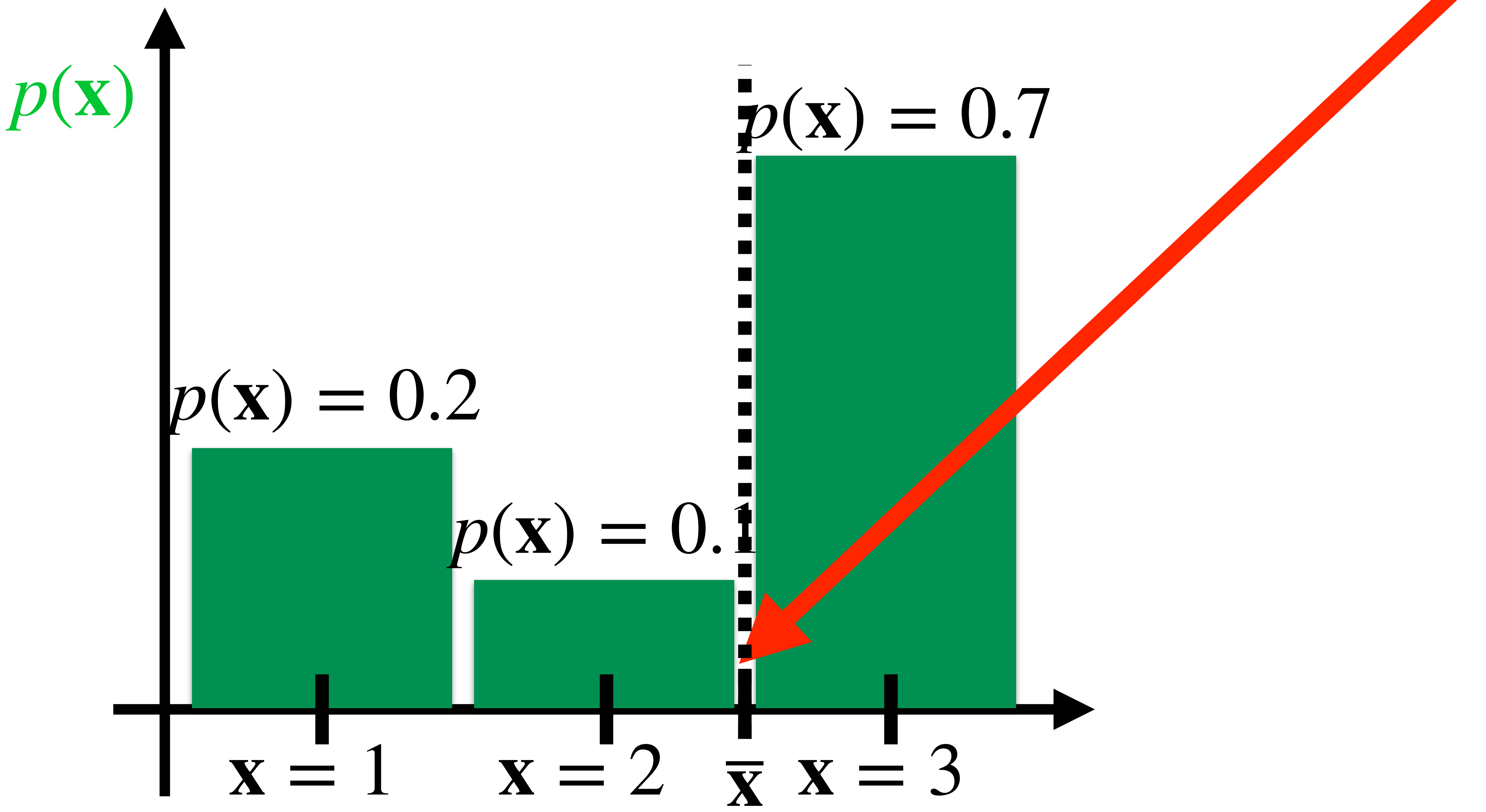
Prerequisites: Mean and average

$$\bar{\mathbf{x}} = \sum_{\mathbf{x}} p(\mathbf{x}) \cdot \mathbf{x} = \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})}[\mathbf{x}] = ??$$



Prerequisites: Mean and average

$$\bar{\mathbf{x}} = \sum_{\mathbf{x}} p(\mathbf{x}) \cdot \mathbf{x} = \mathbb{E}_{\mathbf{x}\sim p(\mathbf{x})}\left[\mathbf{x}\right] = 0.2 \cdot 1 \; + \; 0.1 \cdot 2 \; + \; 0.7 \cdot 3 = 2.5$$



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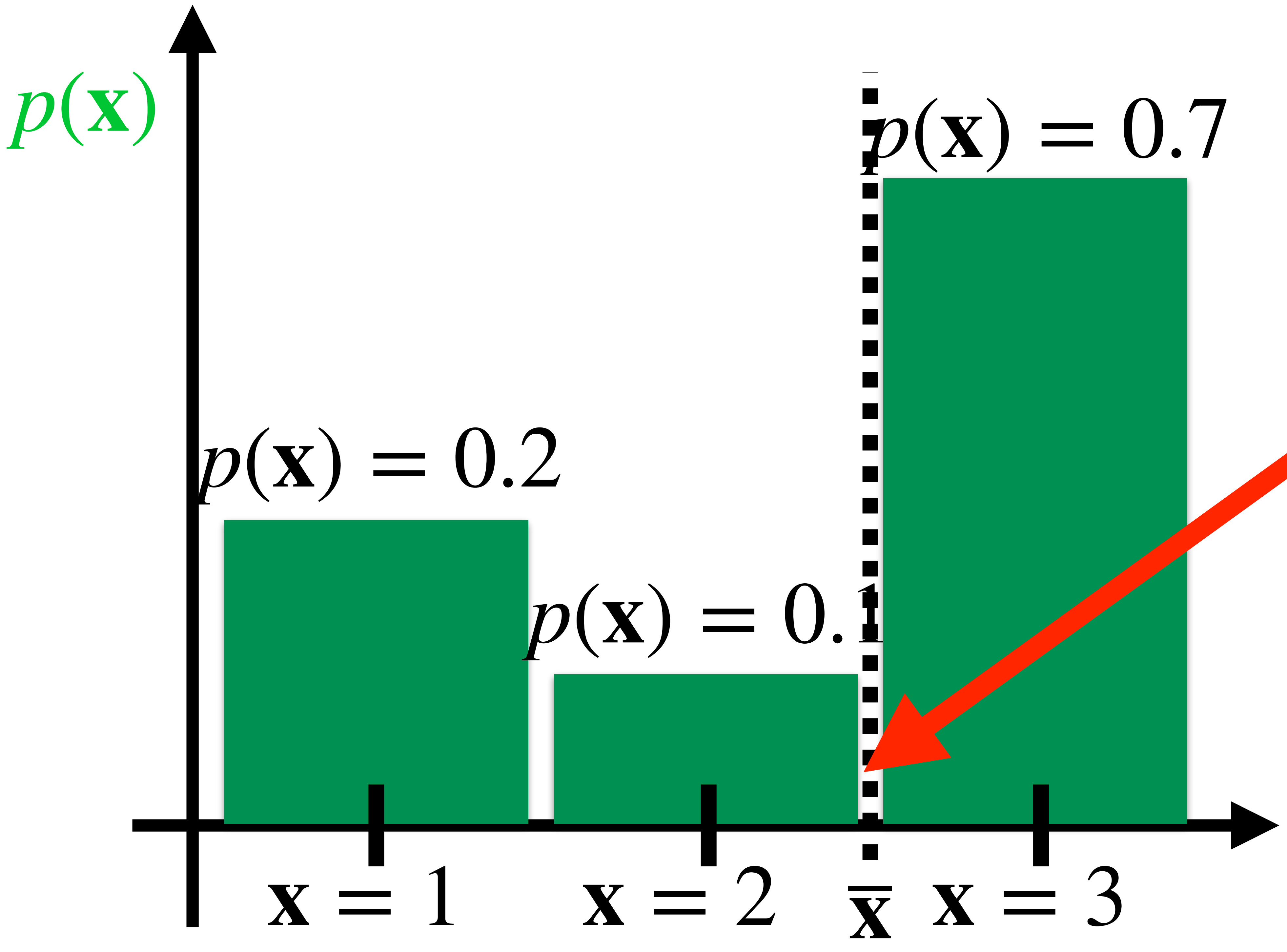
$$\approx \frac{1}{N}\sum_{i} \mathbf{x}_i \ = \frac{1}{10}(1+1+2+3+3+3+3+3+3+3) = 2.5$$

where $\mathbf{x}_i \sim p$



Prerequisites: Mean and average

$$\bar{f} = \sum_{\mathbf{x}} p(\mathbf{x}) \cdot f(\mathbf{x}) = \mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})}\left[f(\mathbf{x})\right] = ??$$

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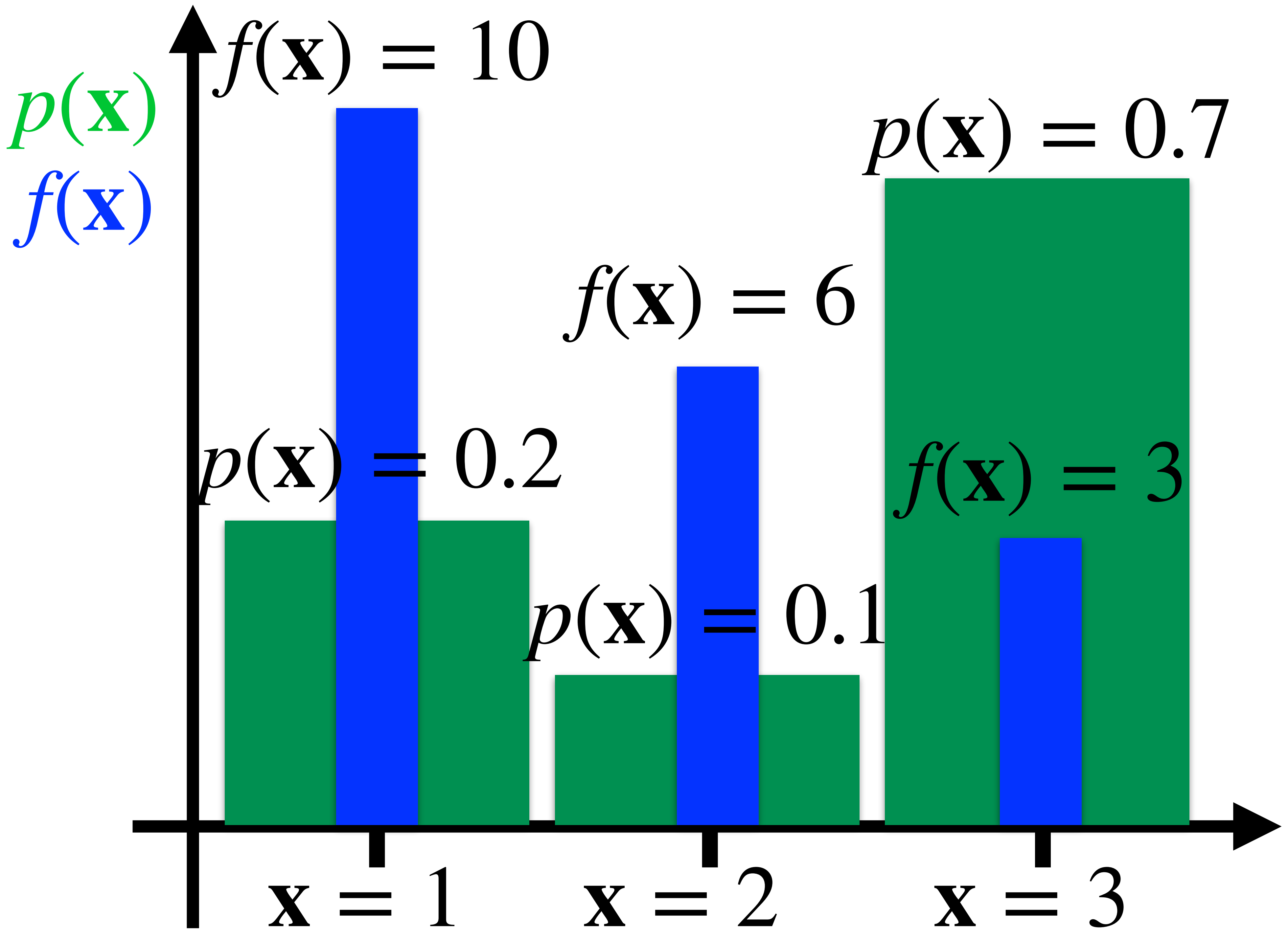
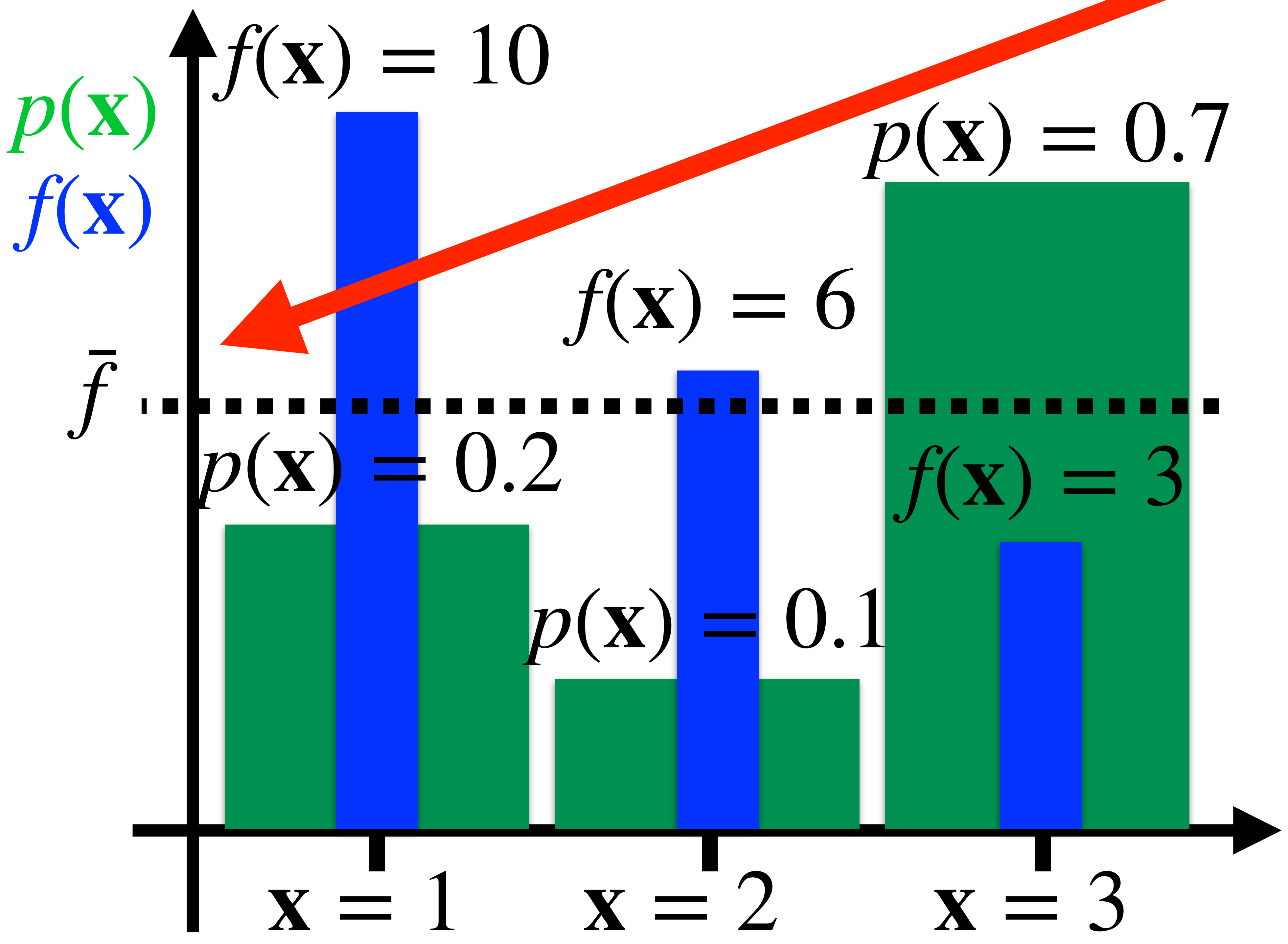


Prerequisites: Mean and average

$$\bar{f} = \sum_{\mathbf{x}} p(\mathbf{x}) \cdot f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}\sim p(\mathbf{x})}\left[f(\mathbf{x})\right] = 0.2 \cdot 10 \ + \ 0.1 \cdot 6 \ + \ 0.7 \cdot 3 = 4.7$$

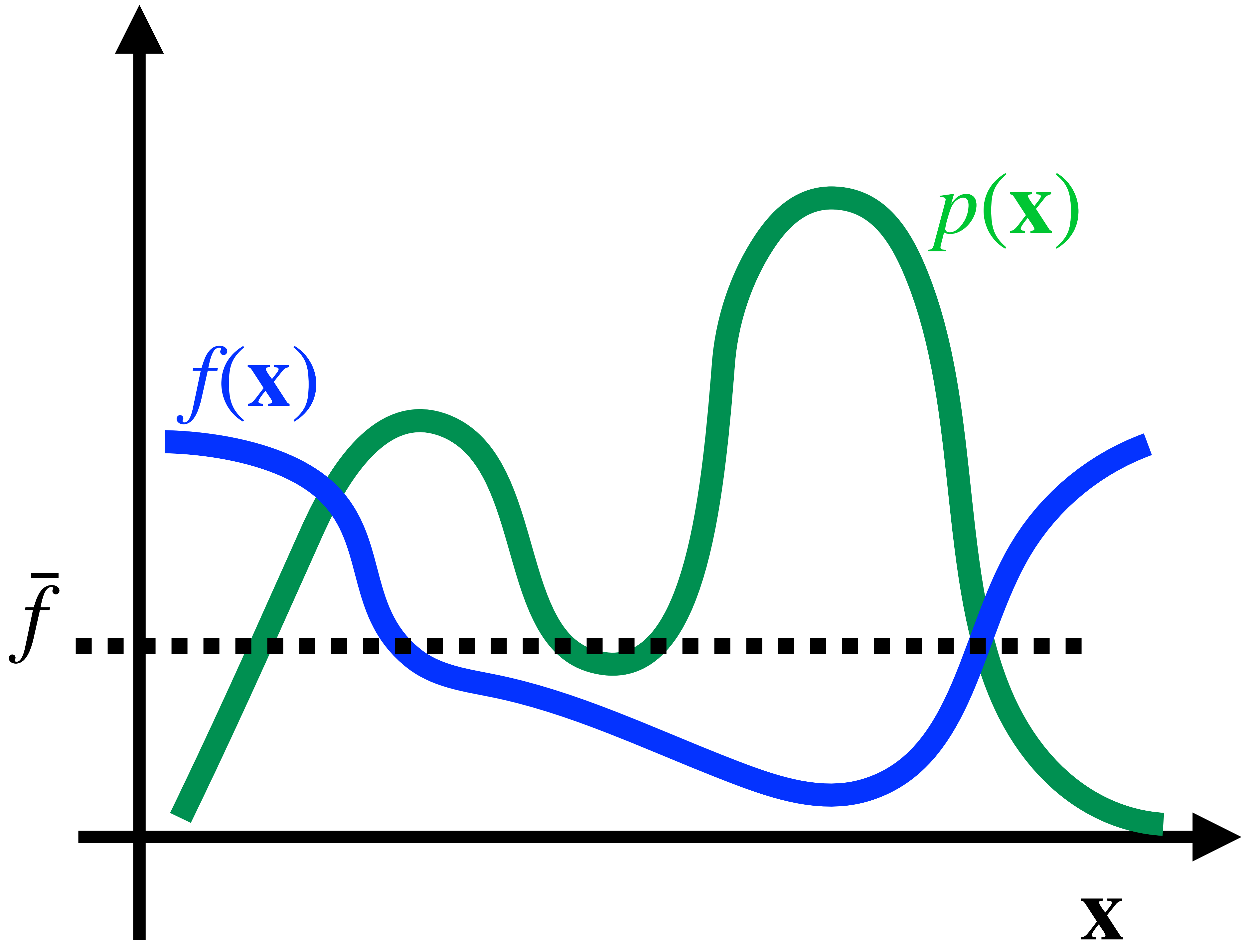
$$\approx \frac{1}{N}\sum_{i} f(\mathbf{x}_i) = \frac{1}{10}(10 + 10 + 6 + 3 + 3 + 3 + 3 + 3 + 3 + 3) = 4.7$$

where $\mathbf{x}_i \sim p$



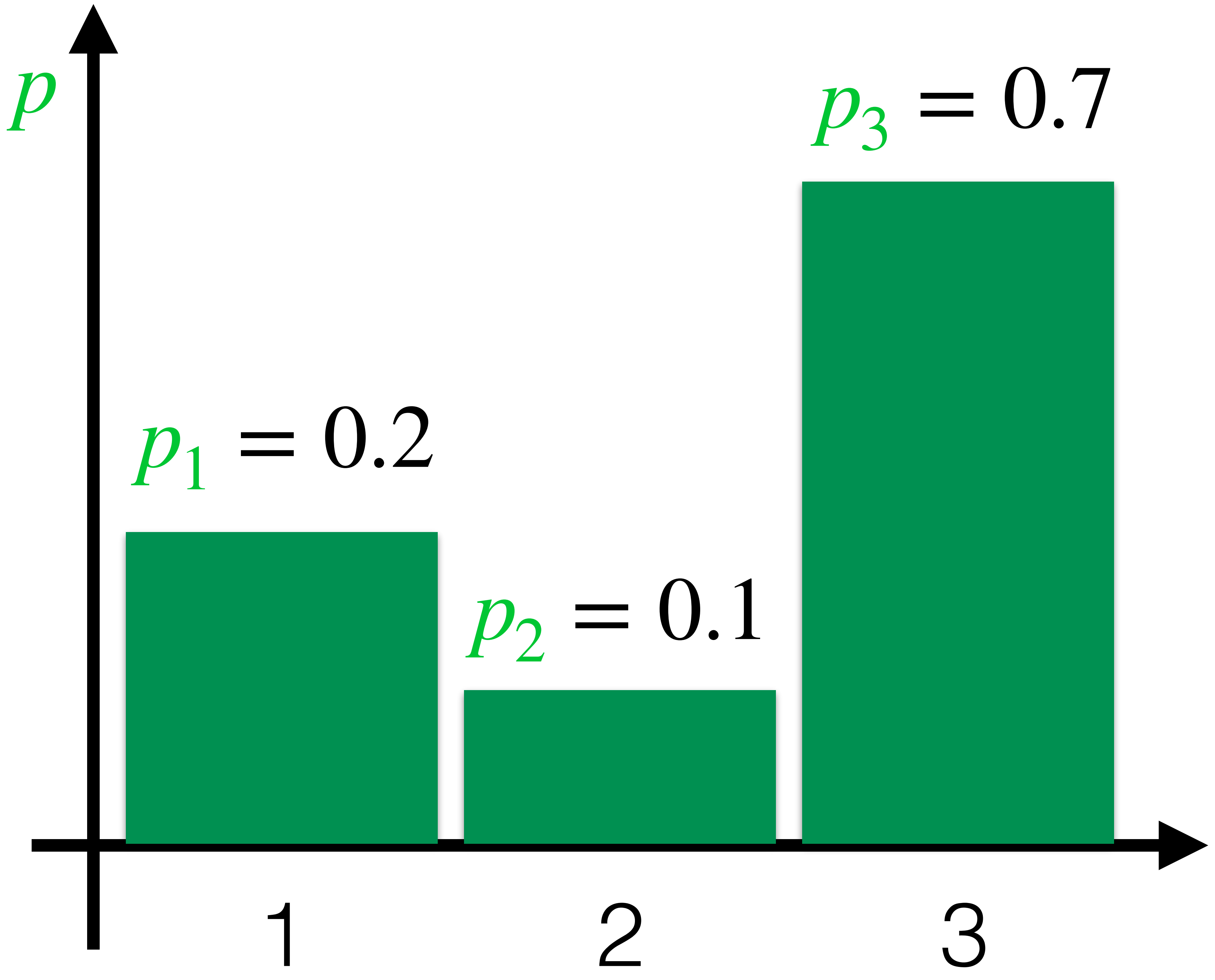
Prerequisites: Mean and average

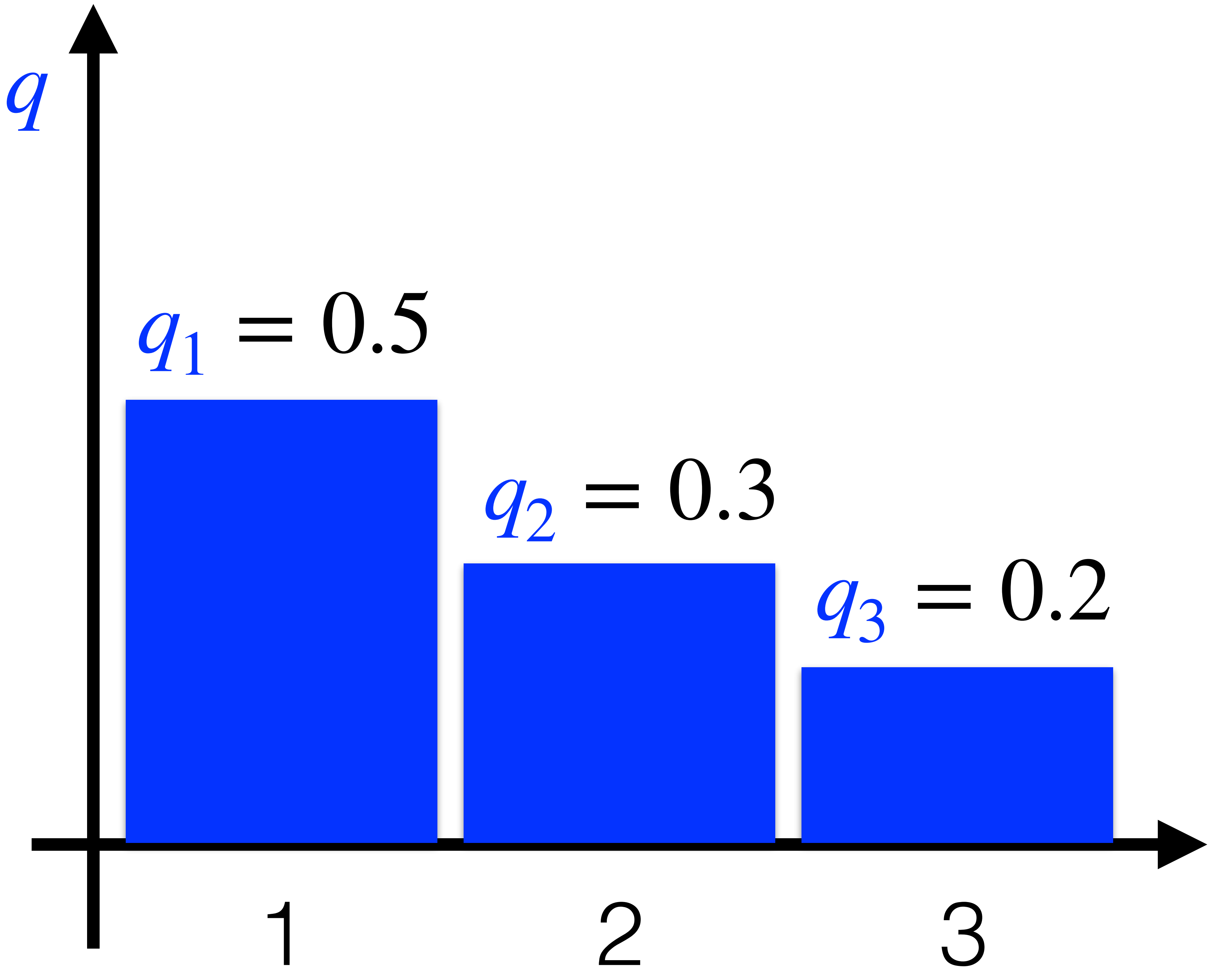
$$\bar{f} = \int_{\mathbf{x}} p(\mathbf{x}) \cdot f(\mathbf{x})\, d\mathbf{x} = \mathbb{E}_{\mathbf{x}\sim p(\mathbf{x})}\left[f(\mathbf{x})\right] \approx \frac{1}{N}\sum_i f(\mathbf{x}_i) \qquad \text{where } \mathbf{x}_i \sim p$$



Prerequisites: KL-divergence

$$D_{KL}(p \parallel q) = \sum_i p_i \cdot \log \frac{p_i}{q_i} = 0.2 \cdot \log \frac{0.2}{0.5} + 0.1 \cdot \log \frac{0.1}{0.3} + 0.7 \cdot \log \frac{0.7}{0.2} = 0.2535$$



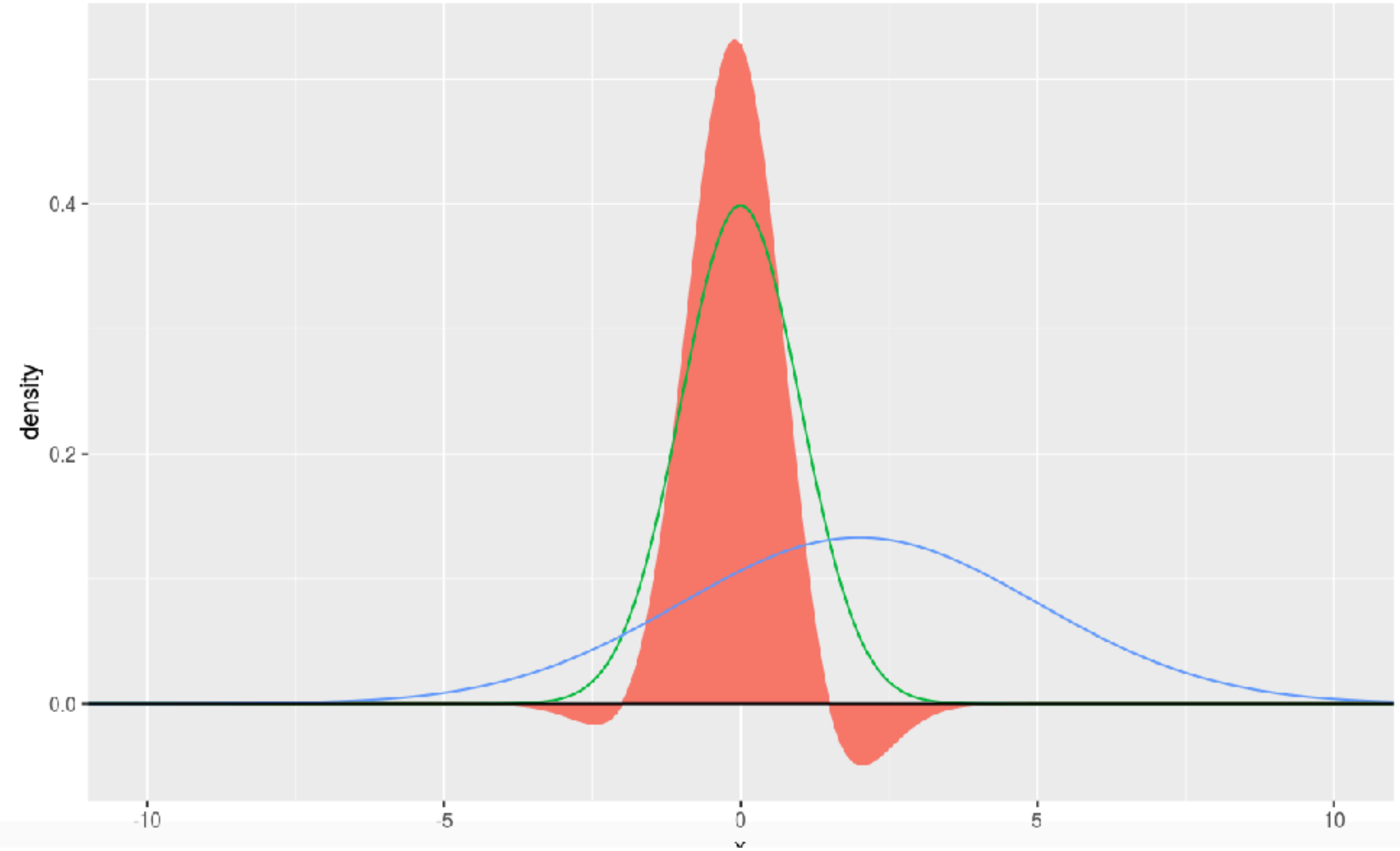


- What if $q_k = 0$ and $p_k > 0$?
- What if $q_k = p_k$ for k=1,2,3 ?
- Is it symmetrical $D_{KL}(p \parallel q) \neq D_{KL}(q \parallel p)$?

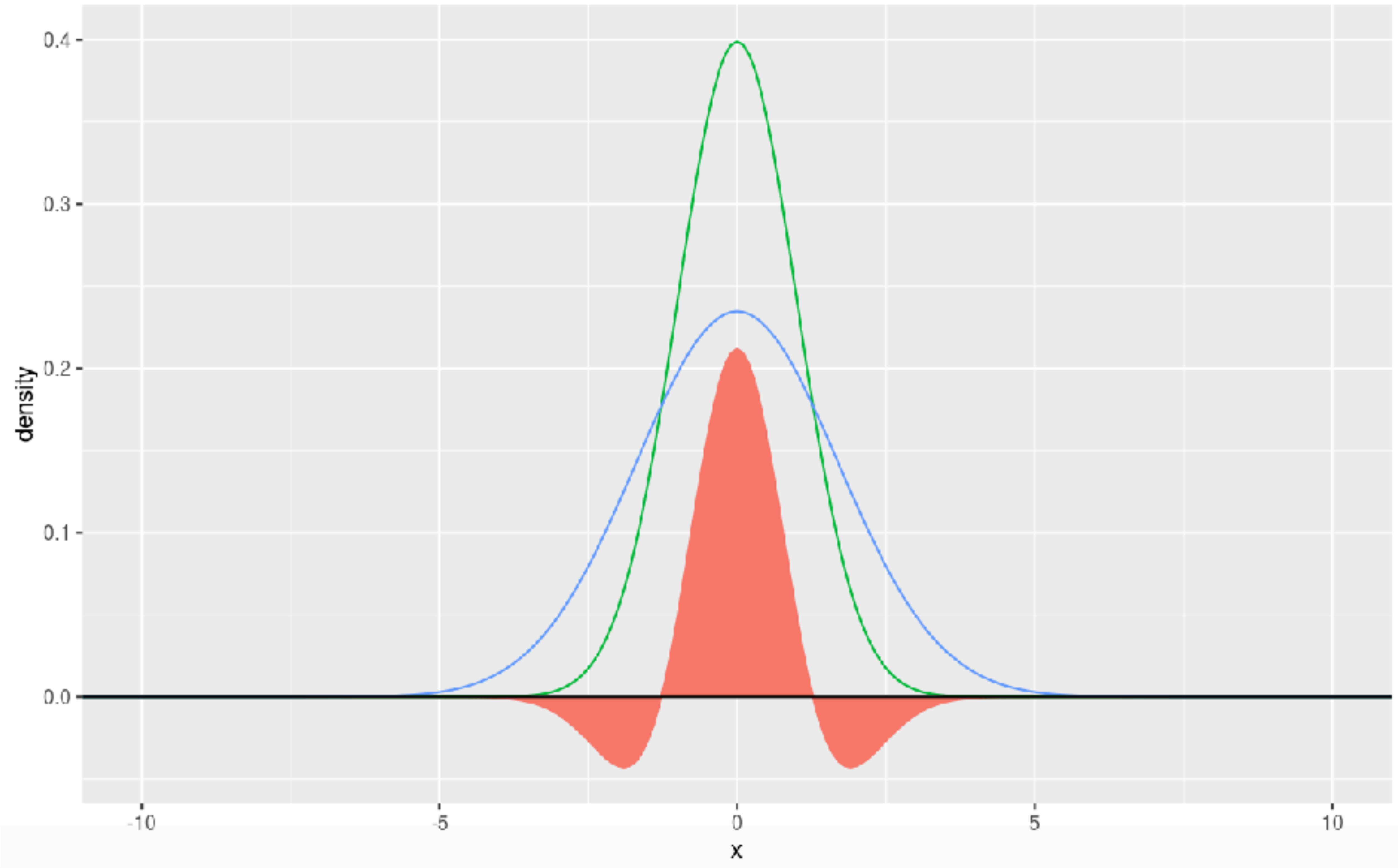
Prerequisites: KL-divergence

$$D_{KL}\left(p(\mathbf{x}) \parallel q(\mathbf{x})\right) = \int_{\mathbf{x}} p(\mathbf{x}) \cdot \log \frac{p(\mathbf{x})}{q(\mathbf{x})} \, d\mathbf{x}$$

$D_{KL}\left(p(\mathbf{x}) \parallel q(\mathbf{x})\right)$ =0.8764



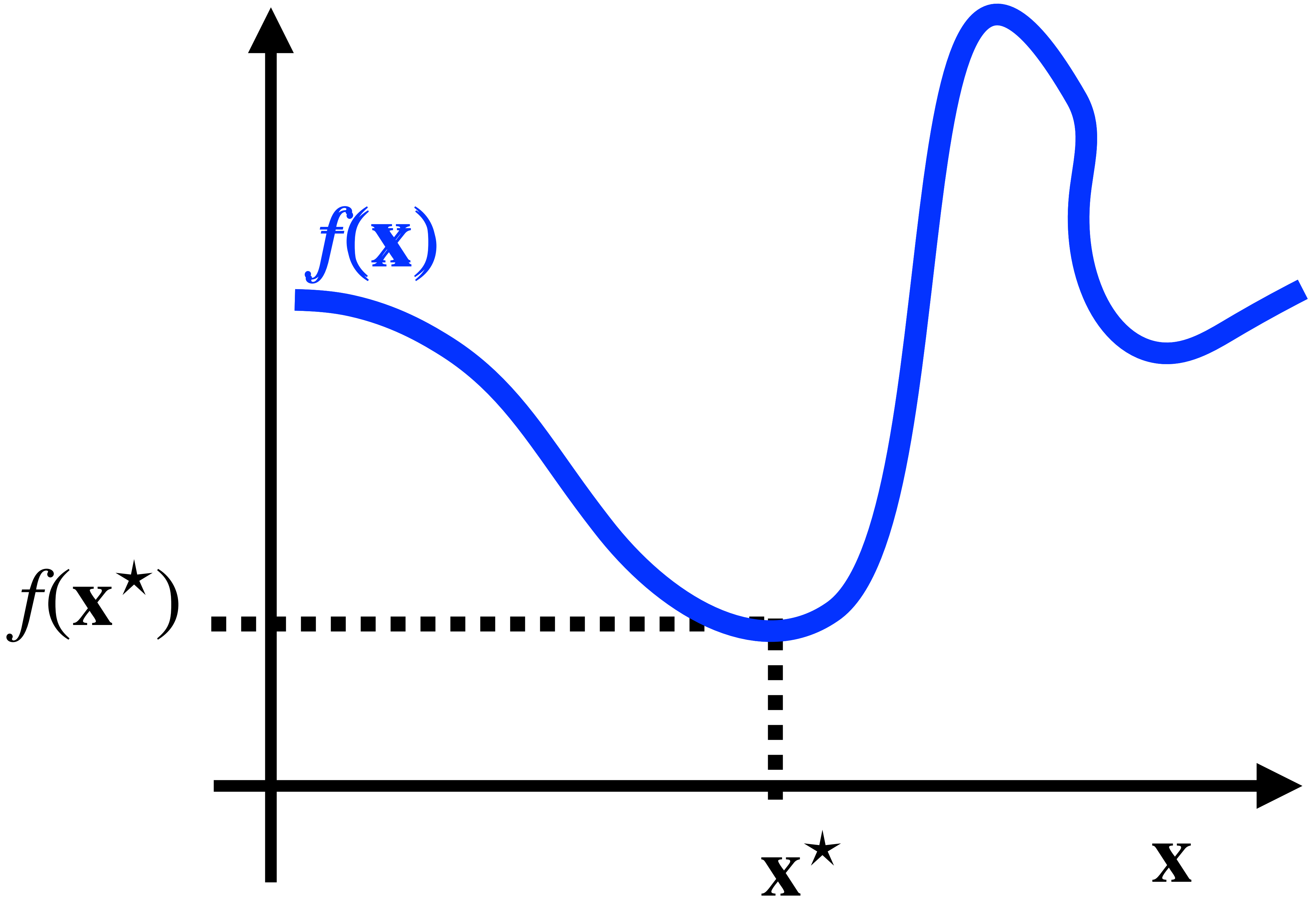
$D_{KL}\left(p(\mathbf{x}) \parallel q(\mathbf{x})\right)$ =0.2036



https://gnarlyware.com/blog/kl-divergence-online-demo/

Prerequisites: argmin

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}} f(\mathbf{x}) = \arg\min_{\mathbf{x}} \log\big(f(\mathbf{x})\big)$$



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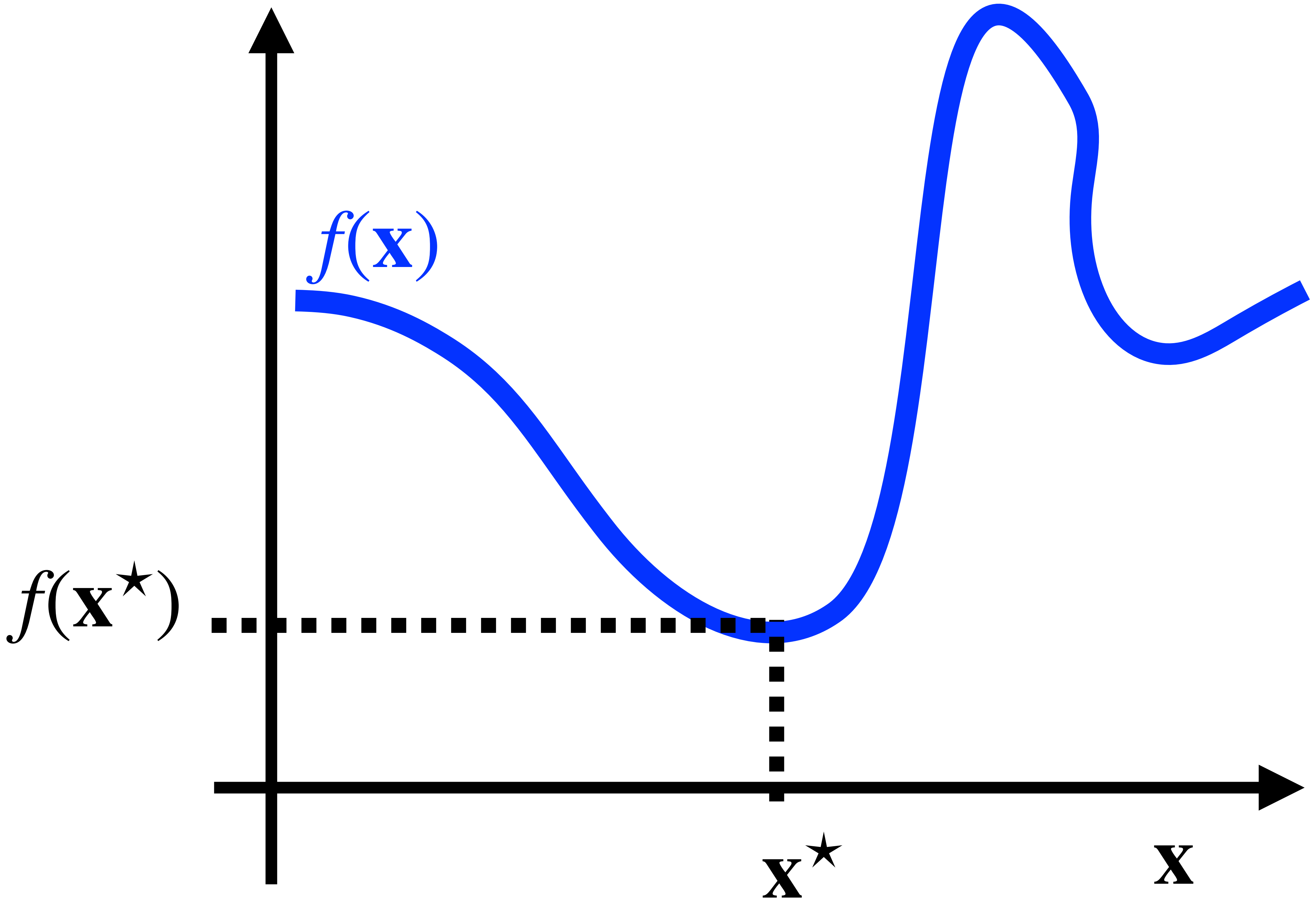

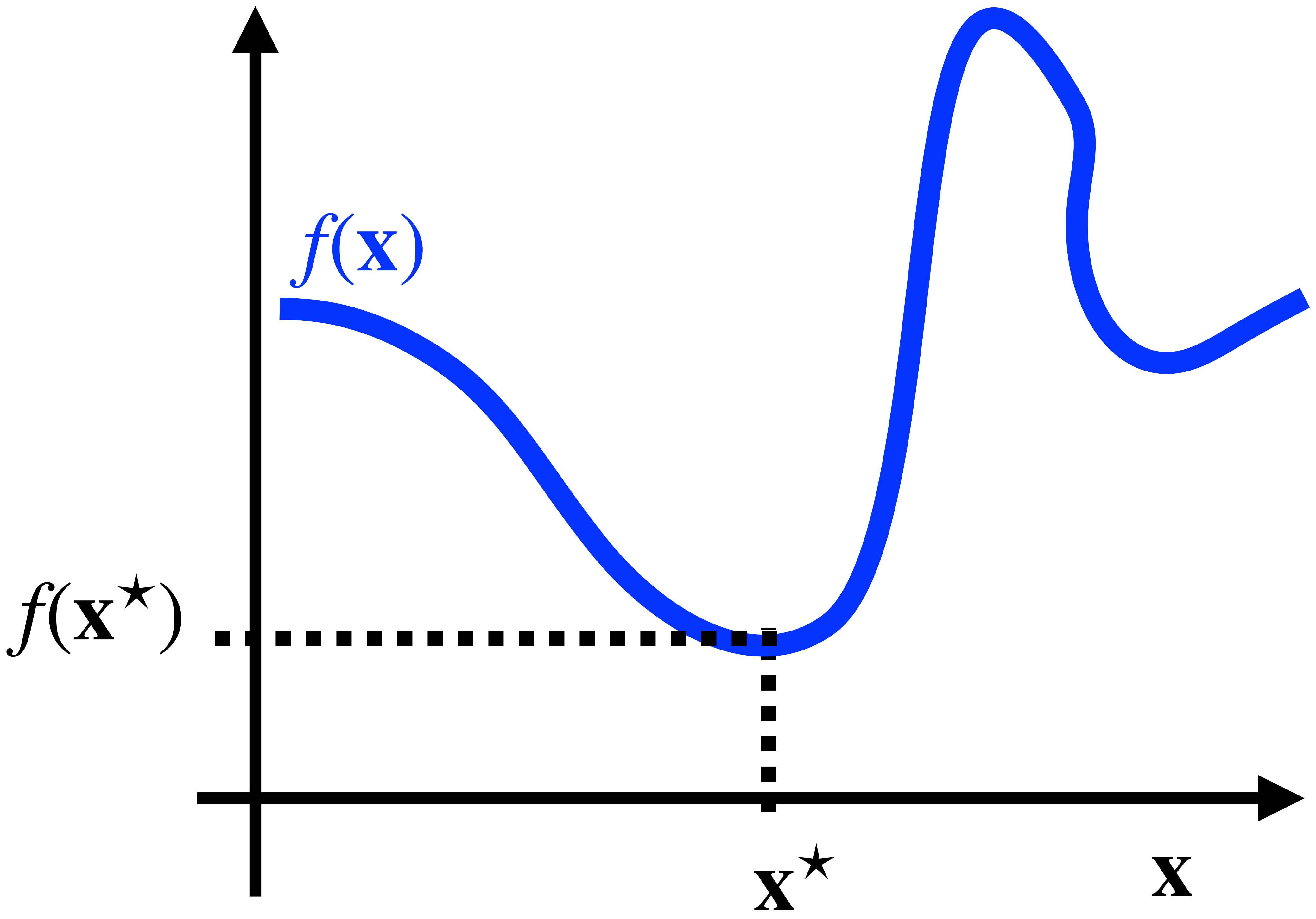

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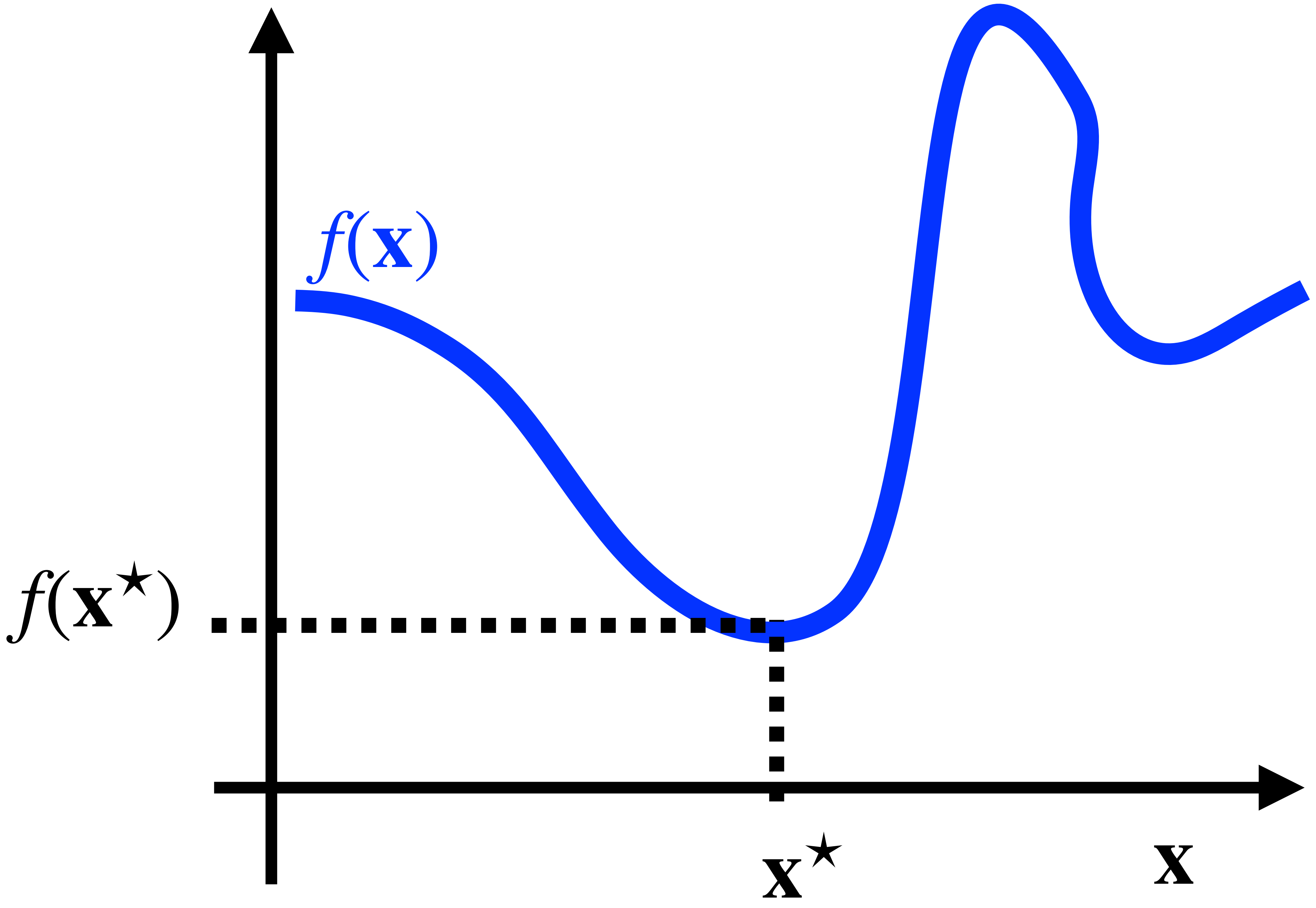
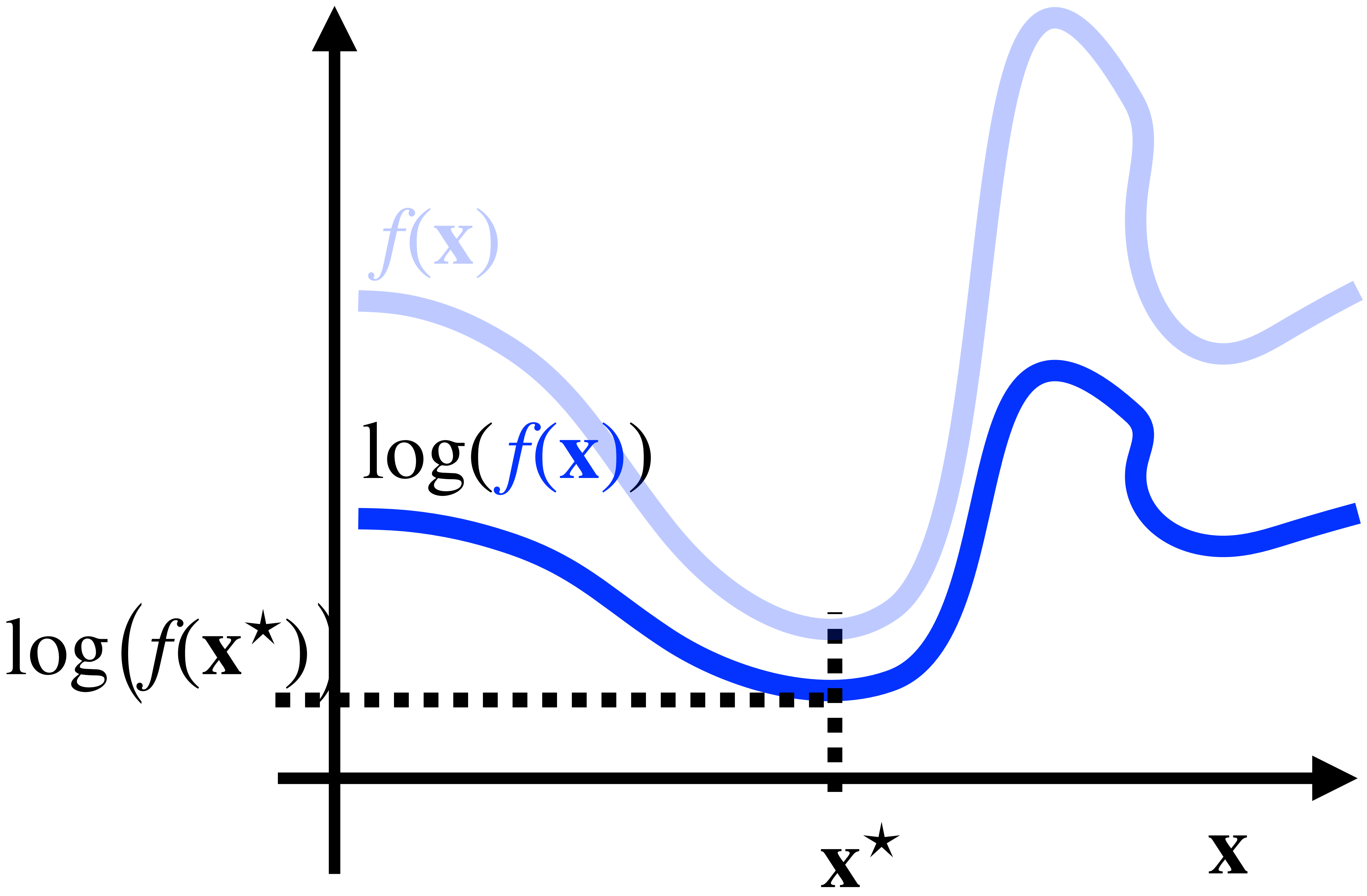


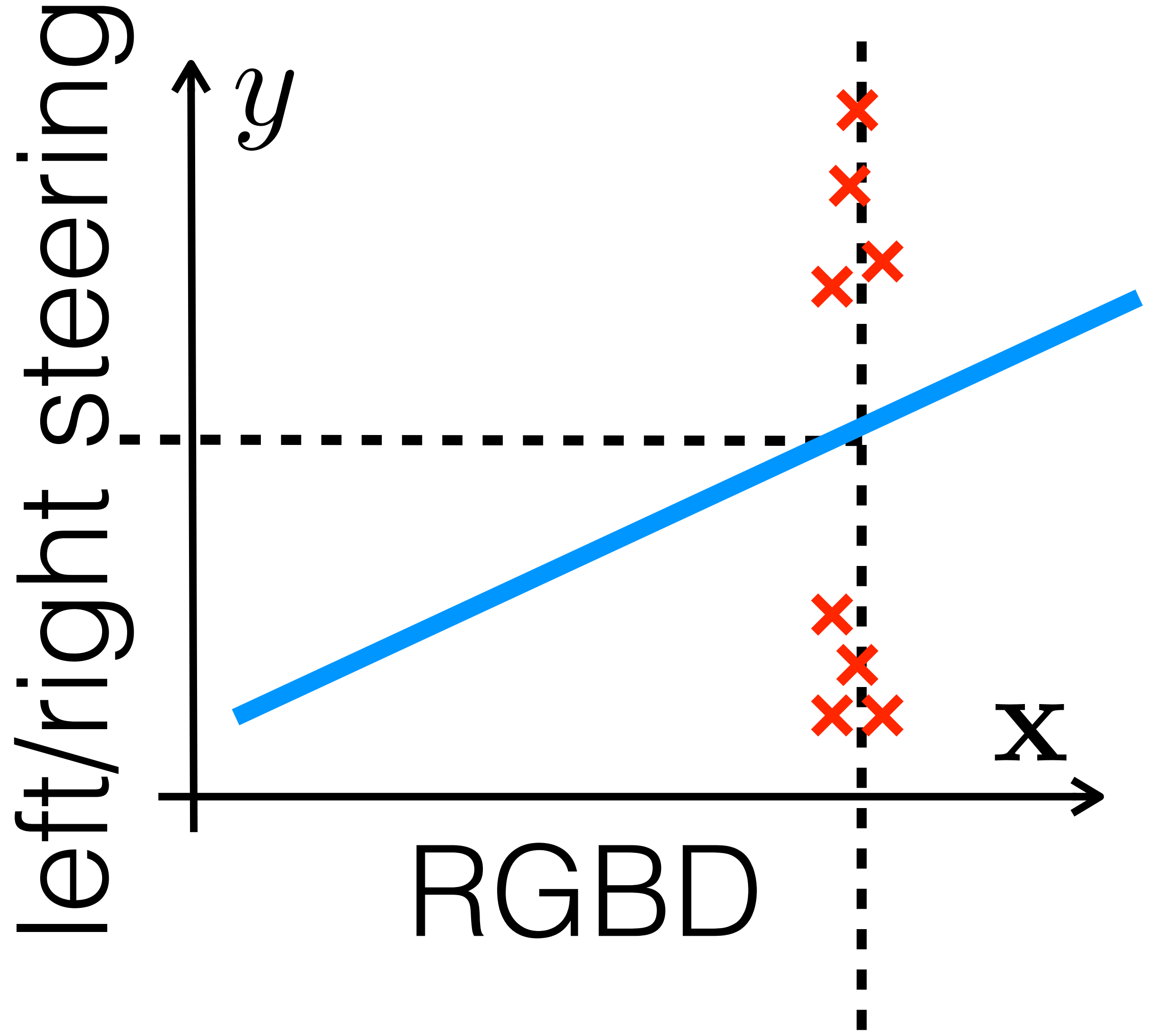


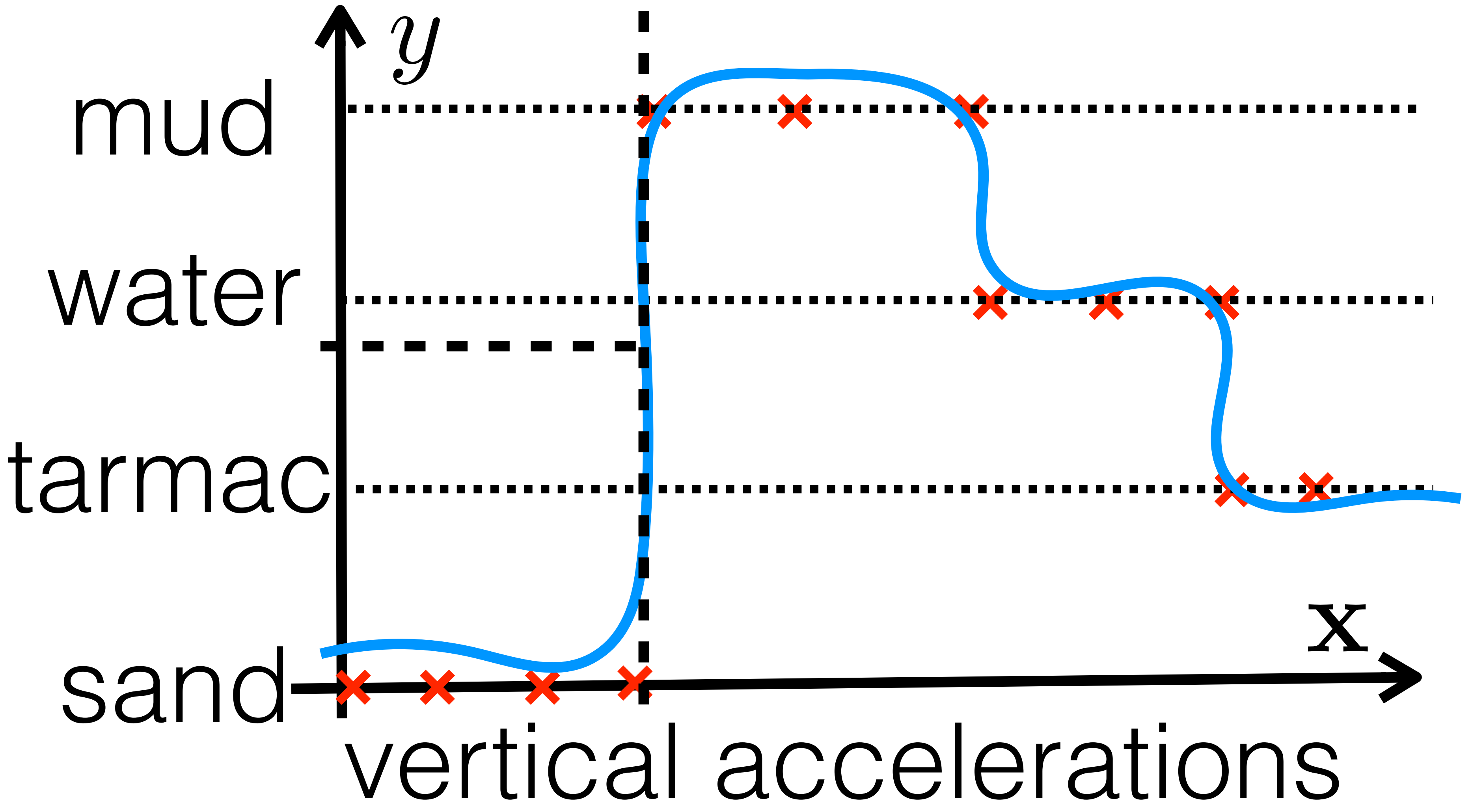
Where does the **loss function** come from?

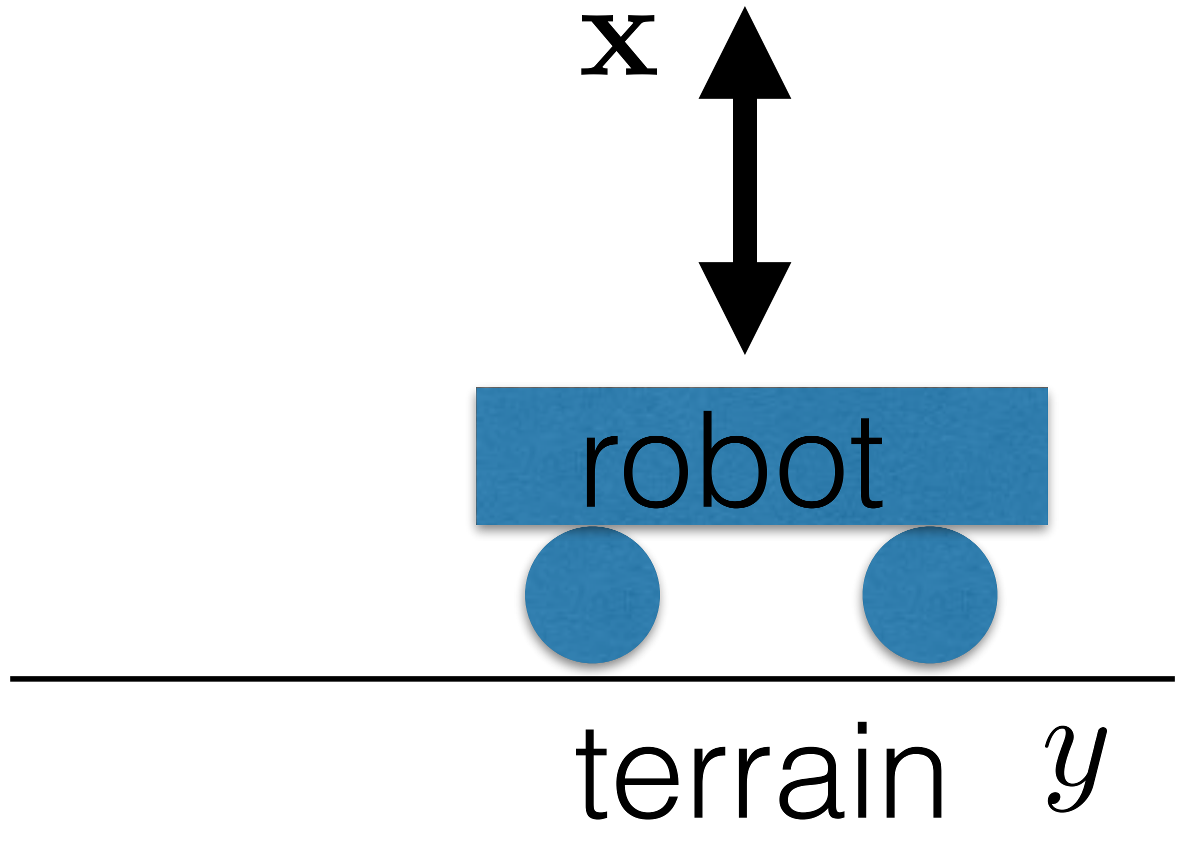
Lec 01: what can go wrong: **inappropriate choice of loss function**

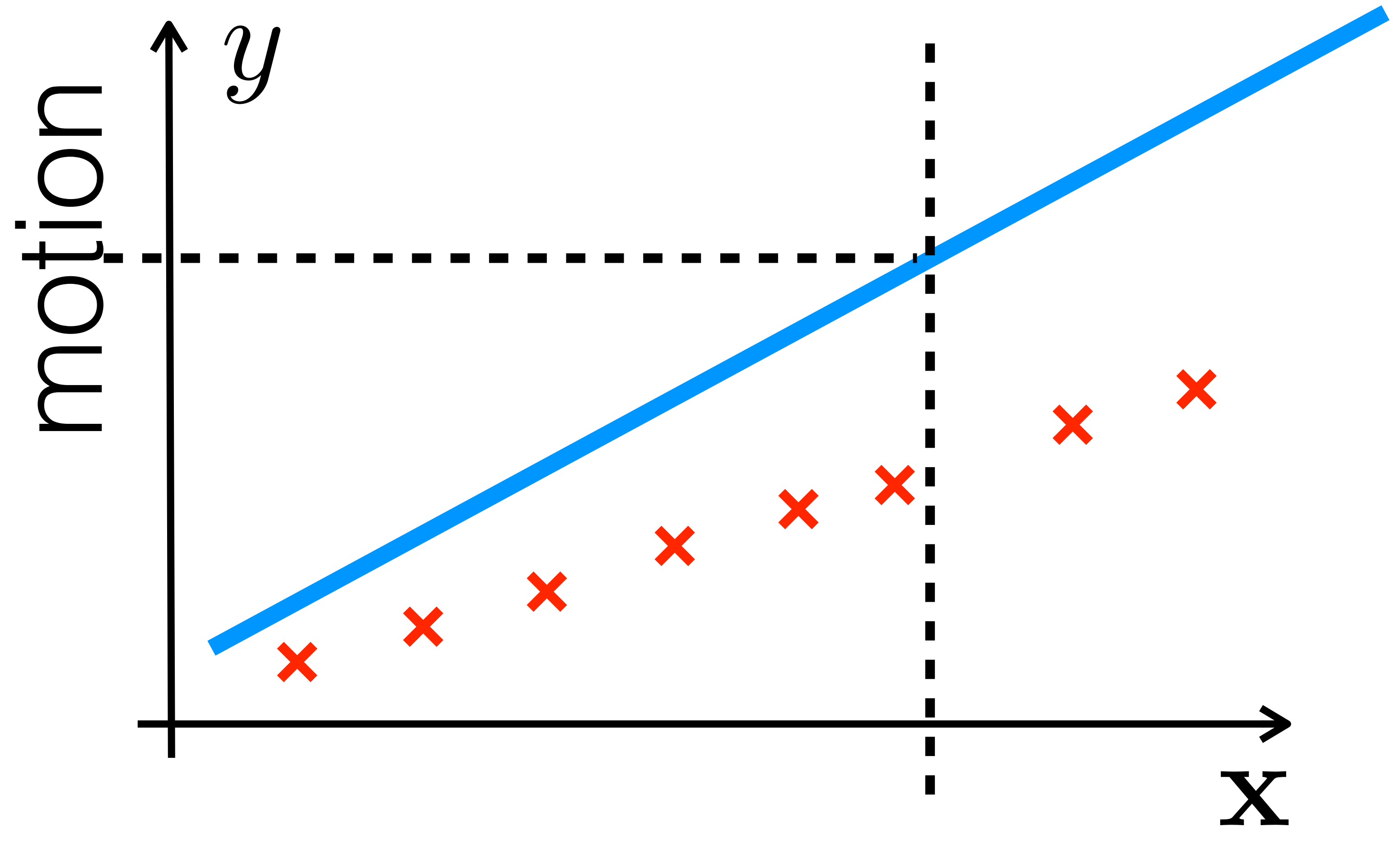
What should I do instead of fitting a curve???

Search for probability distribution of y given x

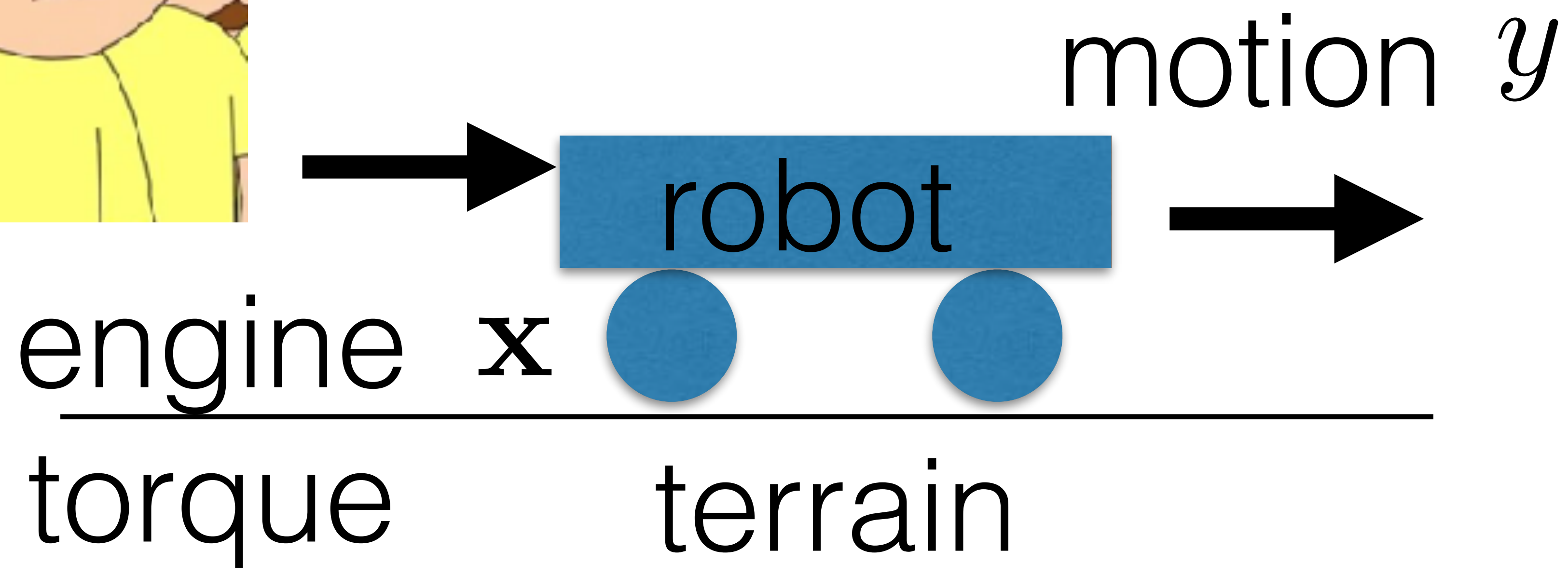






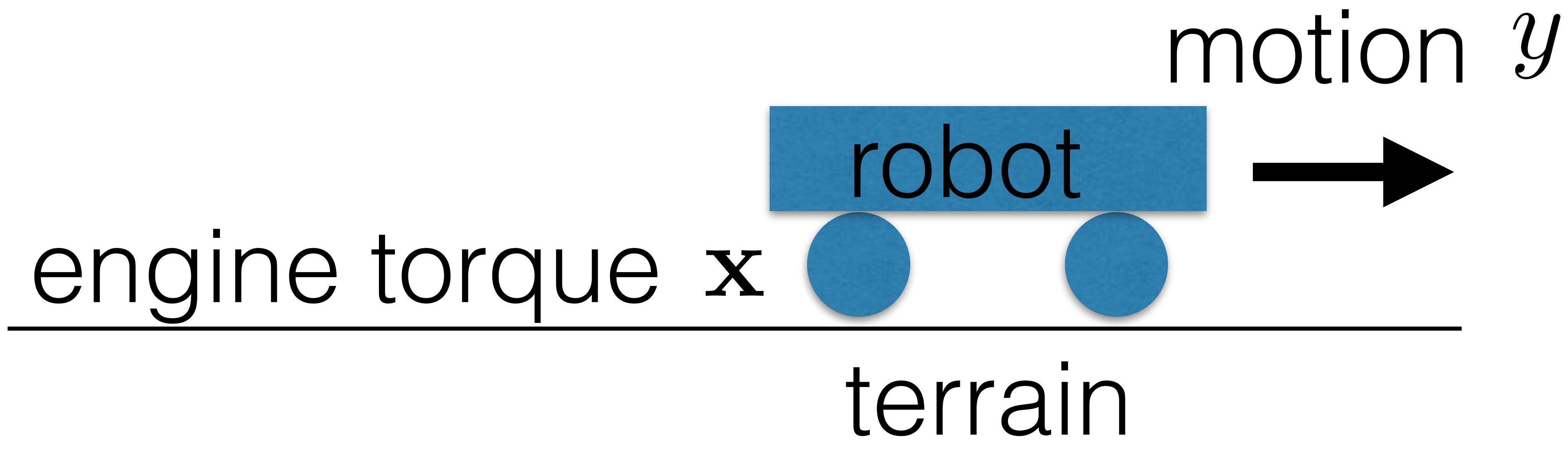


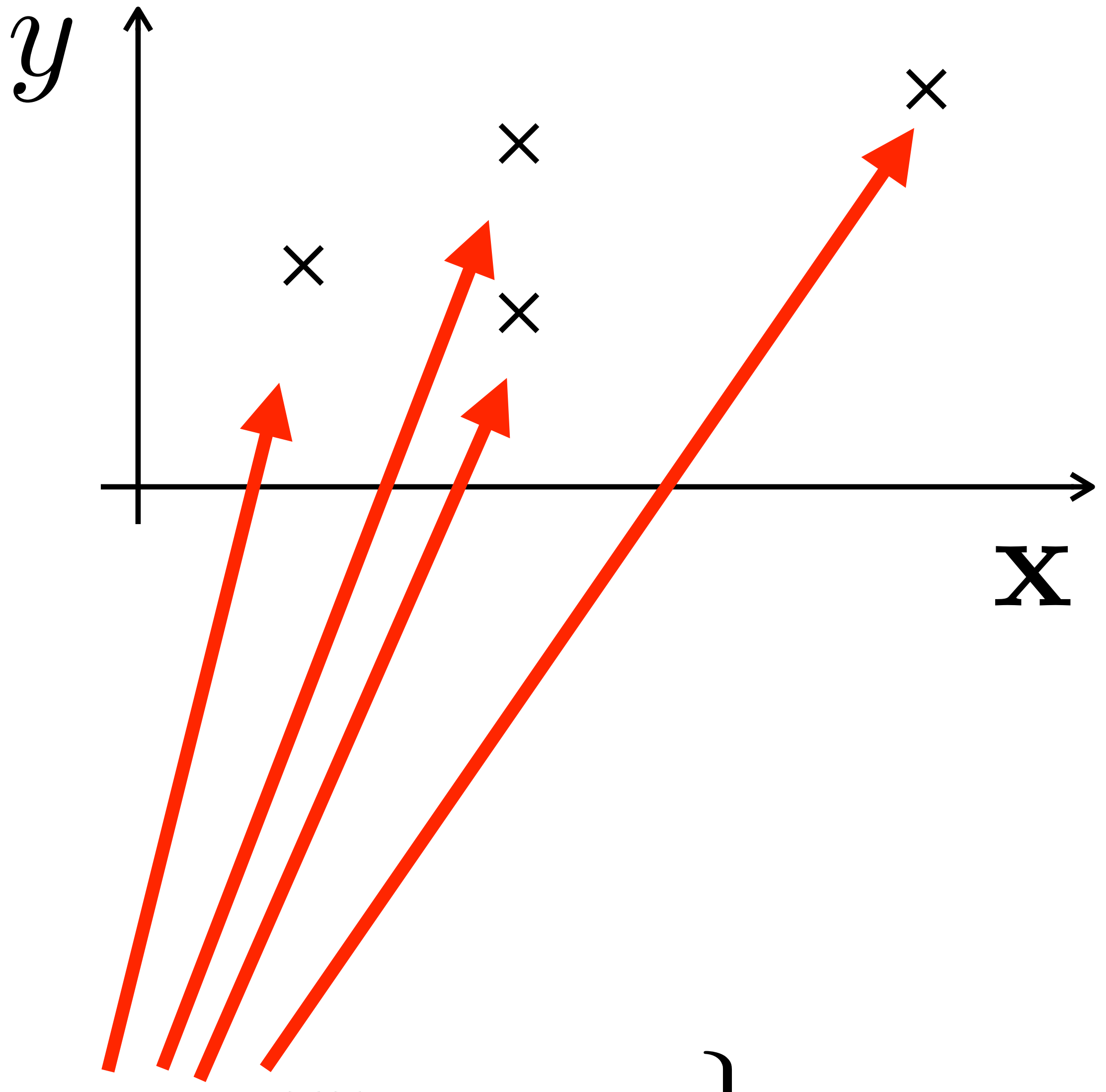
Nature is evil !!!



https://rickandmorty.fandom.com/wiki/Evil_Morty

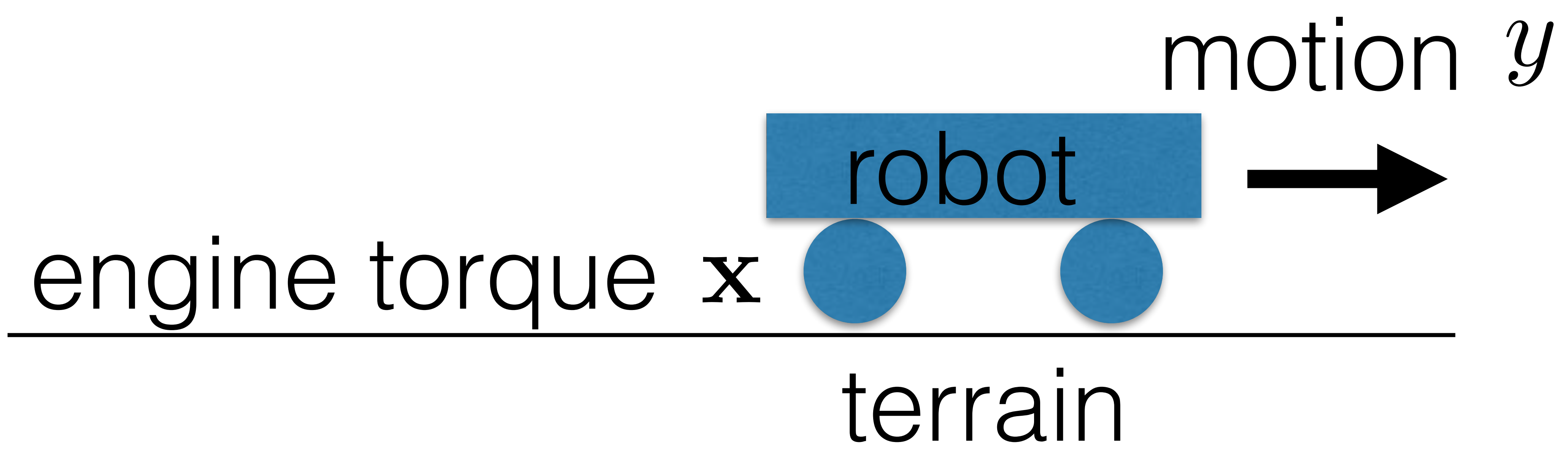
Motivation example: estimation of a motion model

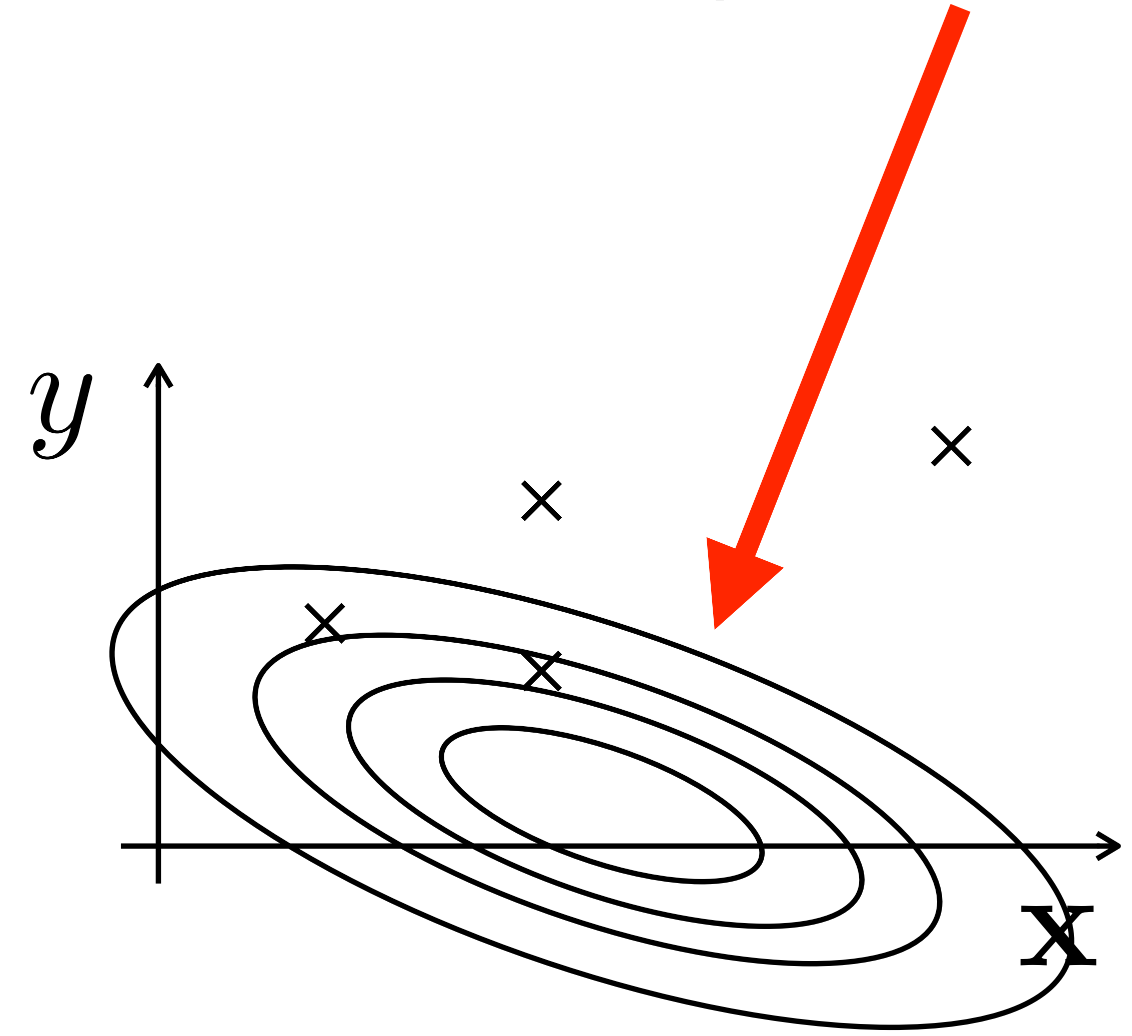




$$\mathcal{D} = \{\mathbf{x}_1, y_1 \ldots \mathbf{x}_N, y_N\}$$

Motivation example: estimation of a motion model





Unknown distribution with parameters **w**

- We search for parameters $\mathbf{w}$ of unknown distribution given measurements $\mathcal{D} = \{\mathbf{x}_1, y_1 \dots \mathbf{x}_N, y_N\}$

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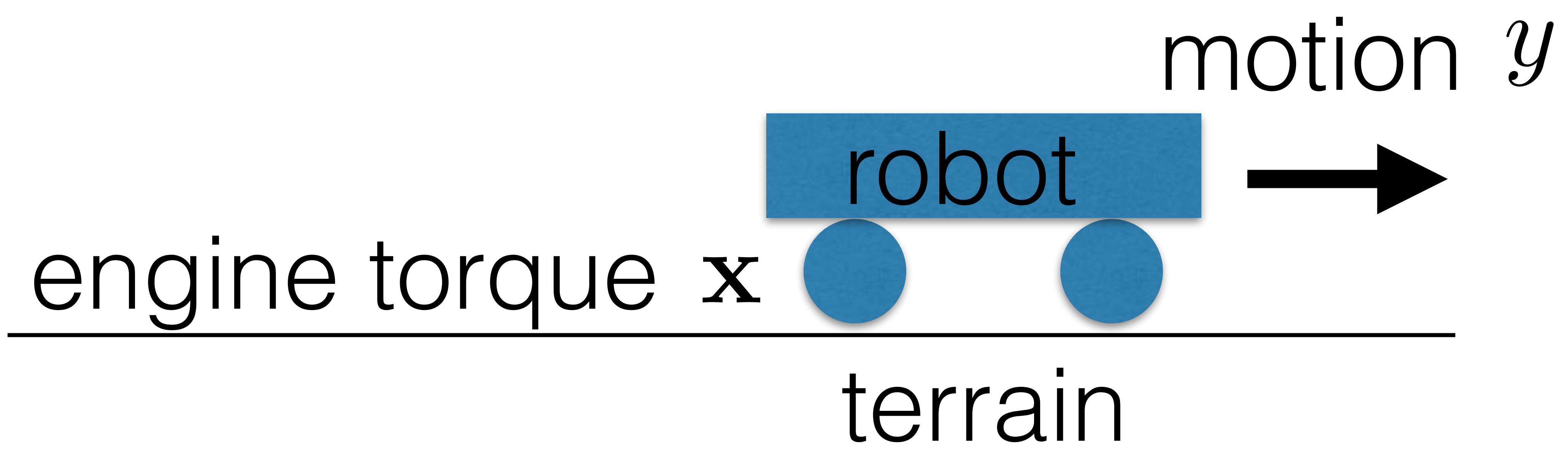

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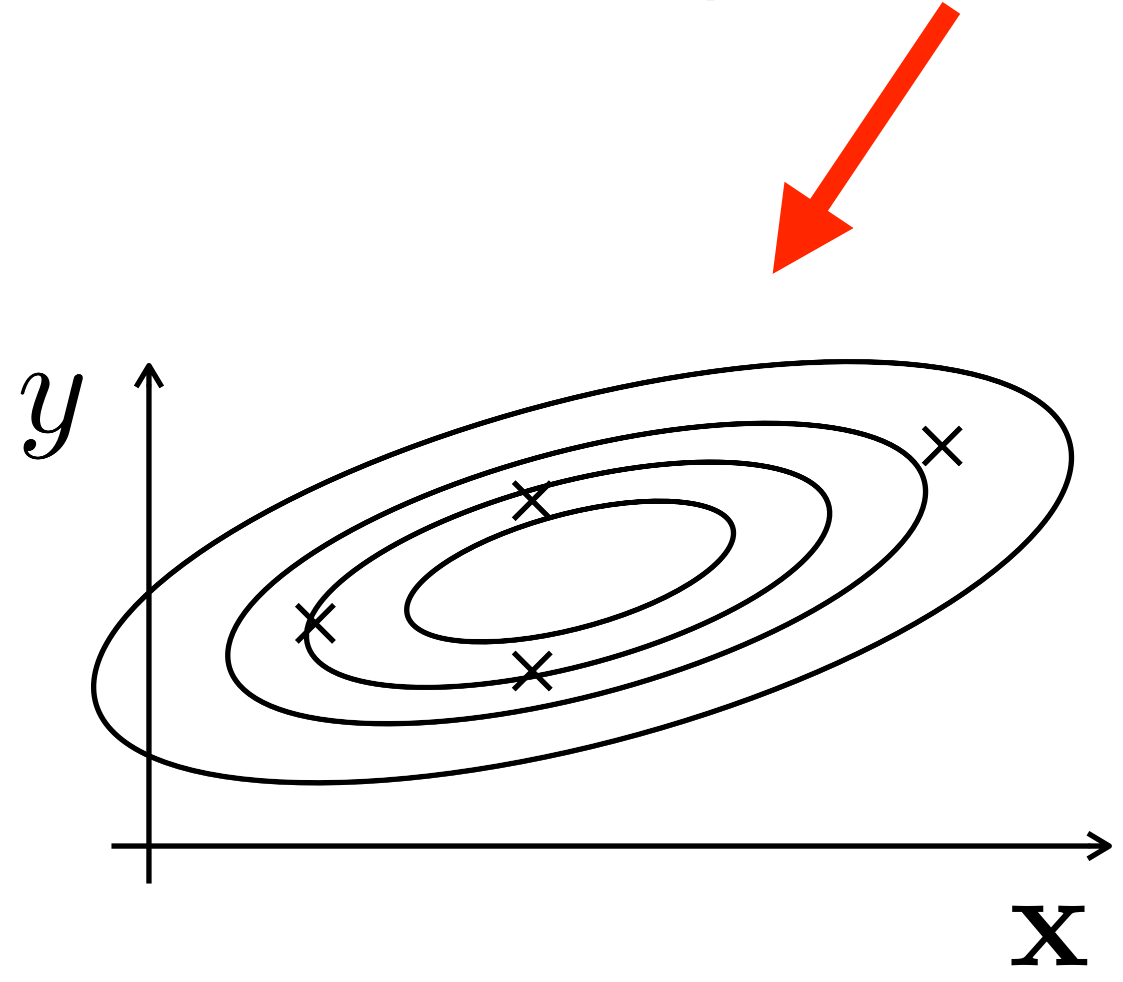

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How should we fit distribution into data?

How should we fit distribution into data?

- Many robotics tasks contain perception problems: given $\mathbf{x}$ estimate y

$\mathbf{x}$

y

🟪 road

🟪 sideway

🟥 pedestrian

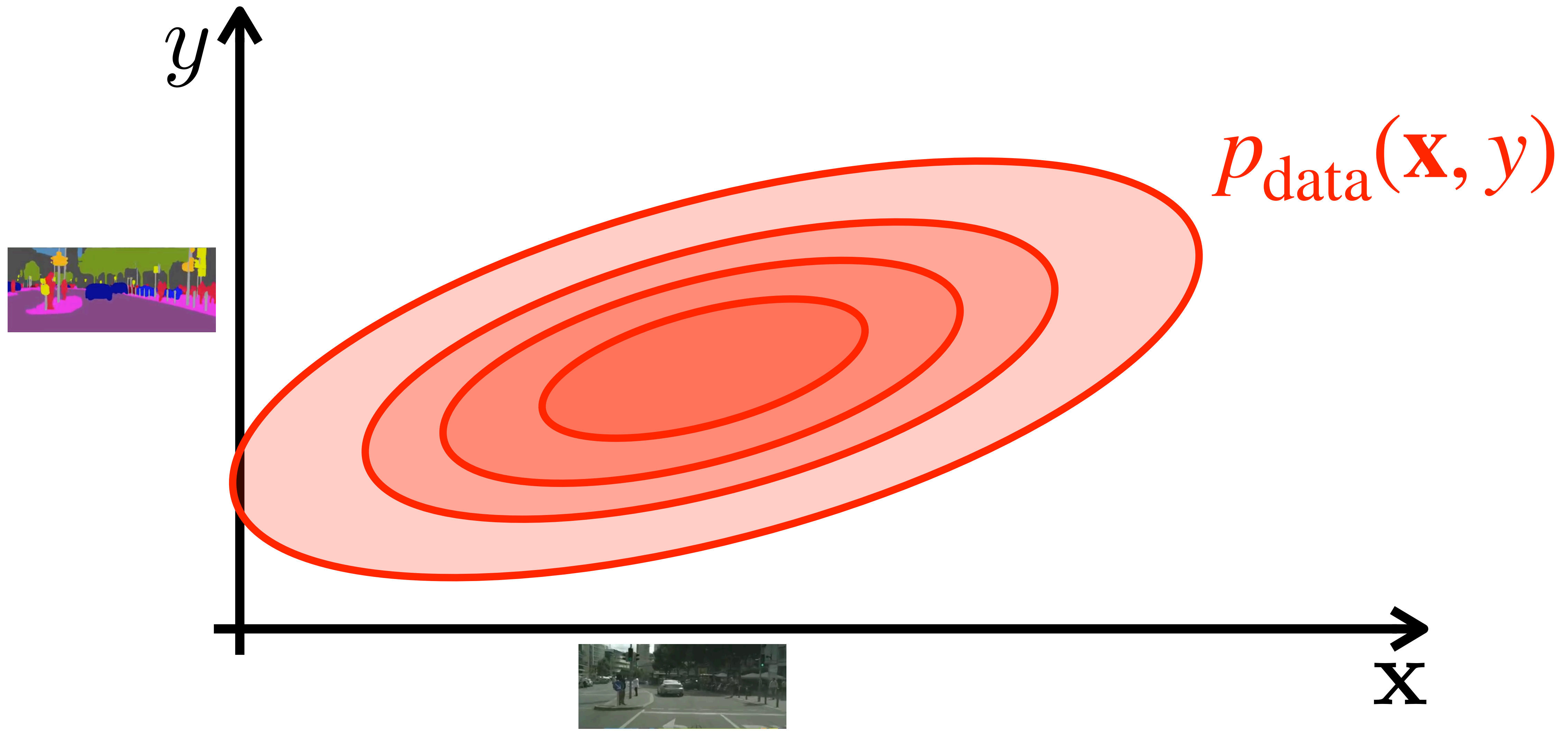
🟨 traffic sign

🟩 trees

🟦 sky

Why should I minimize -log(p)?

- $(\mathbf{x}, y)$-tuples live on unknown distribution $p_{\text{data}}(\mathbf{x}, y)$



Why should I minimize -log(p)?

- $(\mathbf{x}, y)$-tuples live on unknown distribution $p_{\text{data}}(\mathbf{x}, y)$
- We approximate it by $p(\mathbf{x}, y \,|\, \mathbf{w})$
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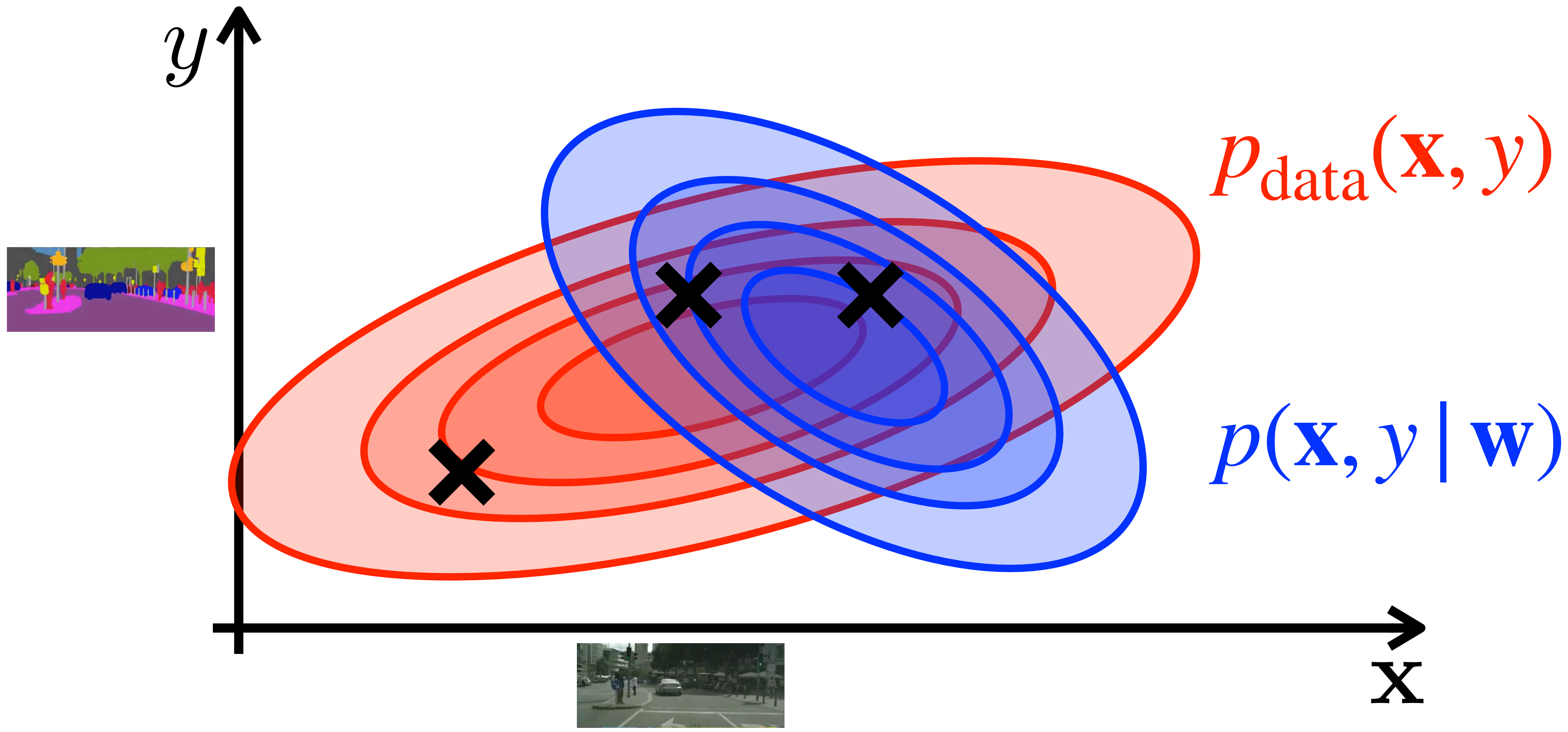
$$= \arg\min_{\mathbf{w}} \mathbb{E}_{(\mathbf{x},y)}\big[-\log p(y \,|\, \mathbf{w}, \mathbf{x})\big] \approx \arg\min_{\mathbf{w}} \frac{1}{N} \sum_{(\mathbf{x}_i,y_i)\,\sim\, p_{\text{data}}(\mathbf{x},y)} \big[-\log p(y_i \,|\, \mathbf{x}_i, \mathbf{w})\big]$$ …KL justification



MLE:

$$= \arg\min_{\mathbf{w}} -\log \prod_{(\mathbf{x}_i,y_i)} p(y_i \,|\, \mathbf{x}_i, \mathbf{w})$$

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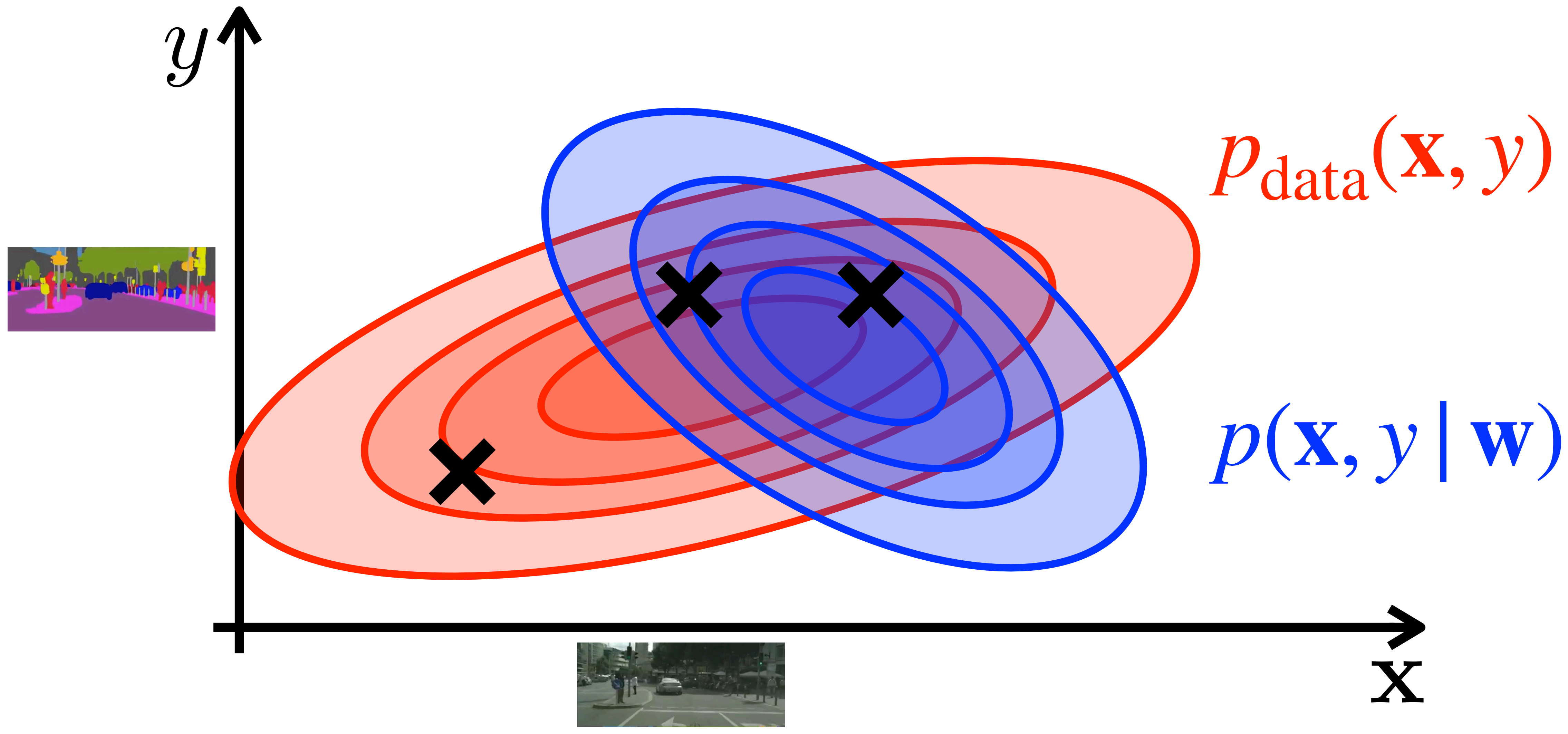

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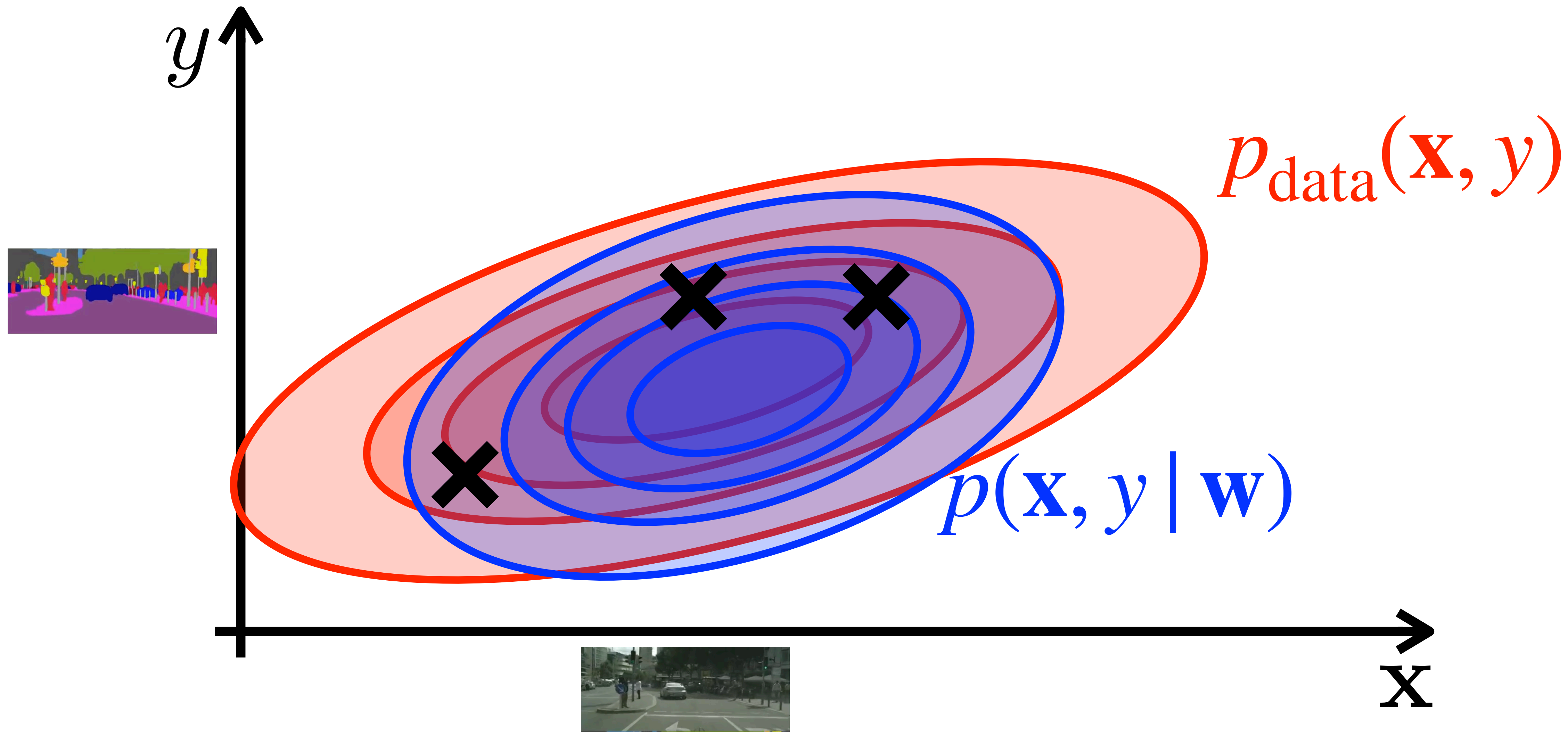
$$= \arg\min_{\mathbf{w}} \mathbb{E}_{(\mathbf{x},y)}\big[-\log p(y \,|\, \mathbf{w}, \mathbf{x})\big] \approx \arg\min_{\mathbf{w}} \frac{1}{N} \sum_{(\mathbf{x}_i,y_i)\,\sim\, p_{\text{data}}(\mathbf{x}, y)} \big[-\log p(y_i \,|\, \mathbf{x}_i, \mathbf{w})\big] \ldots \text{KL justification}$$



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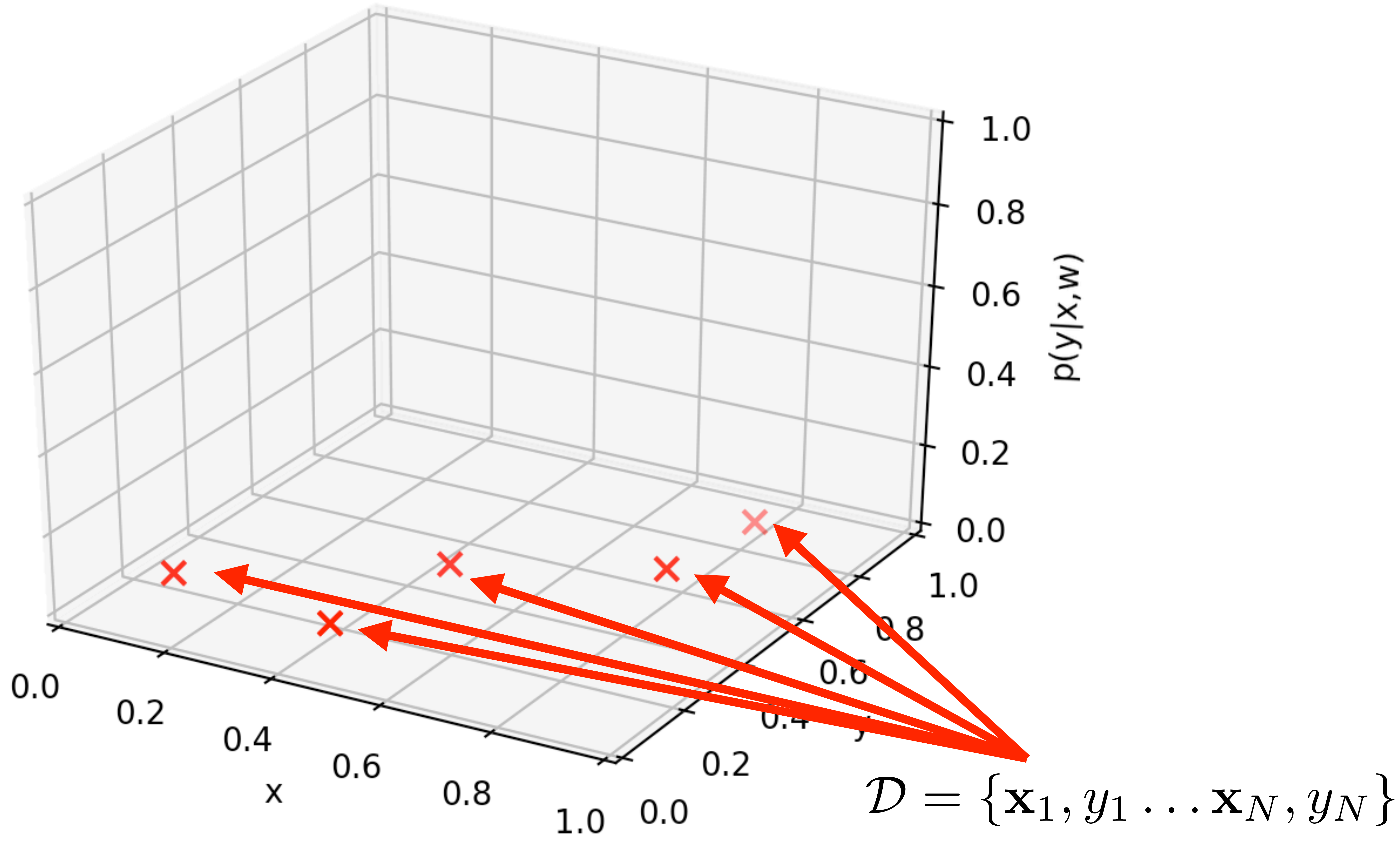
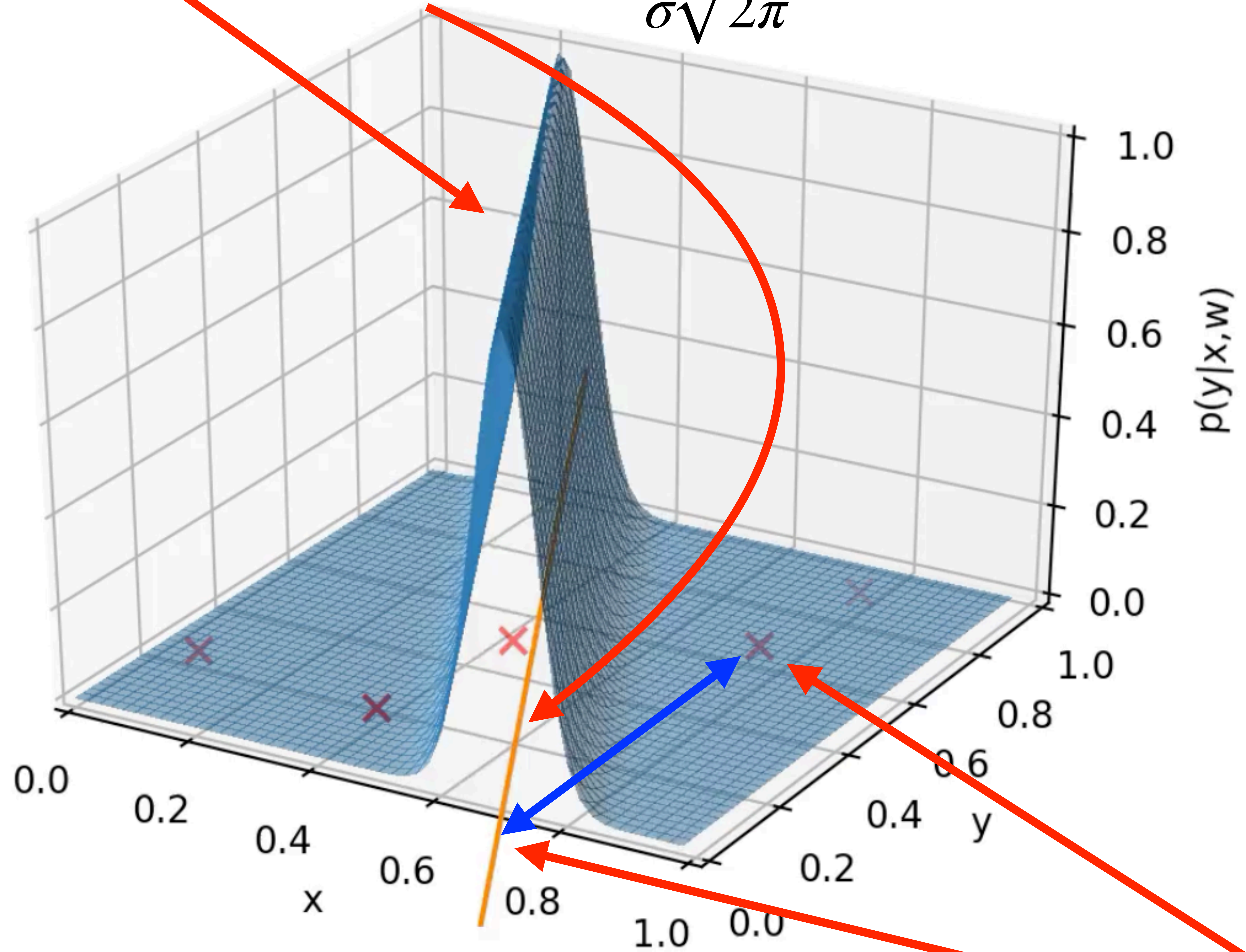
$p(y \mid \mathbf{x}, \mathbf{w}) =$



$\mathcal{D} = \{\mathbf{x}_1, y_1 \ldots \mathbf{x}_N, y_N\}$

$$p(y\,|\,\mathbf{x}, \mathbf{w}) = \mathcal{N}(y;\ w_1x + w_0, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{w_1x+w_0-y}{\sigma}\right)^2}$$



Prove it !!!

$$\mathbf{w}^\star = \arg\max_{\mathbf{w}} \prod_i \mathcal{N}(y_i;\ w_1x_i + w_0, \sigma) = \arg\min_{\mathbf{w}} \sum_i (w_1x_i + w_0 - y_i)^2$$

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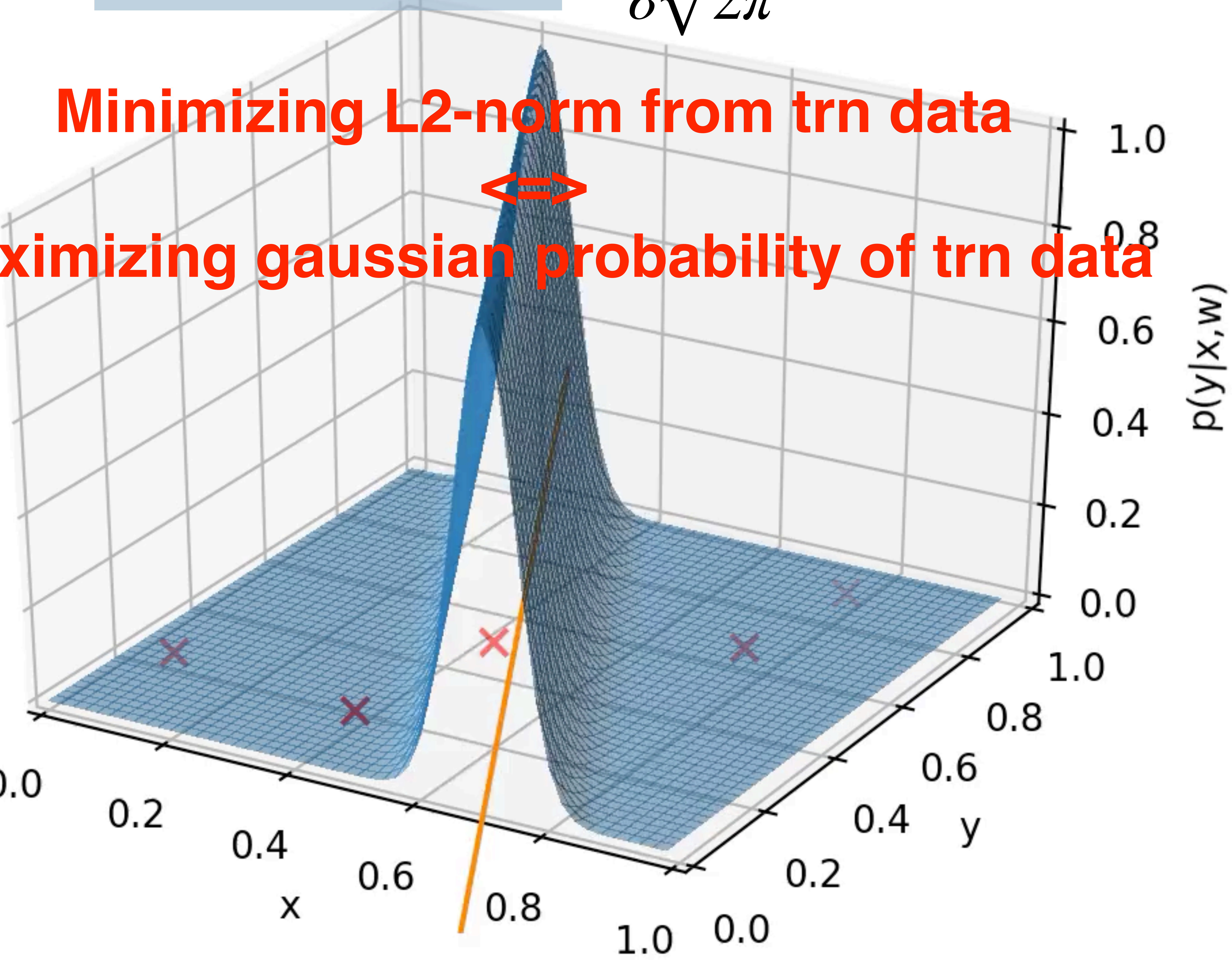


Minimizing L2-norm from trn data

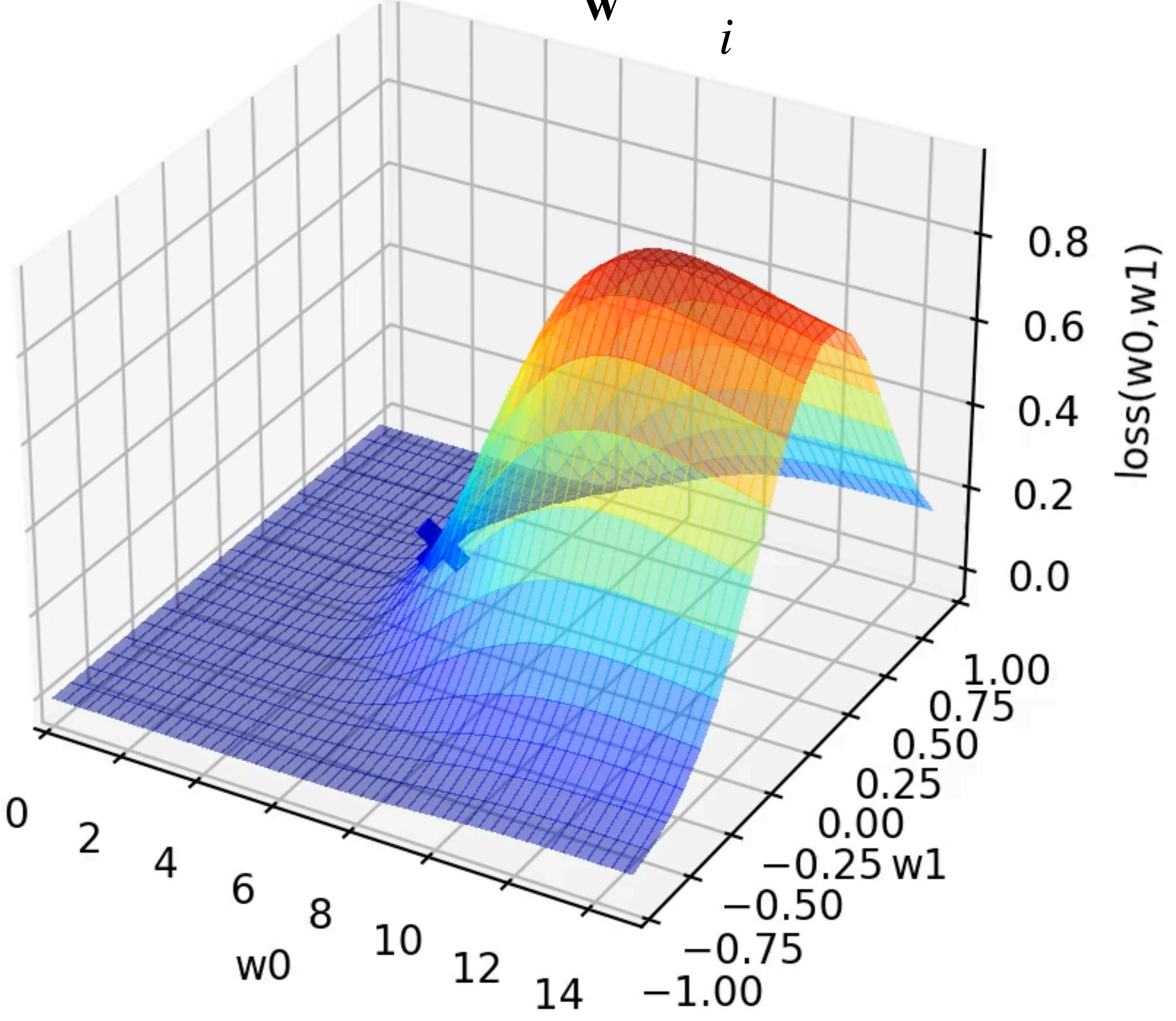
<=>

Maximizing gaussian probability of trn data

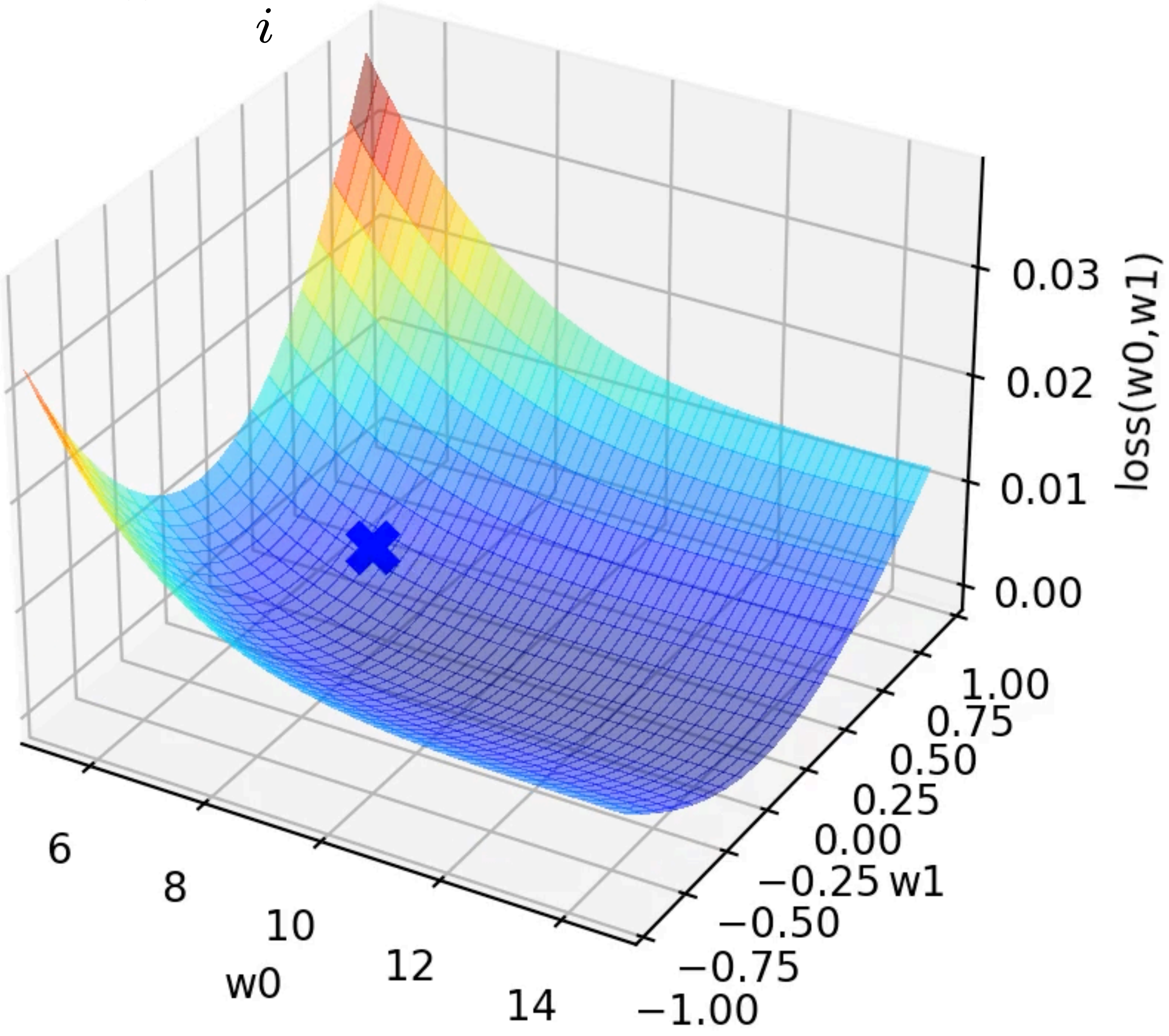
$$\mathbf{w}^{\star} = \arg\max_{\mathbf{w}} \prod_i \mathcal{N}(y_i;\ f(\mathbf{x}_i,\mathbf{w}),\sigma) = \arg\min_{\mathbf{w}} \sum_i (w_1x_i + w_0 - y_i)^2$$

- In what sense is the MLE and the LSQ formulations equivalent?

$$\mathbf{w}^\star = \arg\max_{\mathbf{w}} \prod_i \mathcal{N}(y_i;\, f(\mathbf{x}_i, \mathbf{w}), \sigma) \quad = \arg\min_{\mathbf{w}} \sum_i (f(\mathbf{x}_i, \mathbf{w}) - y_i)^2$$



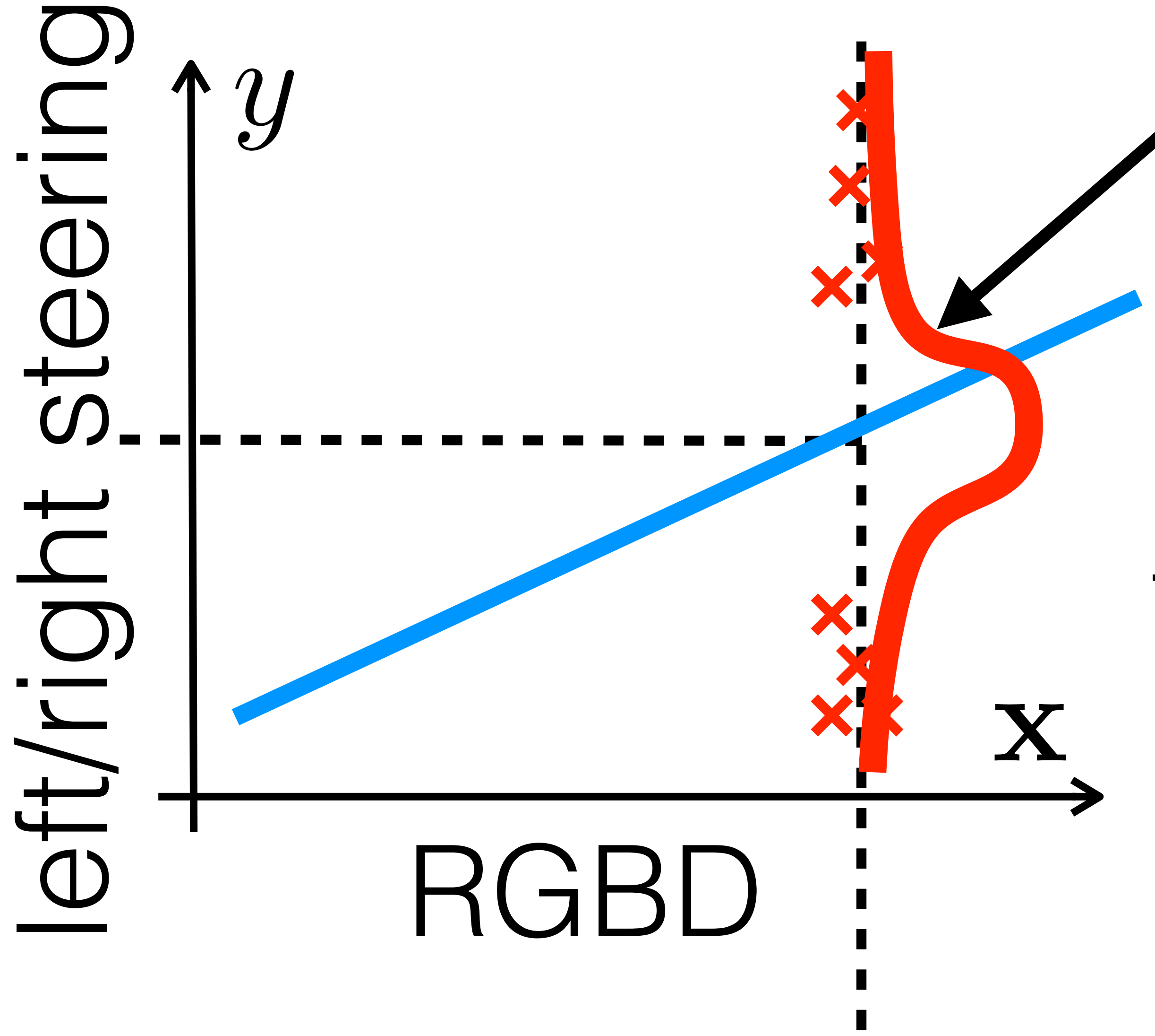
MLE

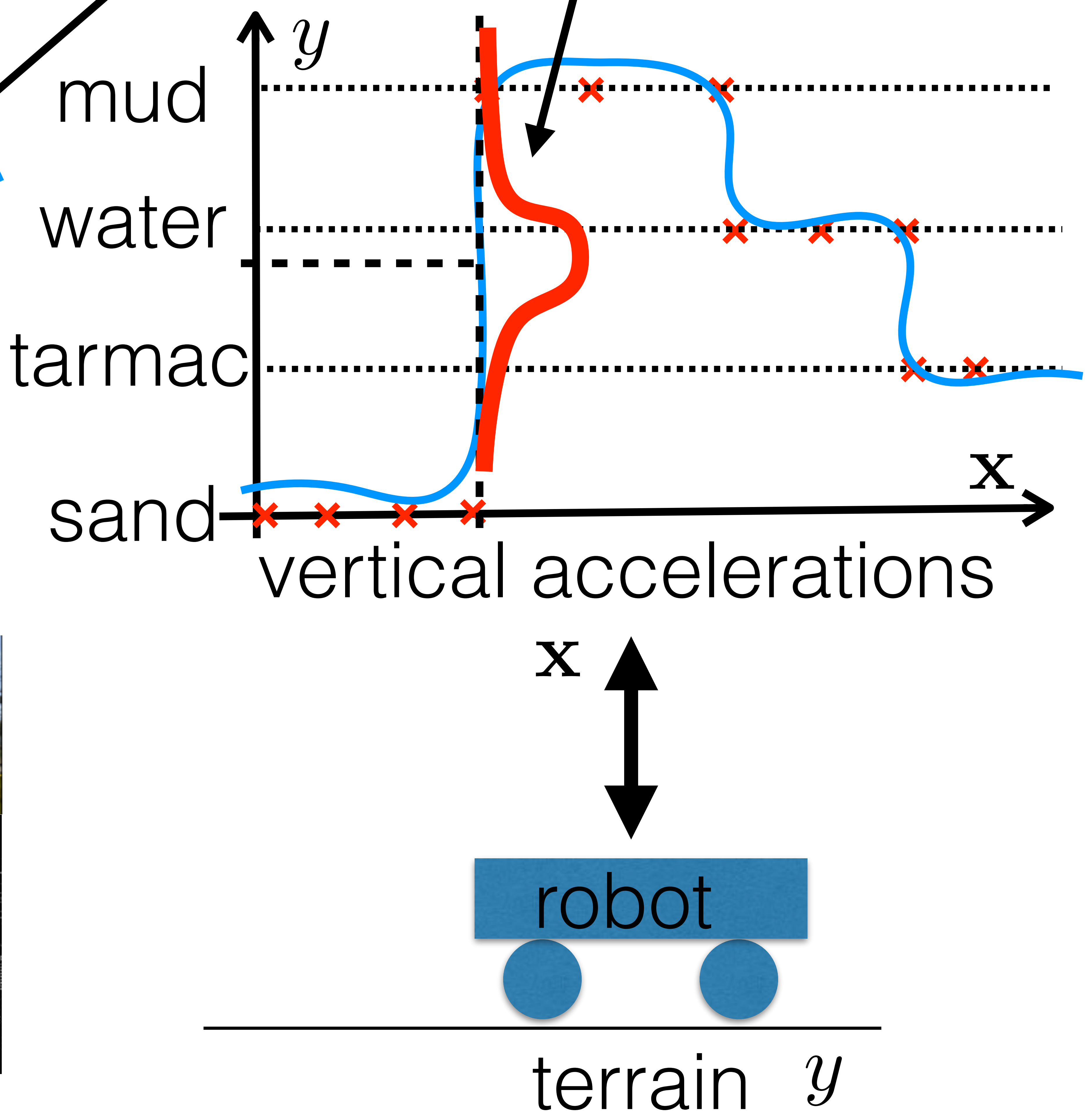


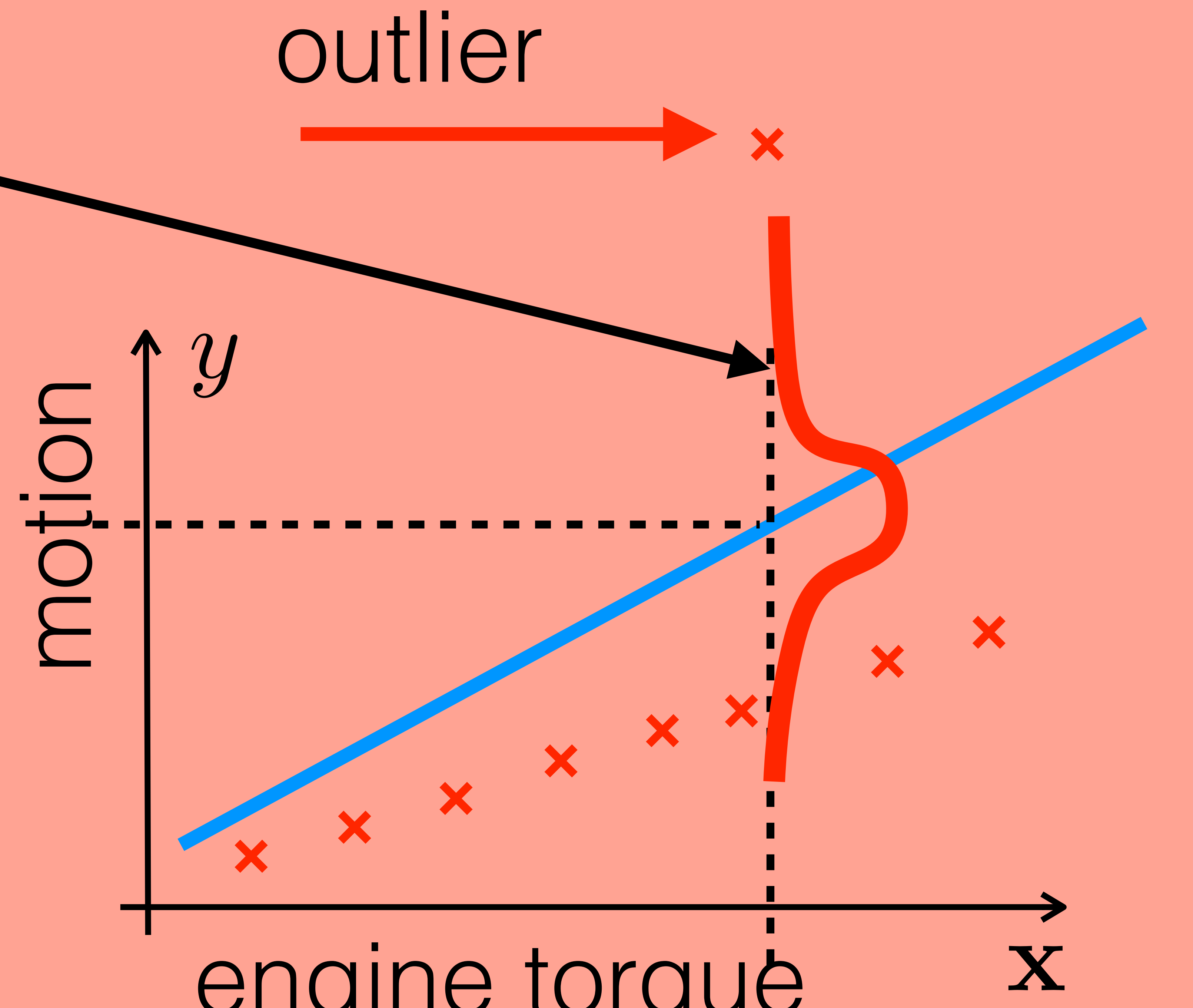
LSQ

What can go wrong: **inappropriate choice of loss function**

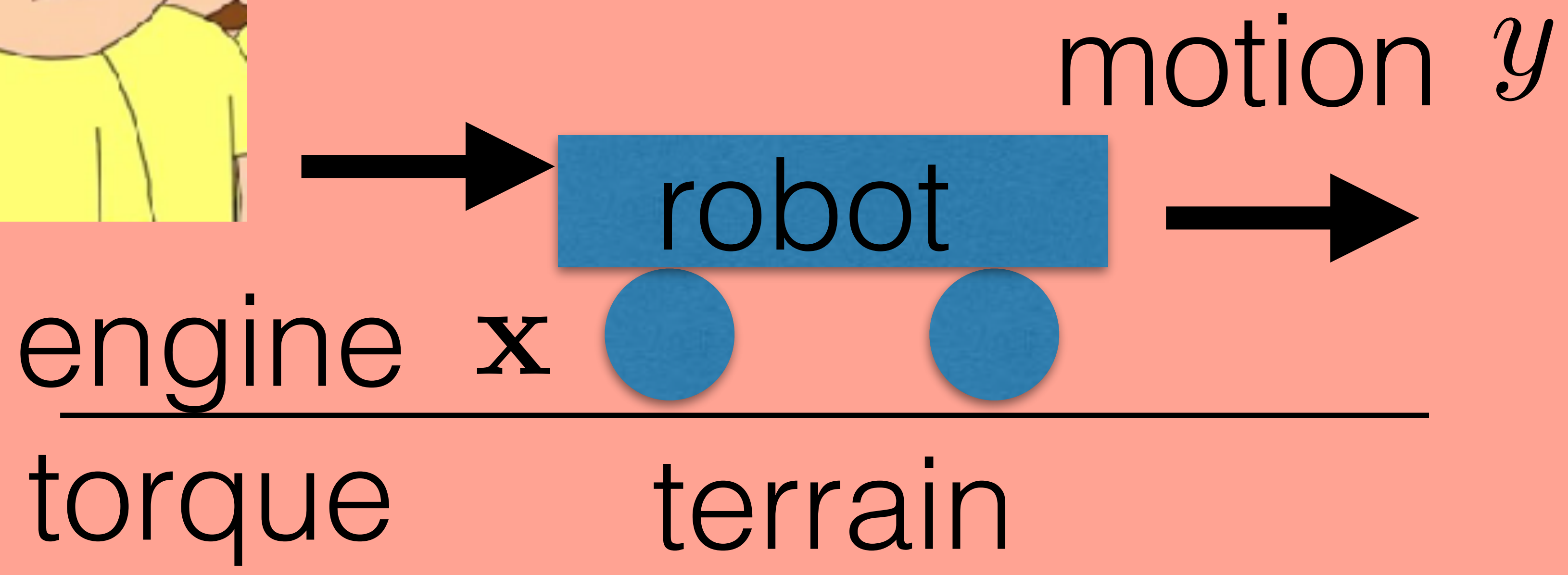
This is the problem!





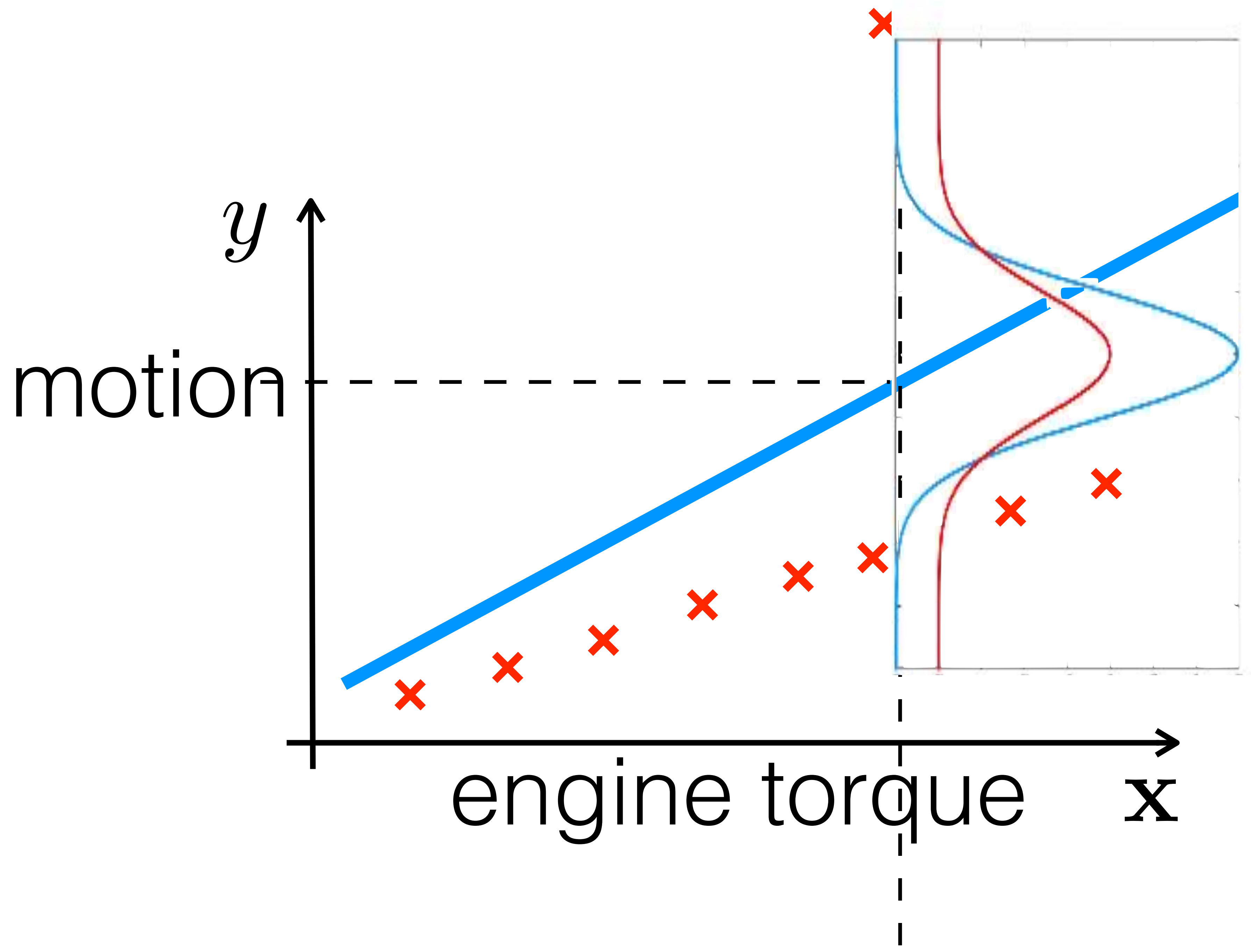


Nature is evil !!!



https://rickandmorty.fandom.com/wiki/Evil_Morty

Robust regression



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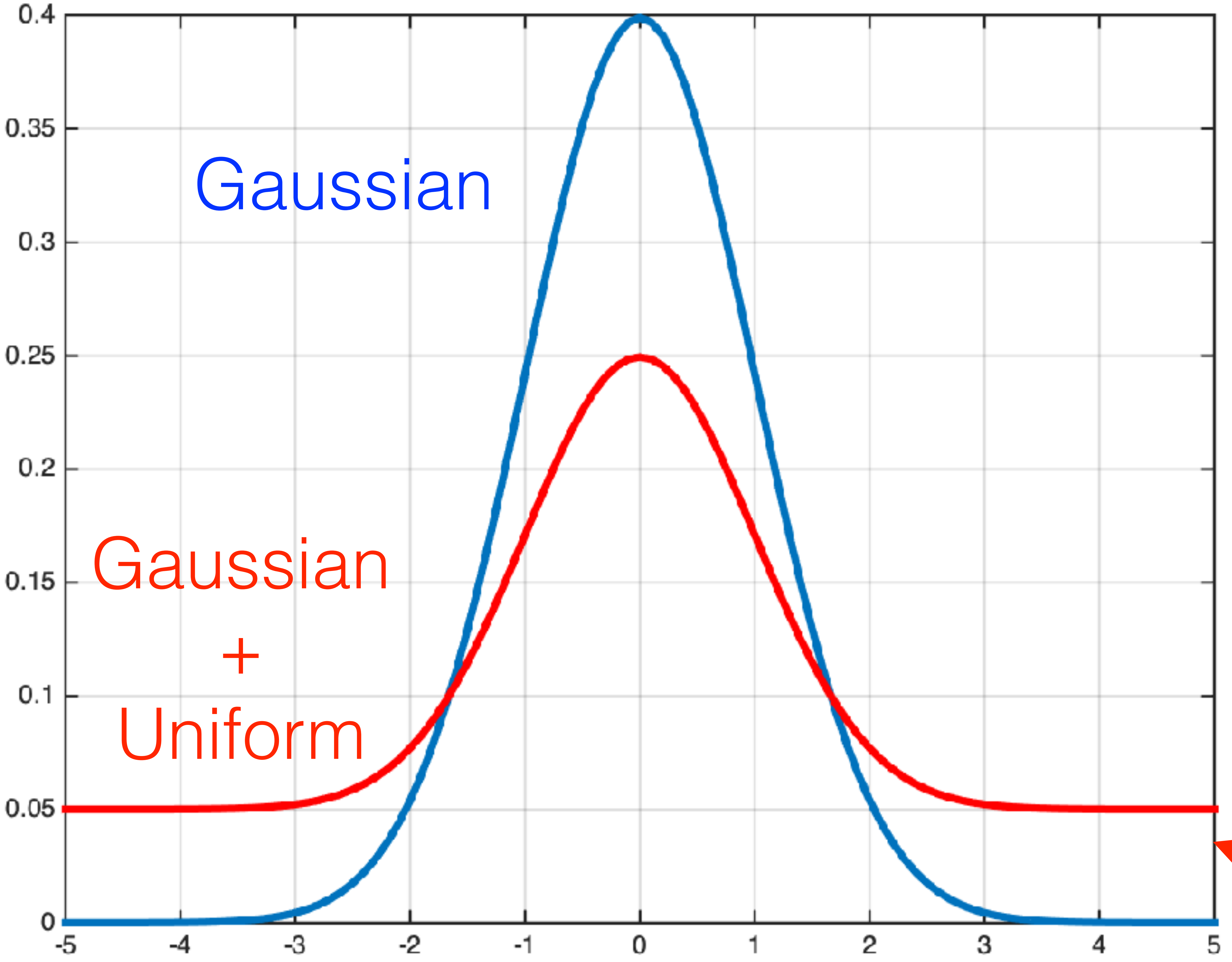
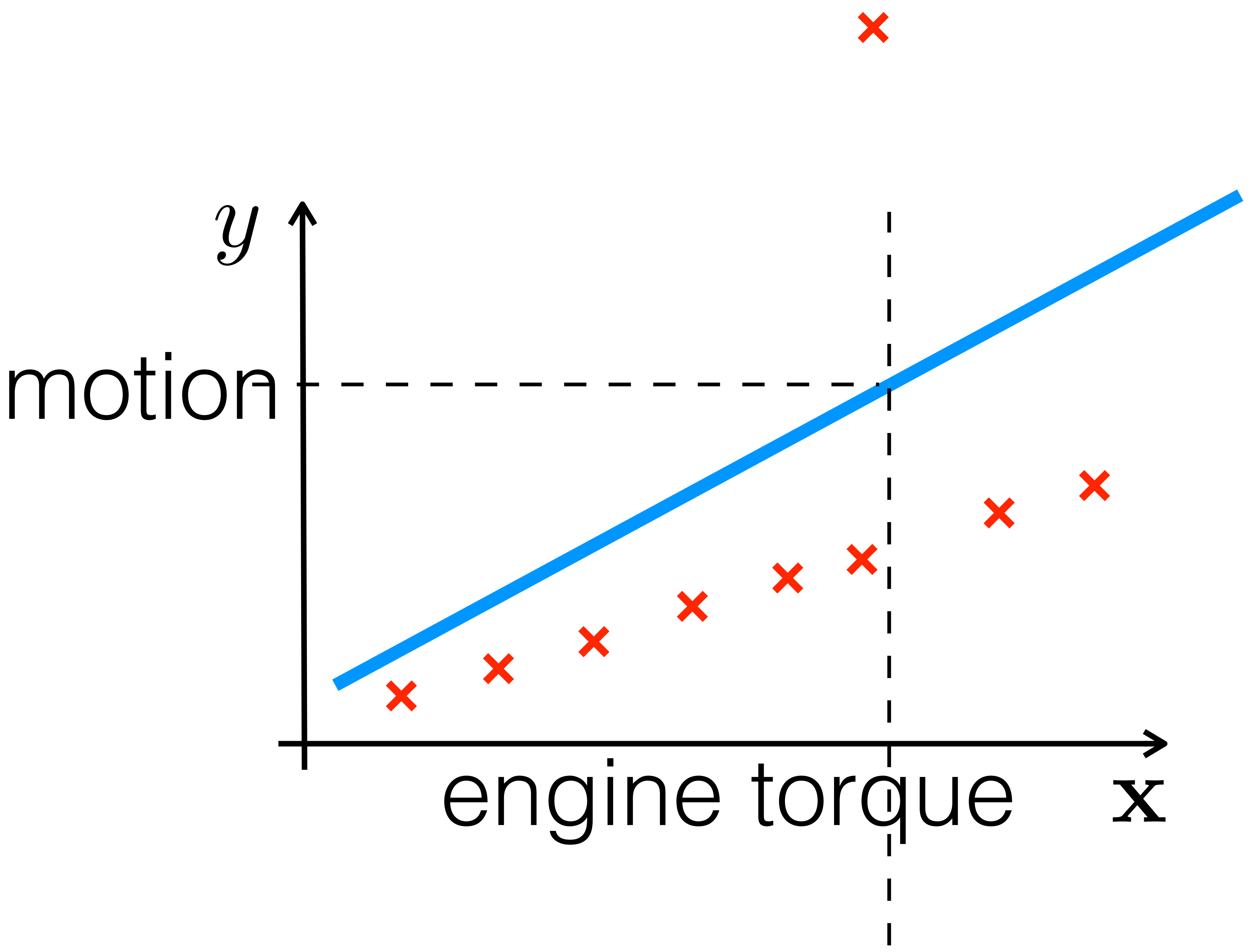


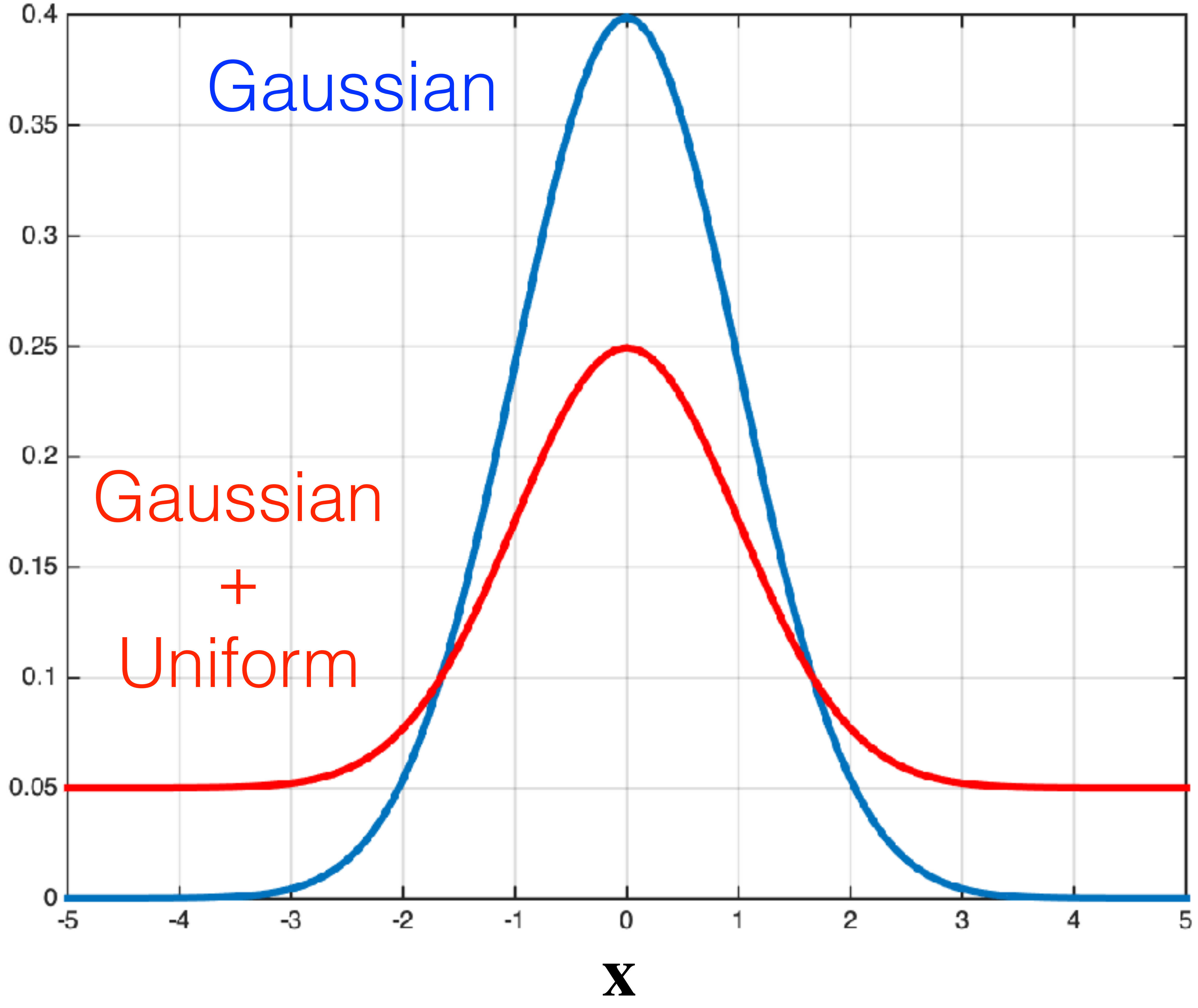


Evil source of uniform noise

Robust regression

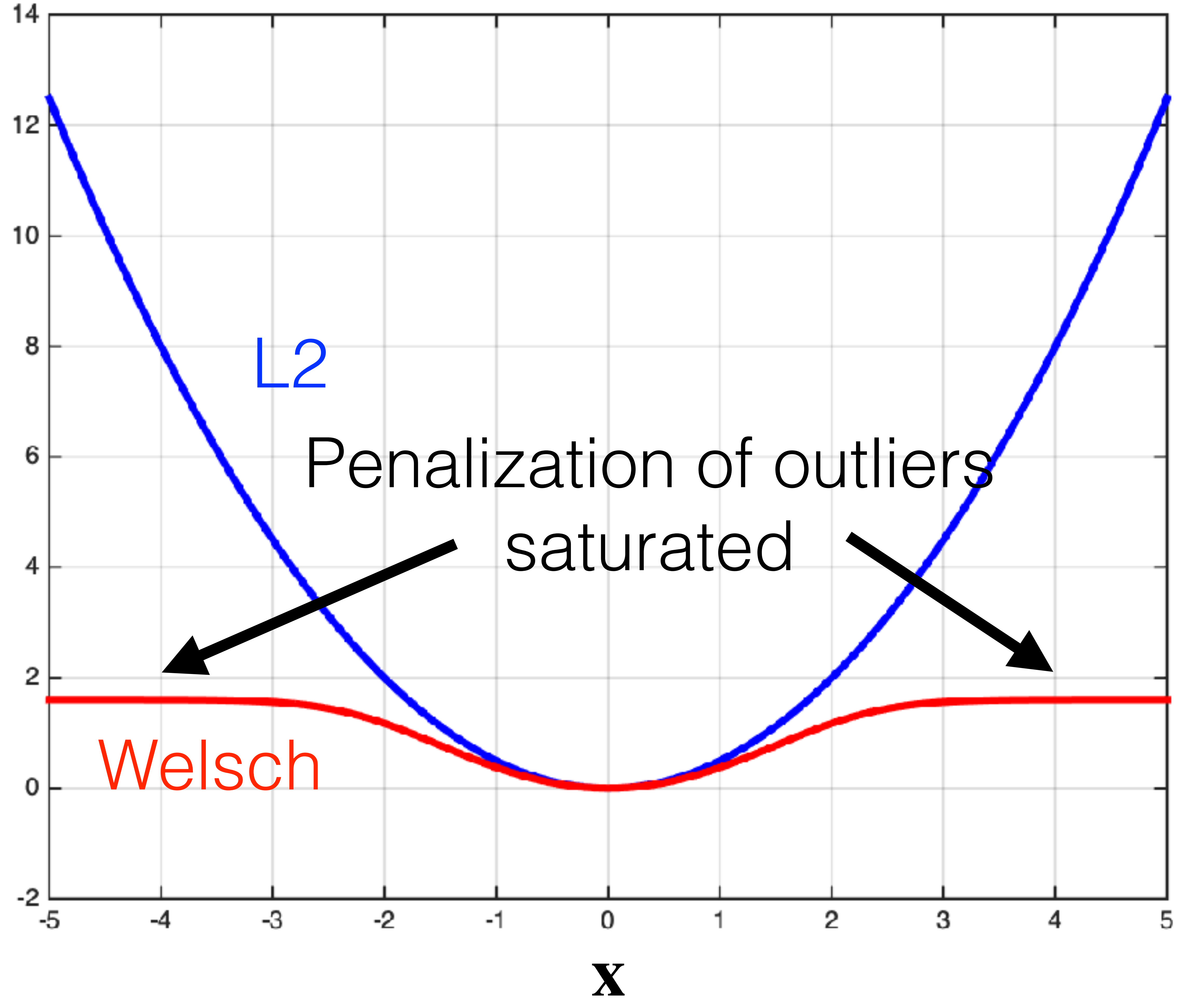
$$p(y \mid \mathbf{x}, \mathbf{w})$$

Probability distributions



$$\mathscr{L}(\mathbf{w}) = -\log(p(y \mid \mathbf{x}, \mathbf{w}))$$

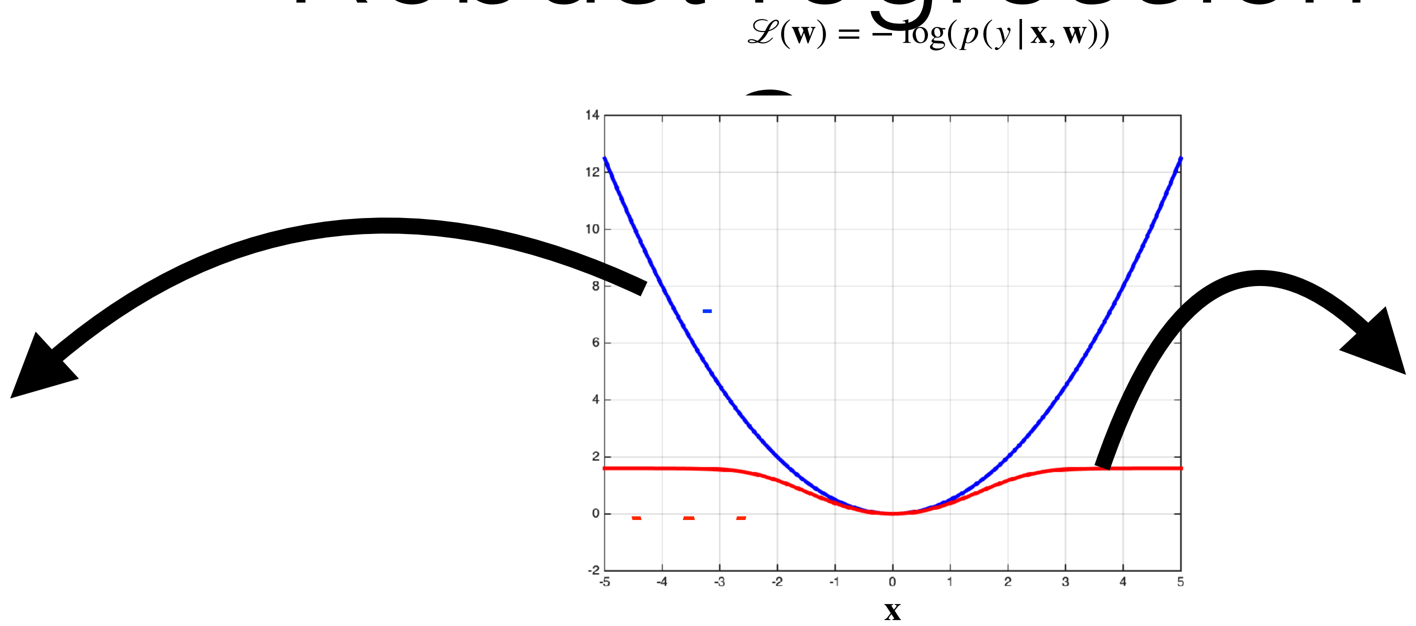
Corresponding losses



Penalization of outliers saturated

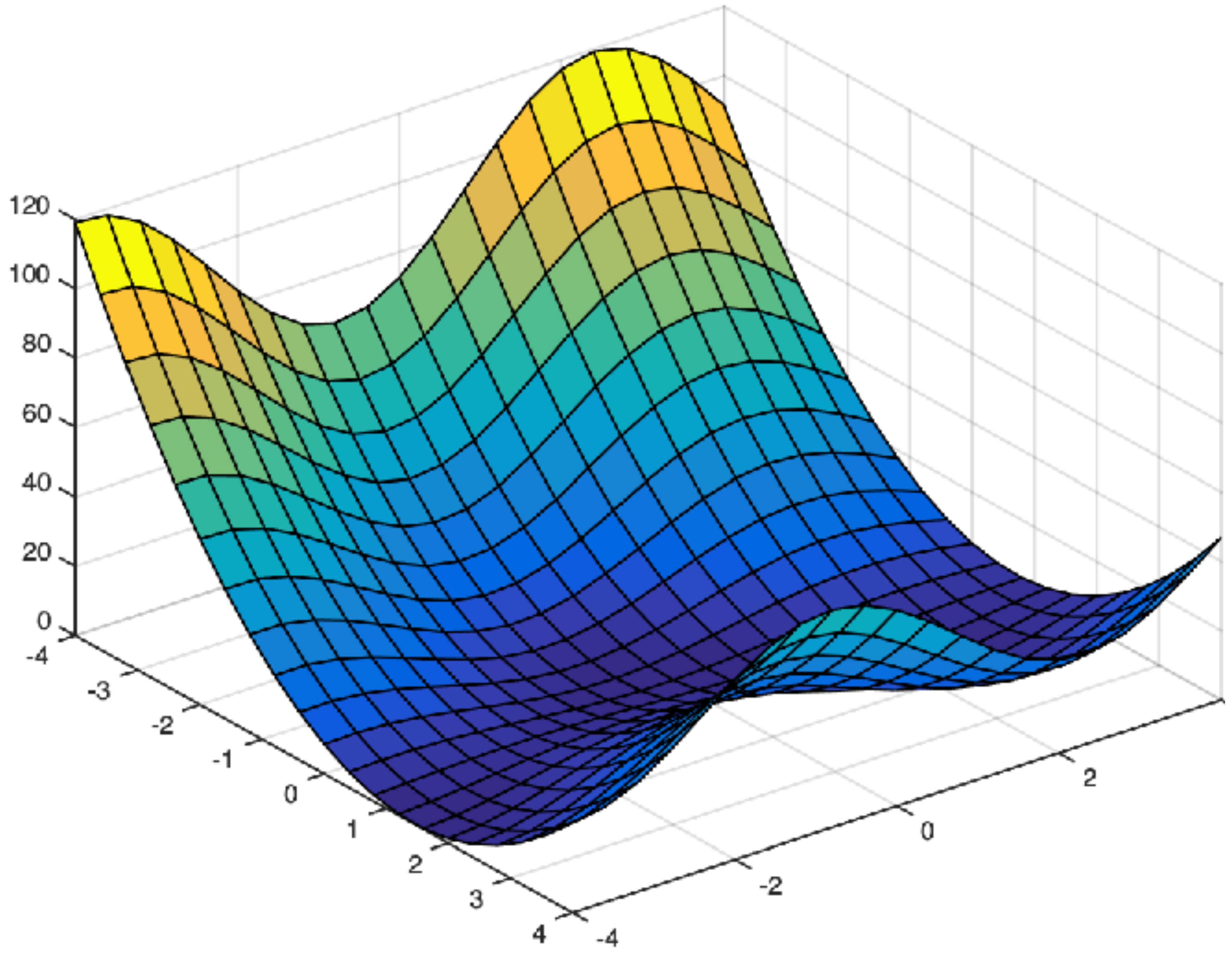
Can you guess where another problem appears?

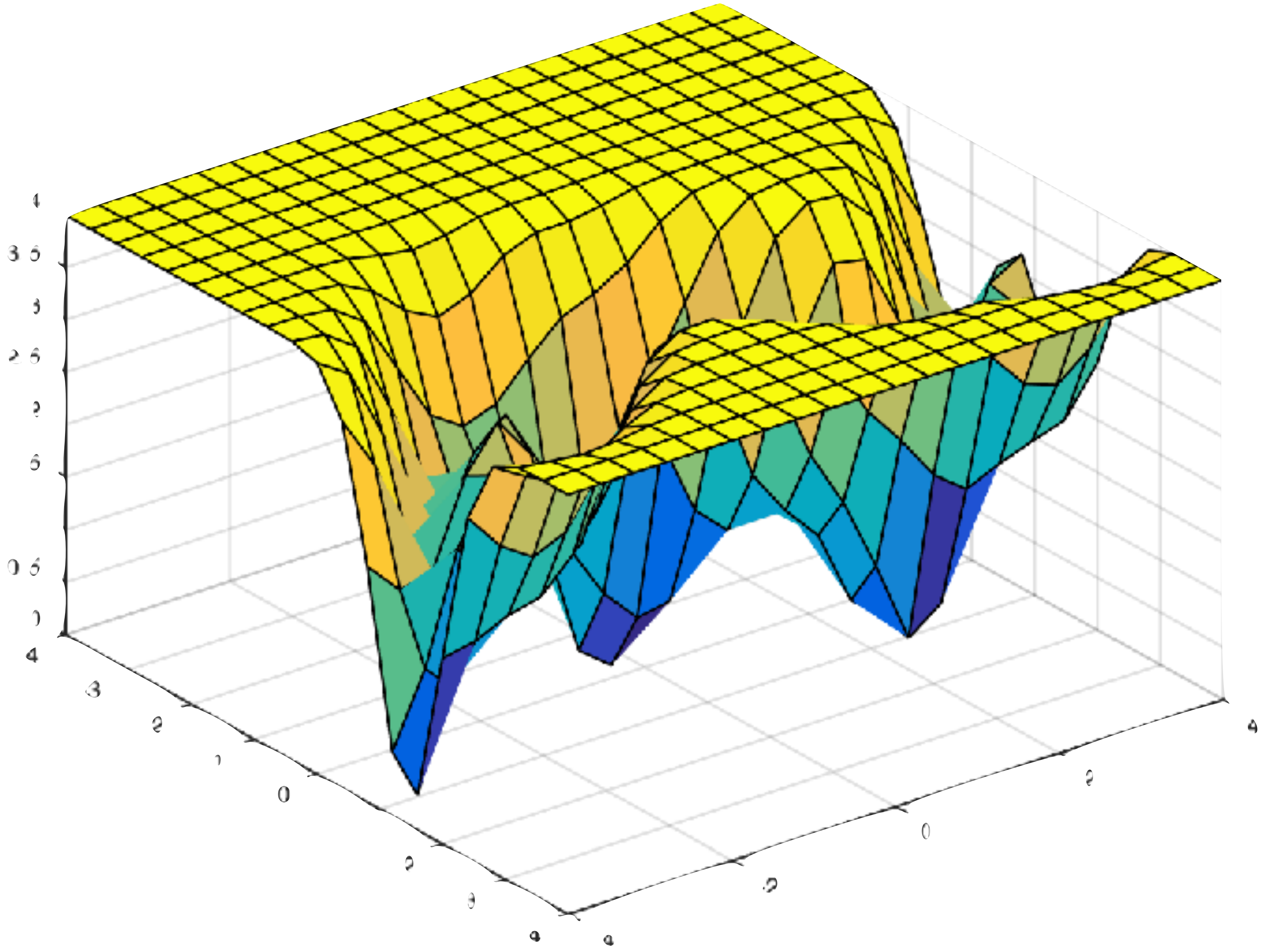
Robust regression



L2 landscape

Welsch landscape



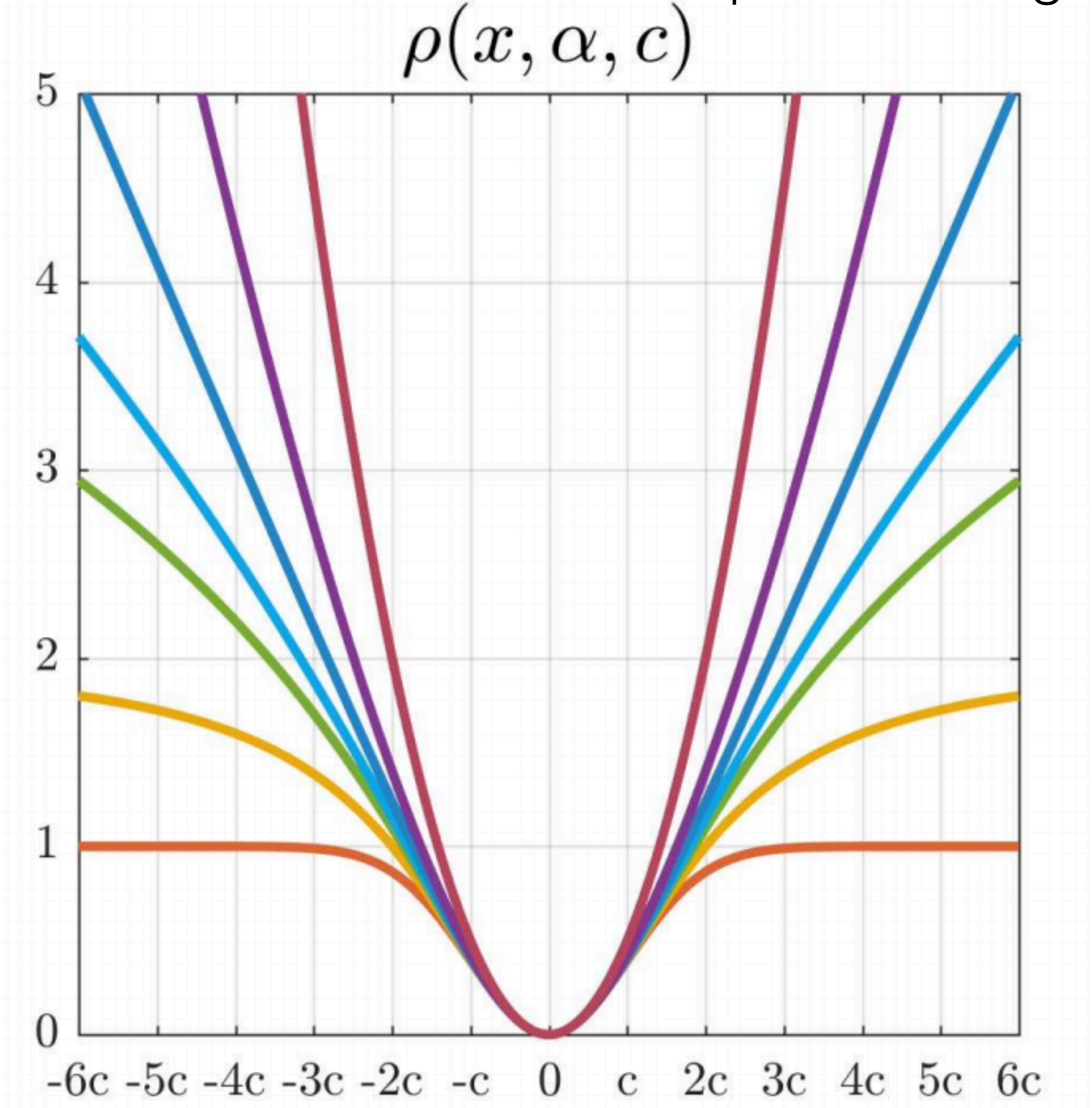


- **Uniform noise omitted**
  => GD-friendly landscape
- Gradient length encodes distance
- Easy to optimize

- **Uniform noise modelled**
  => GD-unfriendly landscape
- Non-convex: Large narrow plateaus with zero gradient
- Good initialization required

Shape of robust regression functions [Barron CVPR 2019]

https://arxiv.org/abs/1701.03077



$$\rho(x, \alpha, c) = \frac{|\alpha - 2|}{\alpha} \left(\left(\frac{(x/c)^2}{|\alpha - 2|} + 1 \right)^{\alpha/2} - 1 \right)$$

Trade-off:

Robustness to uniform noise (outliers)
vs
Optimization-friendly landscape

The best what you can do:

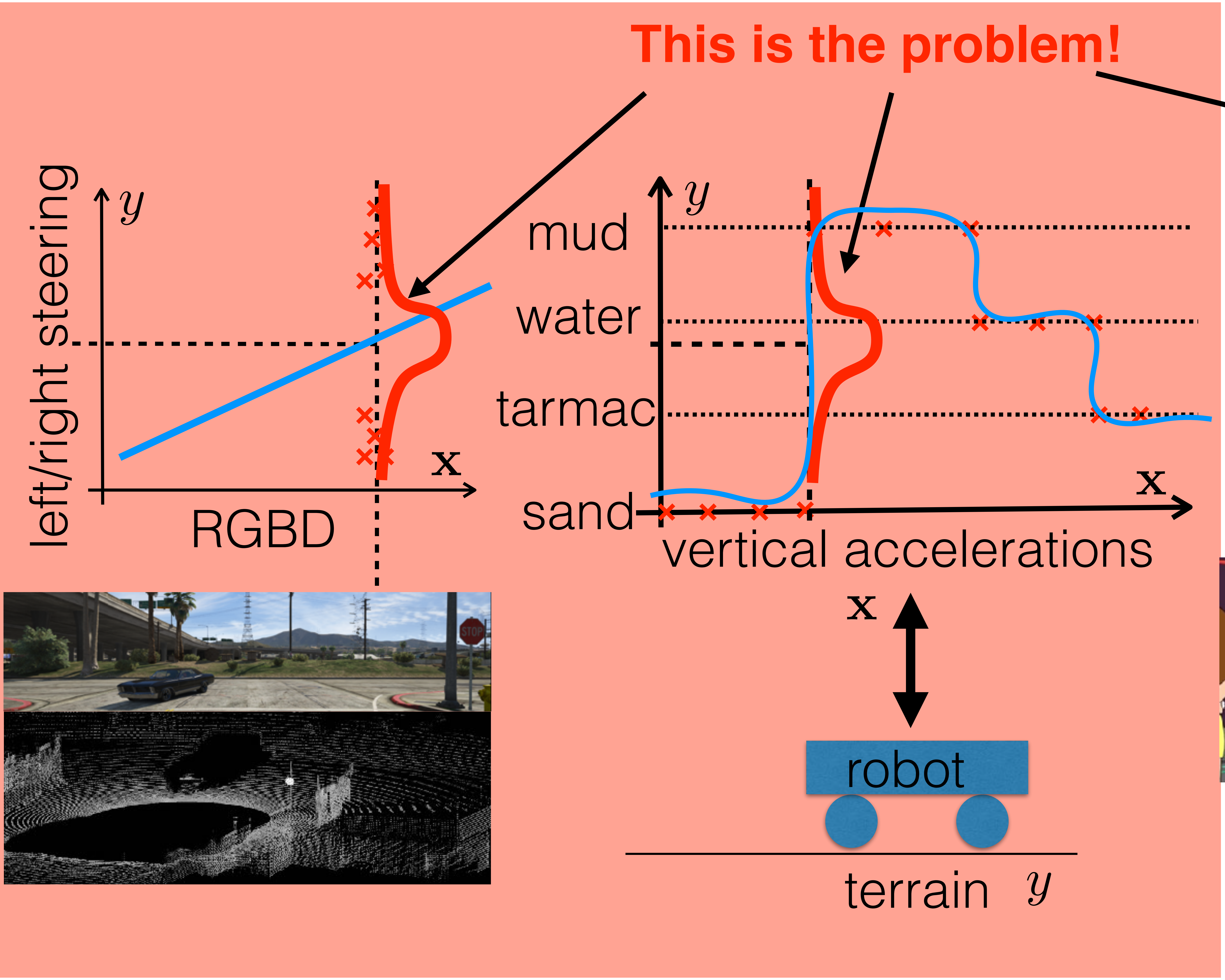
What can go wrong: **inappropriate choice of loss function**

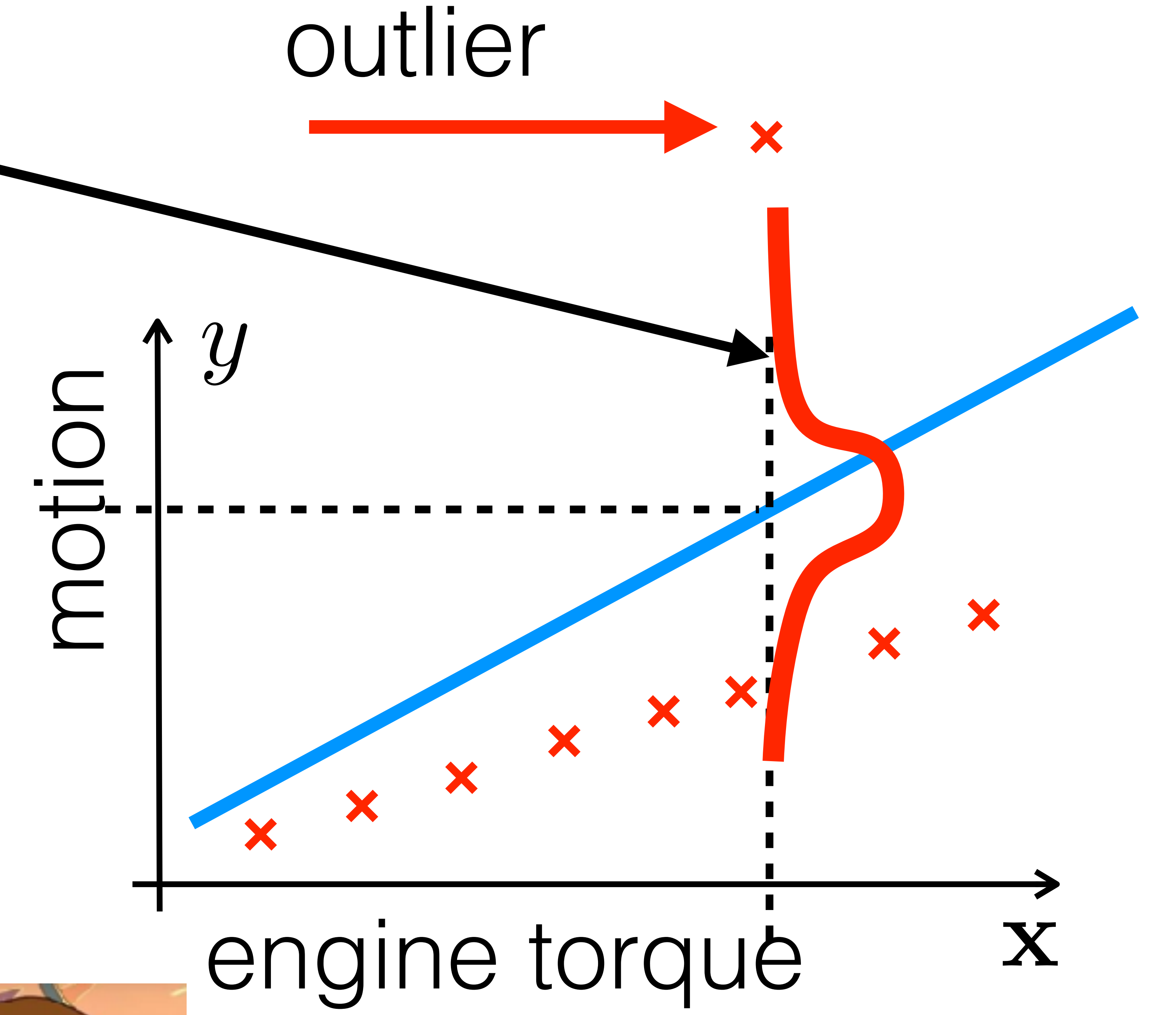
This is the problem!

Nature is evil !!!

https://rickandmorty.fandom.com/wiki/Evil_Morty

What can go wrong: **inappropriate choice of loss function**





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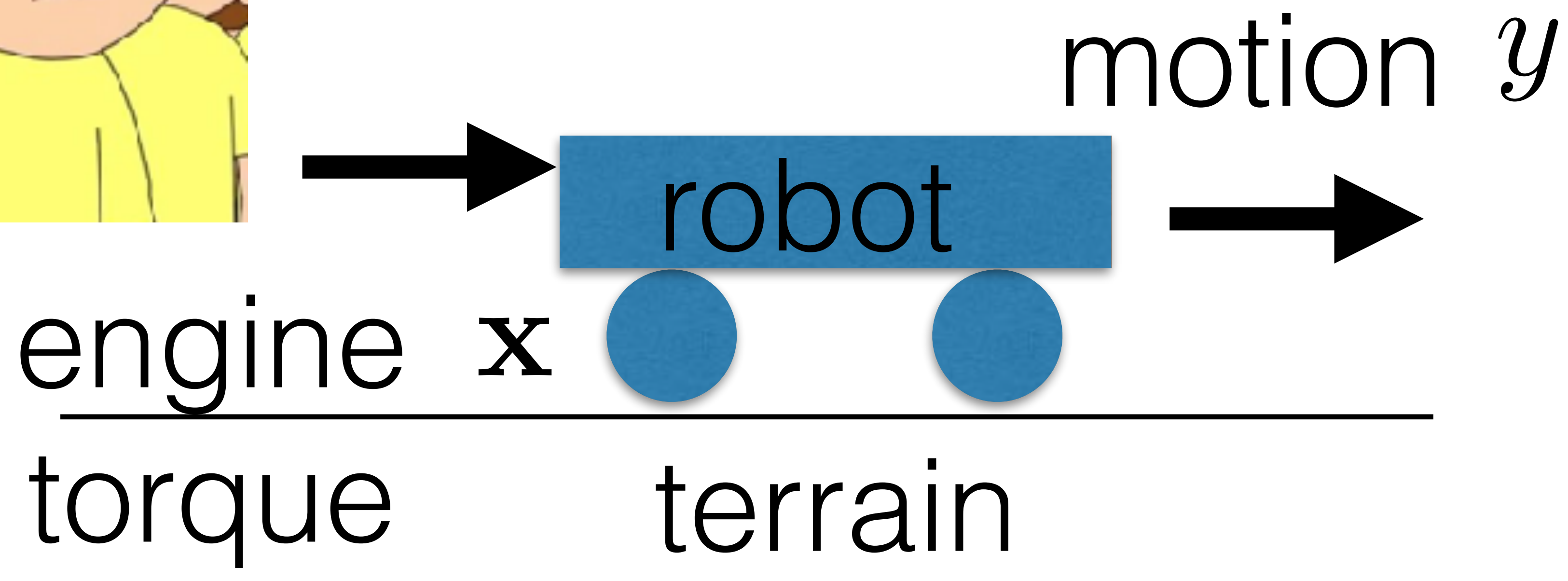
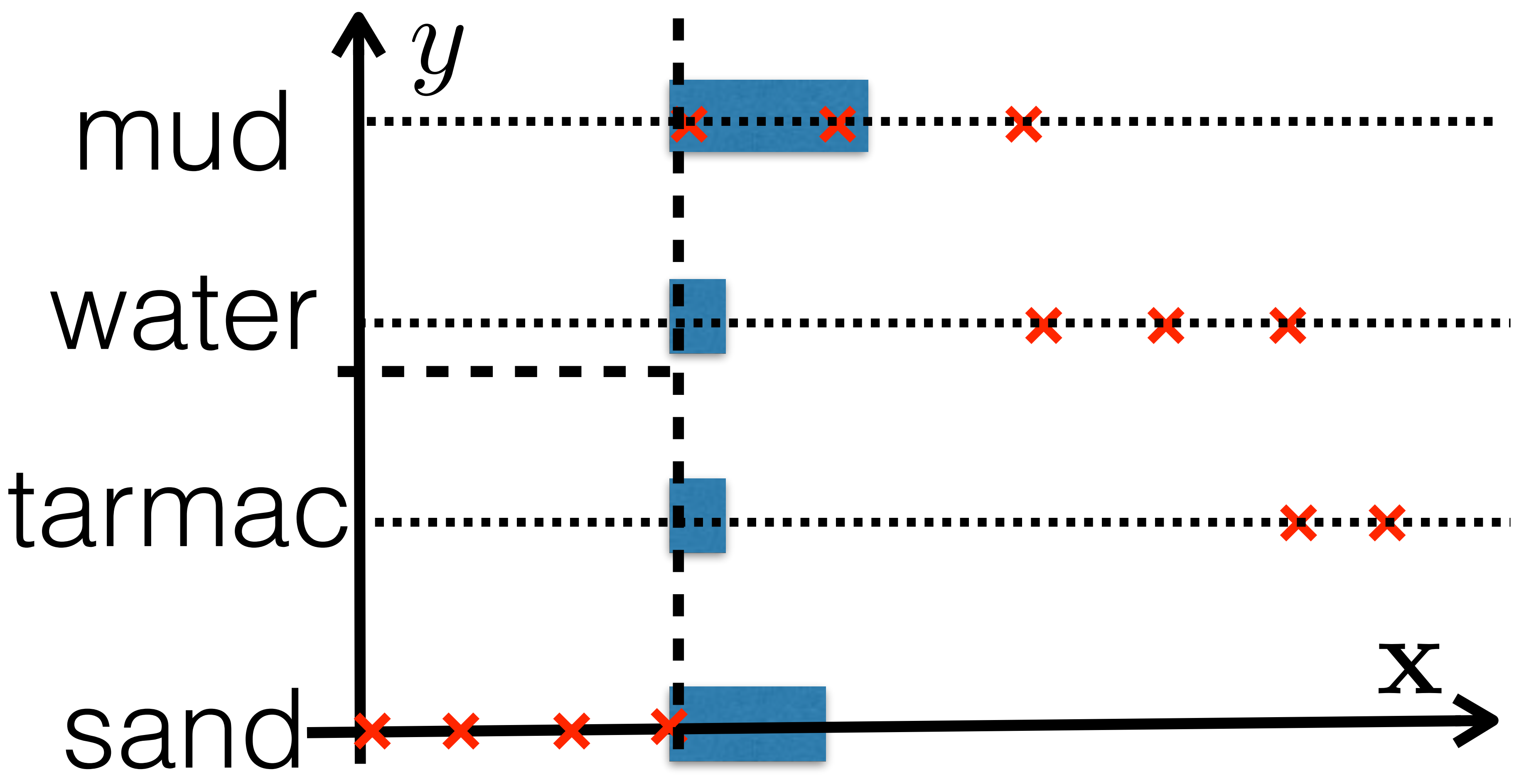
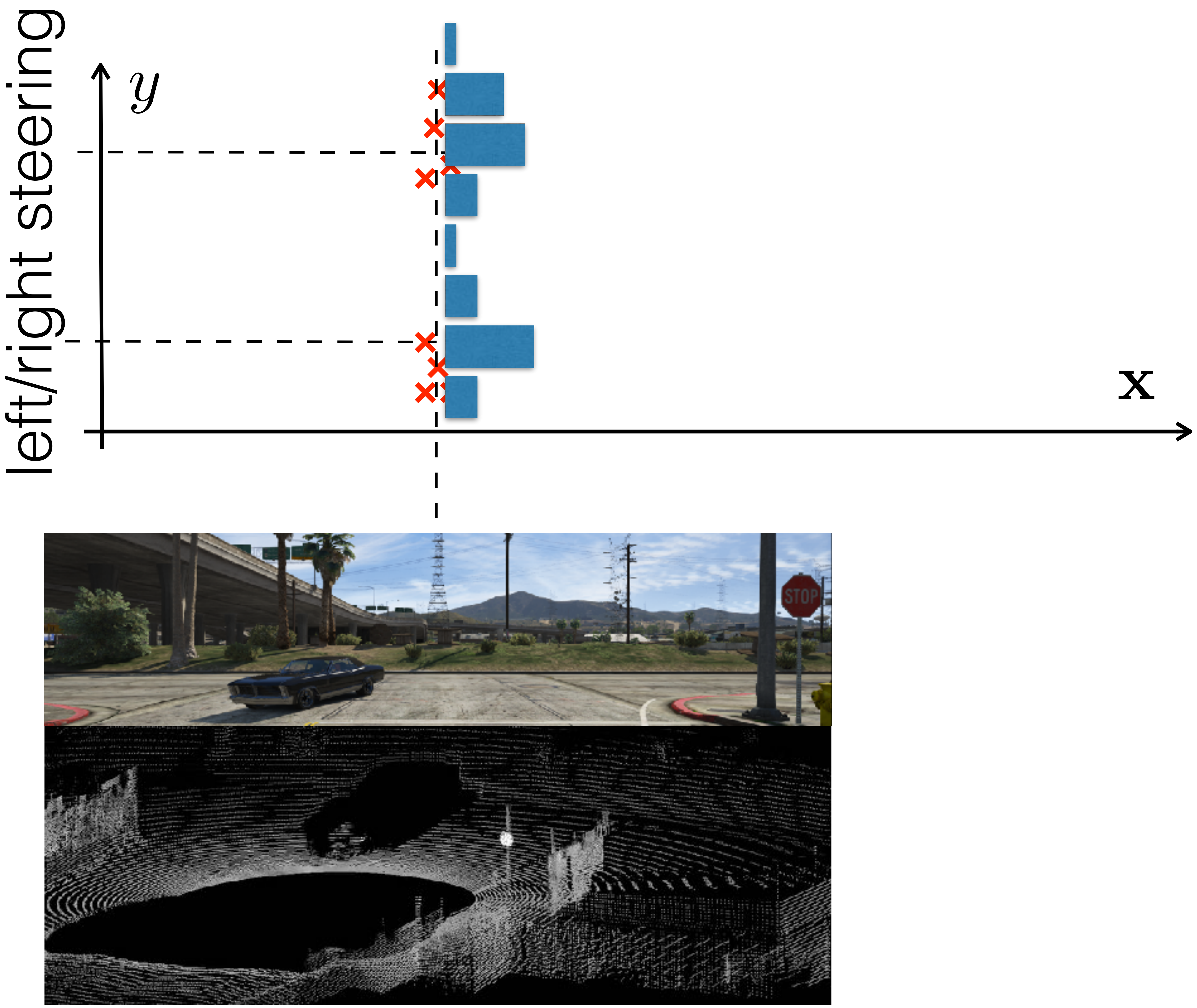


https://rickandmorty.fandom.com/wiki/Evil_Morty

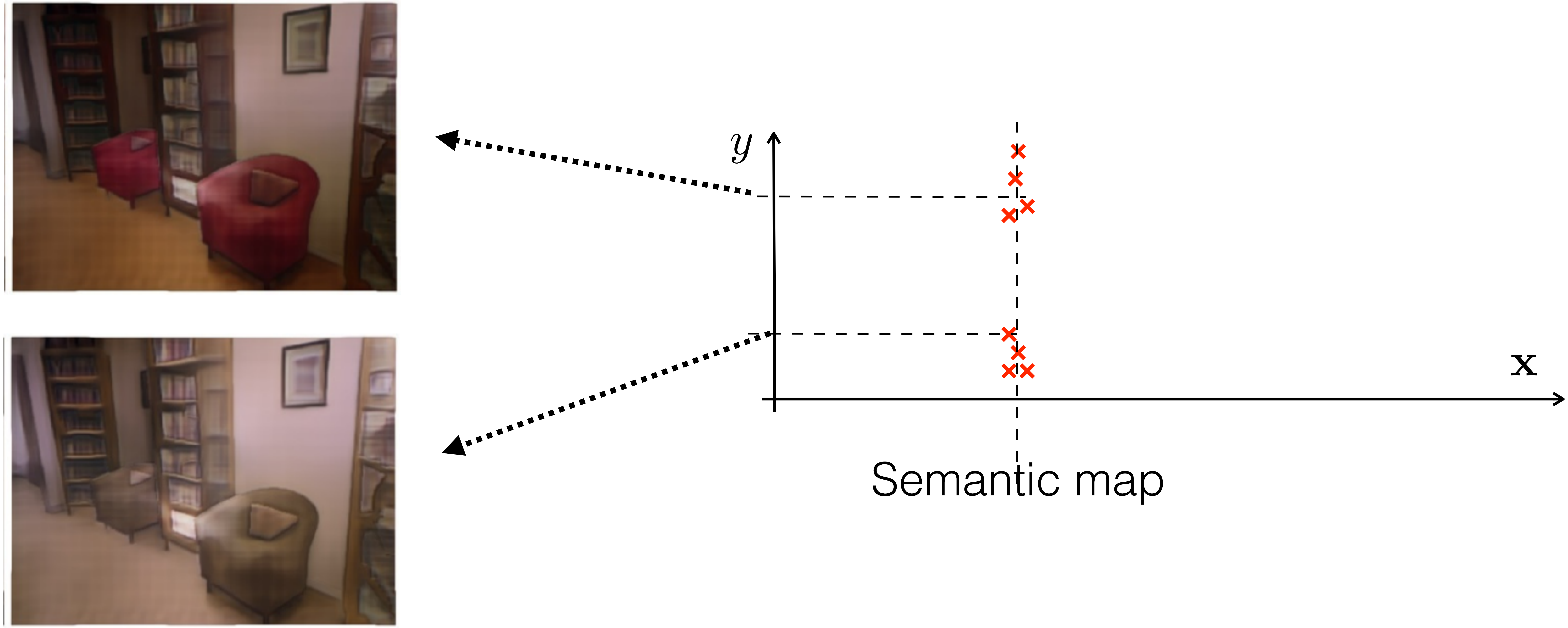
Work-around 1: discretize y-domain and treat the problem as classifcation





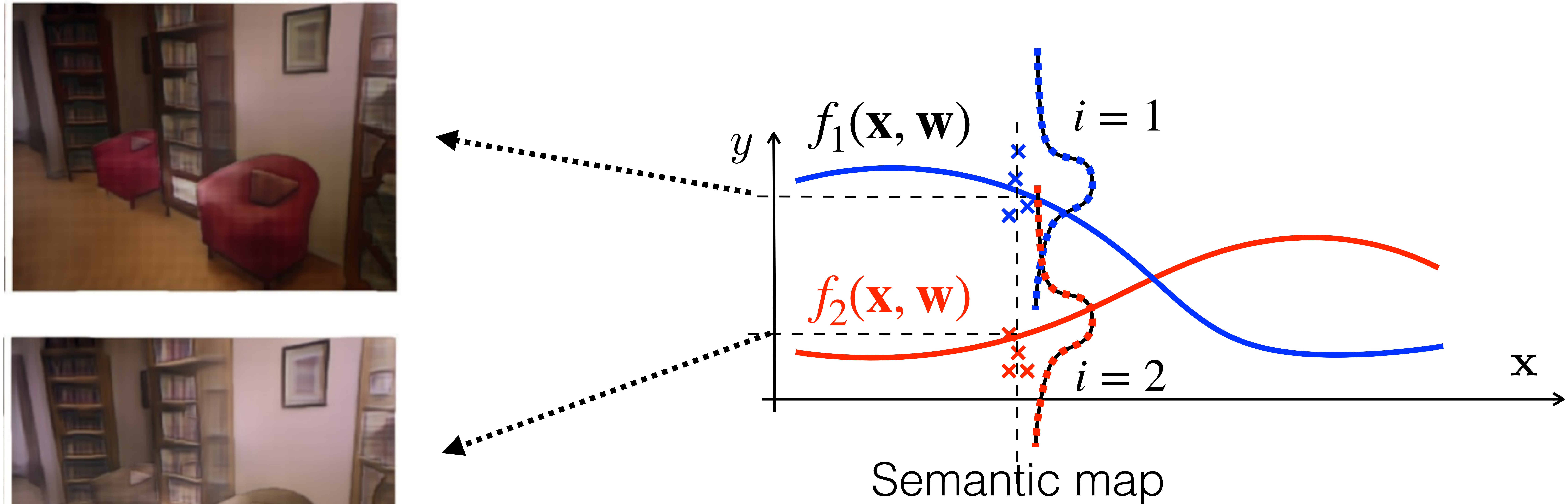
Can I use it always?

Tractable only for low-dim y



What if y are images?

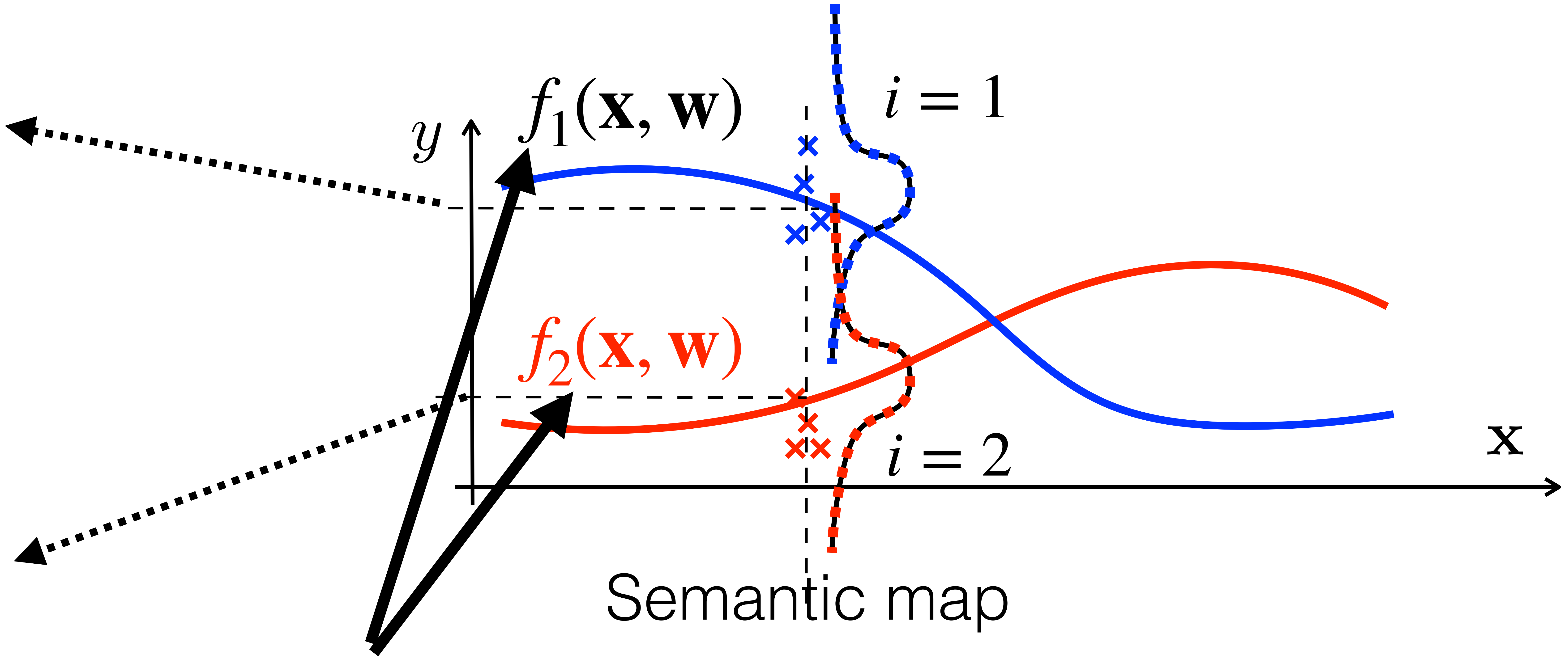
Work-around 2: allow multiple hypothesis



Multiple choice loss
[Microsoft, NIPS, 2012], [Koltun, ICCV, 2017]

$$\mathscr{L}(\mathbf{w}) = \min_i ||f_i(\mathbf{x}, \mathbf{w}) - y||$$

Work-around 2: allow multiple hypothesis



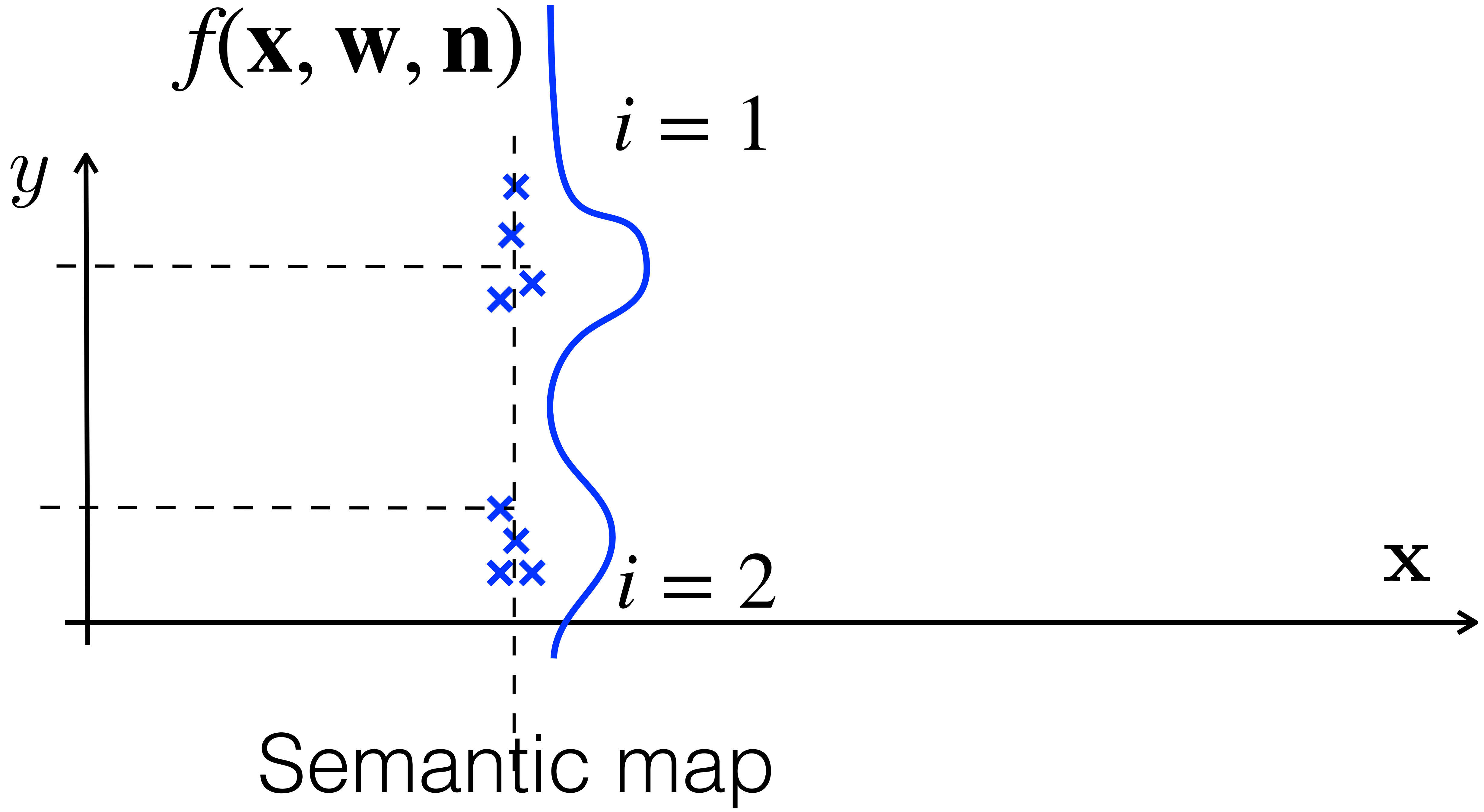
Problem 1: number of hypothesis may grow exponentially

Multiple choice loss
[Microsoft, NIPS, 2012], [Koltun, ICCV, 2017]

$$\mathscr{L}(\mathbf{w}) = \min_i \|f_i(\mathbf{x}, \mathbf{w}) - y\|$$

Problem 2: Measuring similarity of images

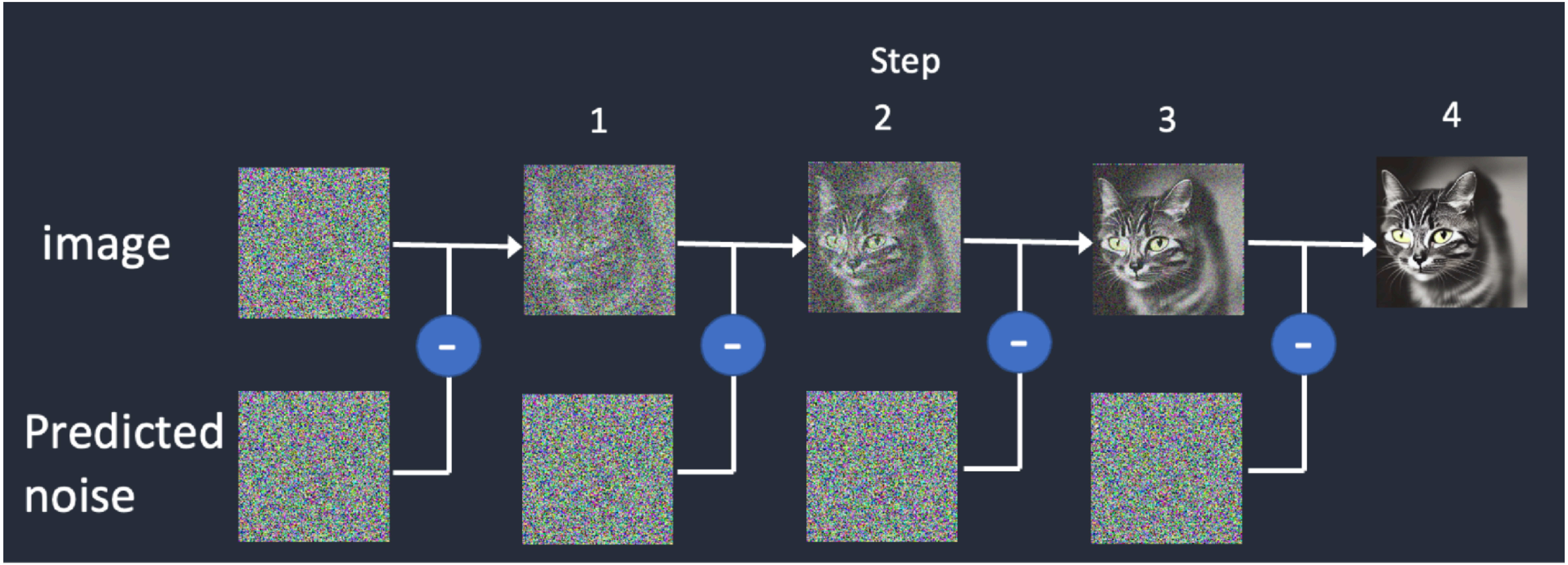
Work-around 3: use generative model



Generative models: GANs [Goodfellow 2016], VAE,

Diffusion models [Ho, NIPS 2020]

Work-around 3: use generative model



Summary loss

- -log(p)-loss stems from fitting the network parameterized p(y|x,w) distr. into data
- Maximum Likelihood = Minimum KL-divergence = Minimum -log(p)-loss
- Different distributions suffer from optimization issues (zero gradients, sensitivity to good initialization, local optima, …)

Where does the **overfitting** come from?

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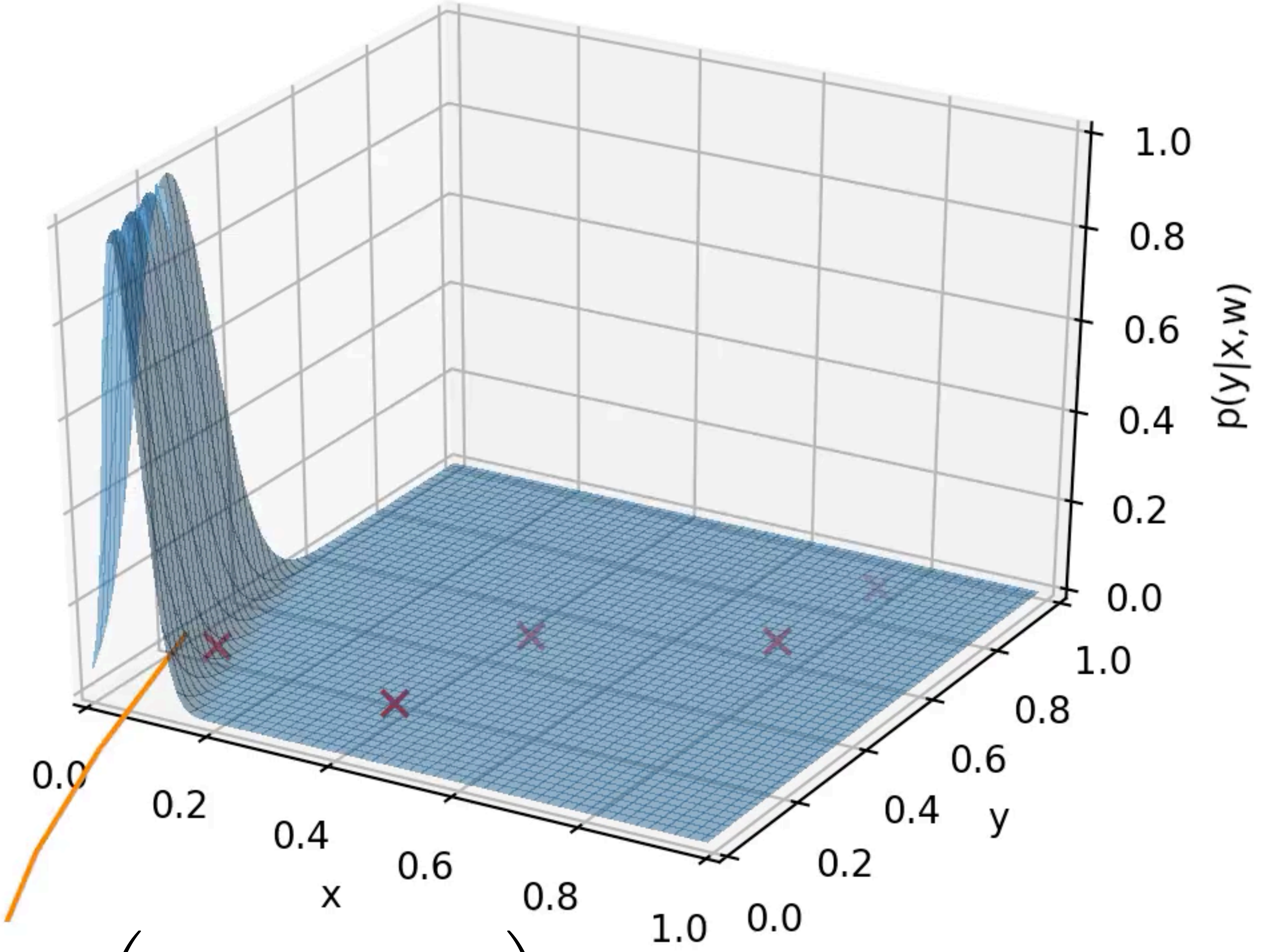

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \left(\prod_i p(y_i|\mathbf{x}_i, \mathbf{w}) \right) = \arg\min_{\mathbf{w}} \sum_i (w_2 x_i^2 + w_1 x_i + w_0 - y_i)^2$$

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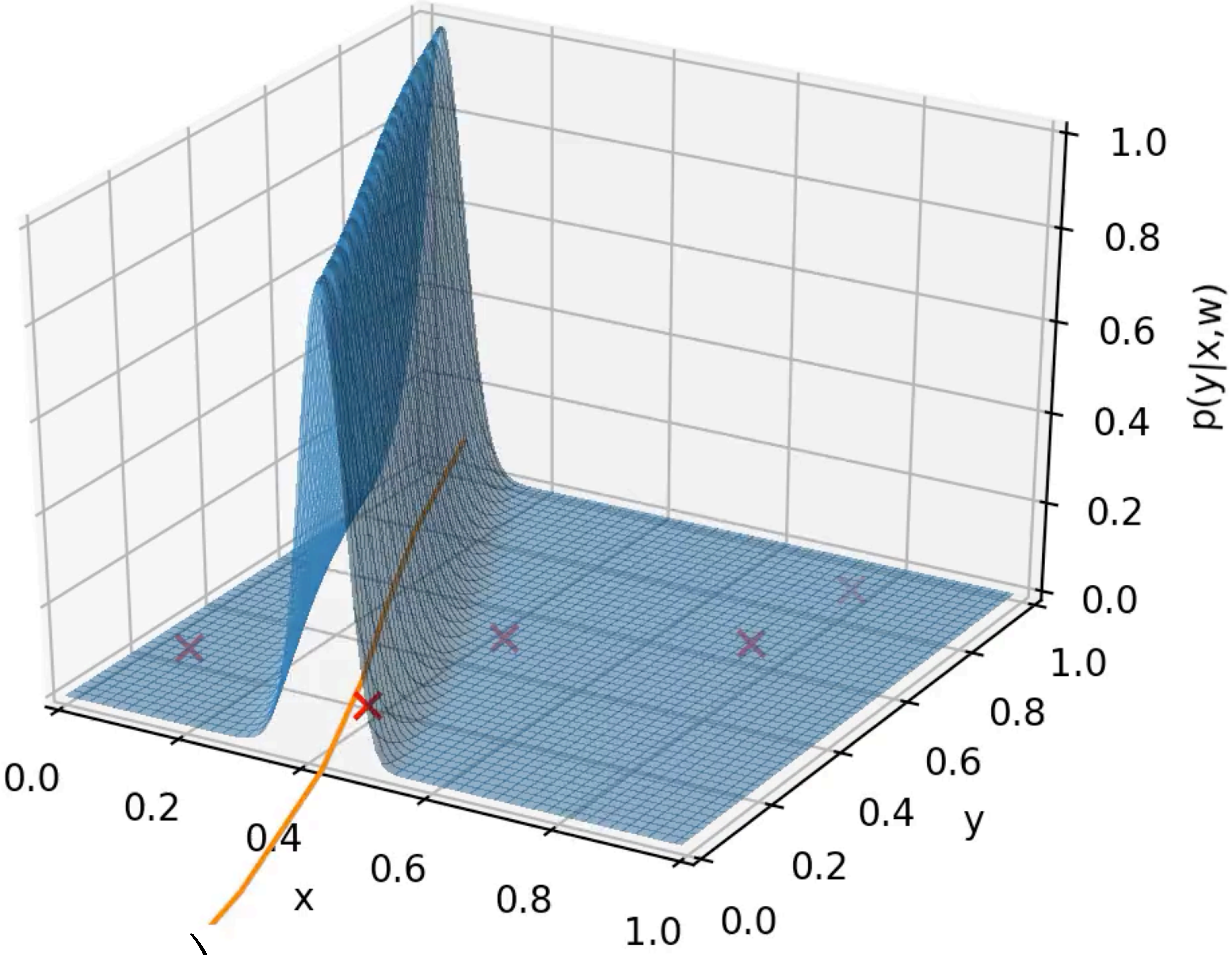

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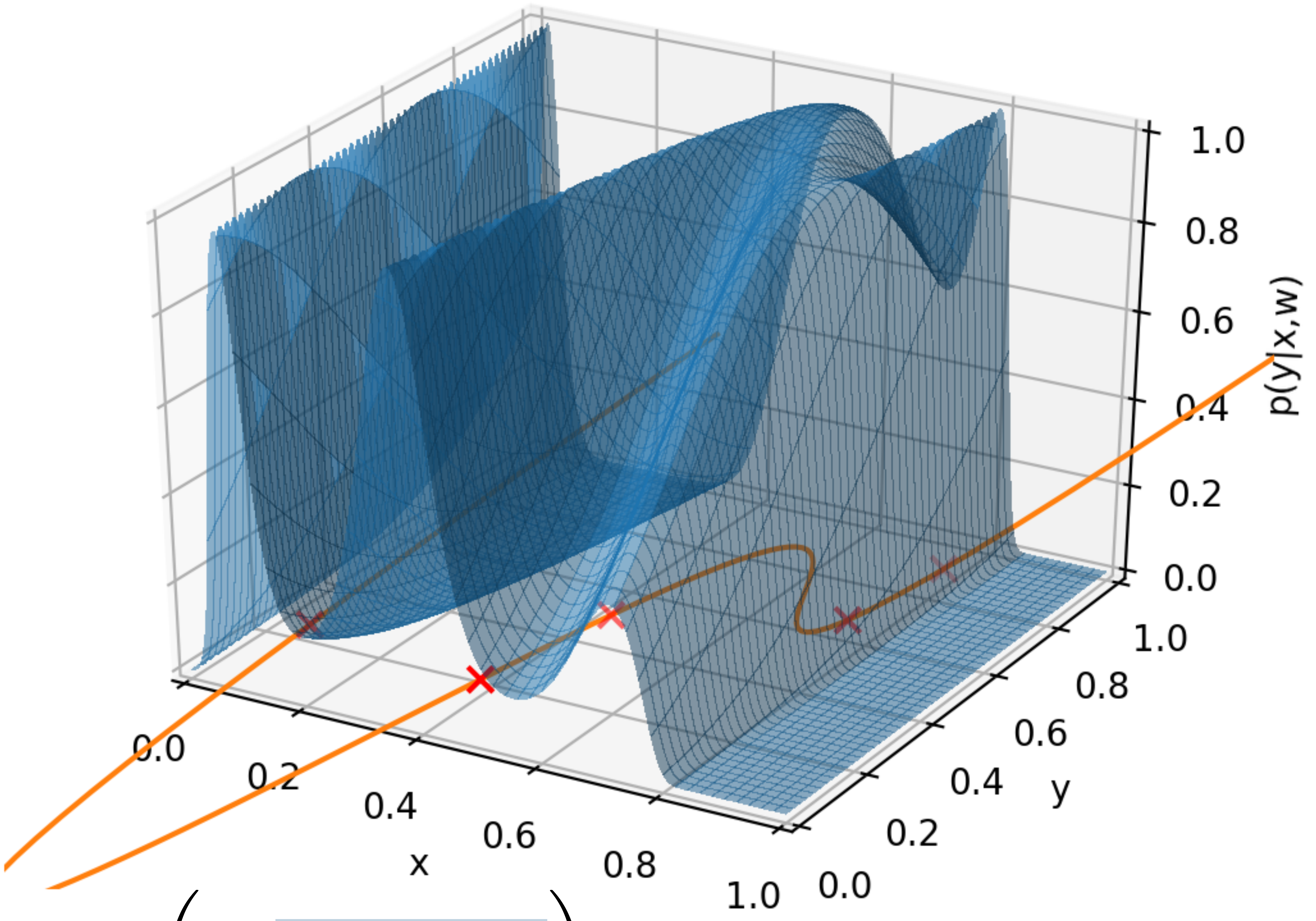

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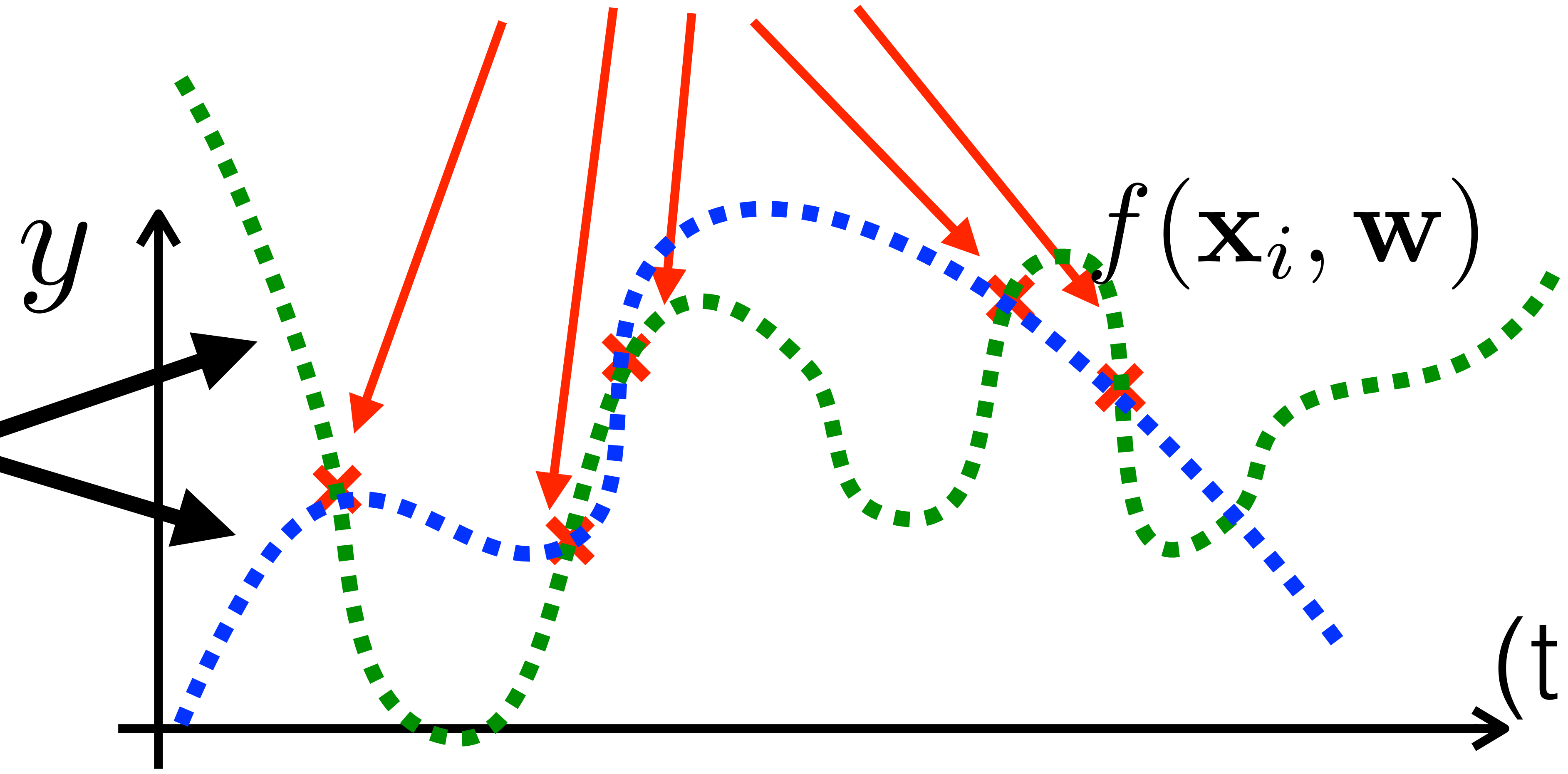
Where does the **overfitting** come from?

William of Ockham
(1287-1347)

leprechauns can be
involved in any explanation

observations

Many stories consistent with
the broken vase observation



The space of possible
stories is too wild

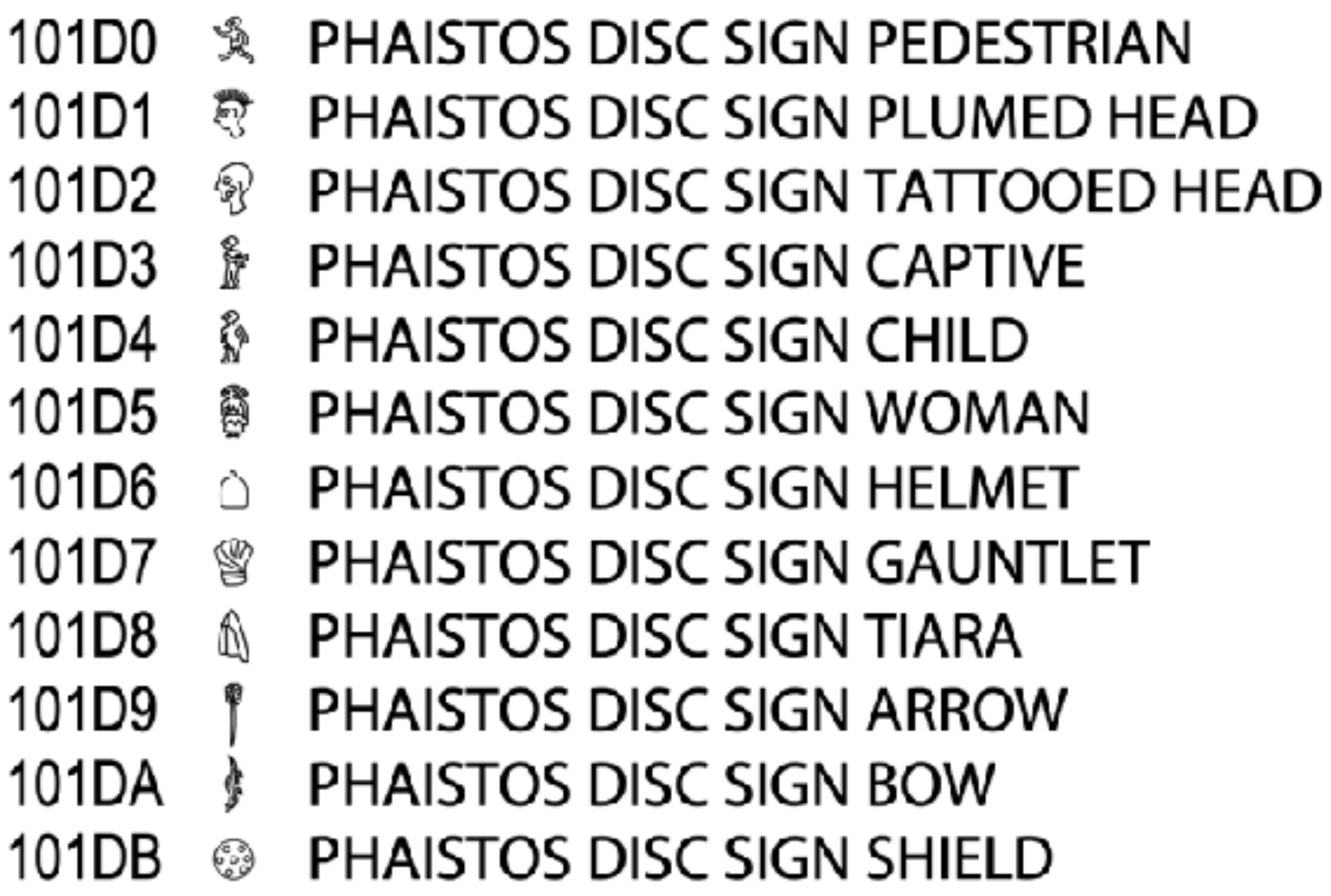
=> Use the simplest
(the most apriori probable)

https://en.wikipedia.org/wiki/Occam%27s_razor

Where does the **overfitting** come from?

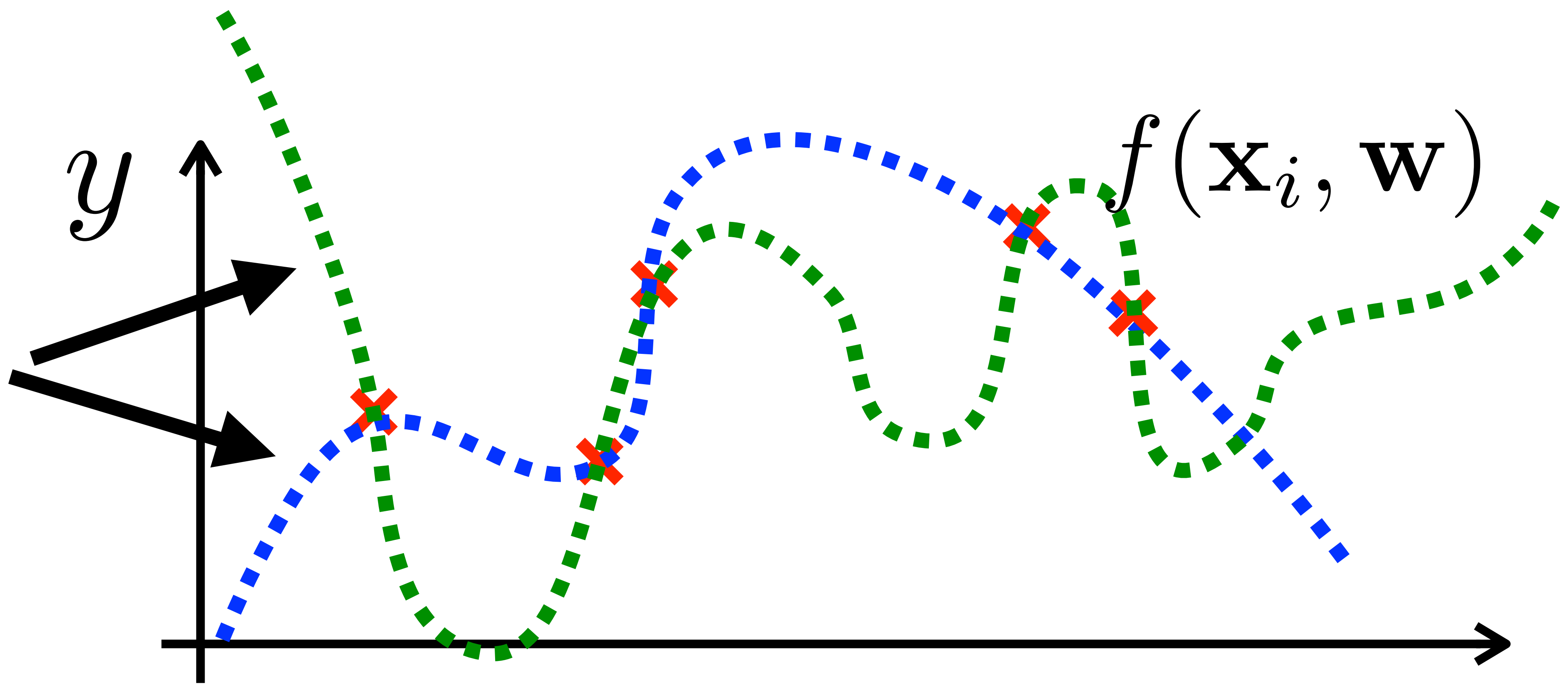
Phaistos disc

Unicode



101D0	𐇐	PHAISTOS DISC SIGN PEDESTRIAN
101D1	𐇑	PHAISTOS DISC SIGN PLUMED HEAD
101D2	𐇒	PHAISTOS DISC SIGN TATTOOED HEAD
101D3	𐇓	PHAISTOS DISC SIGN CAPTIVE
101D4	𐇔	PHAISTOS DISC SIGN CHILD
101D5	𐇕	PHAISTOS DISC SIGN WOMAN
101D6	𐇖	PHAISTOS DISC SIGN HELMET
101D7	𐇗	PHAISTOS DISC SIGN GAUNTLET
101D8	𐇘	PHAISTOS DISC SIGN TIARA
101D9	𐇙	PHAISTOS DISC SIGN ARROW
101DA	𐇚	PHAISTOS DISC SIGN BOW
101DB	𐇛	PHAISTOS DISC SIGN SHIELD

Many stories consistent with sequence of visual symbols



The space of possible stories is too wild

https://en.wikipedia.org/wiki/Phaistos_Disc

Where does the **overfitting** come from?

- We fit $p(\mathbf{x}, y \,|\, \mathbf{w})$ into unknown distribution $p_{\text{data}}(\mathbf{x}, y)$:

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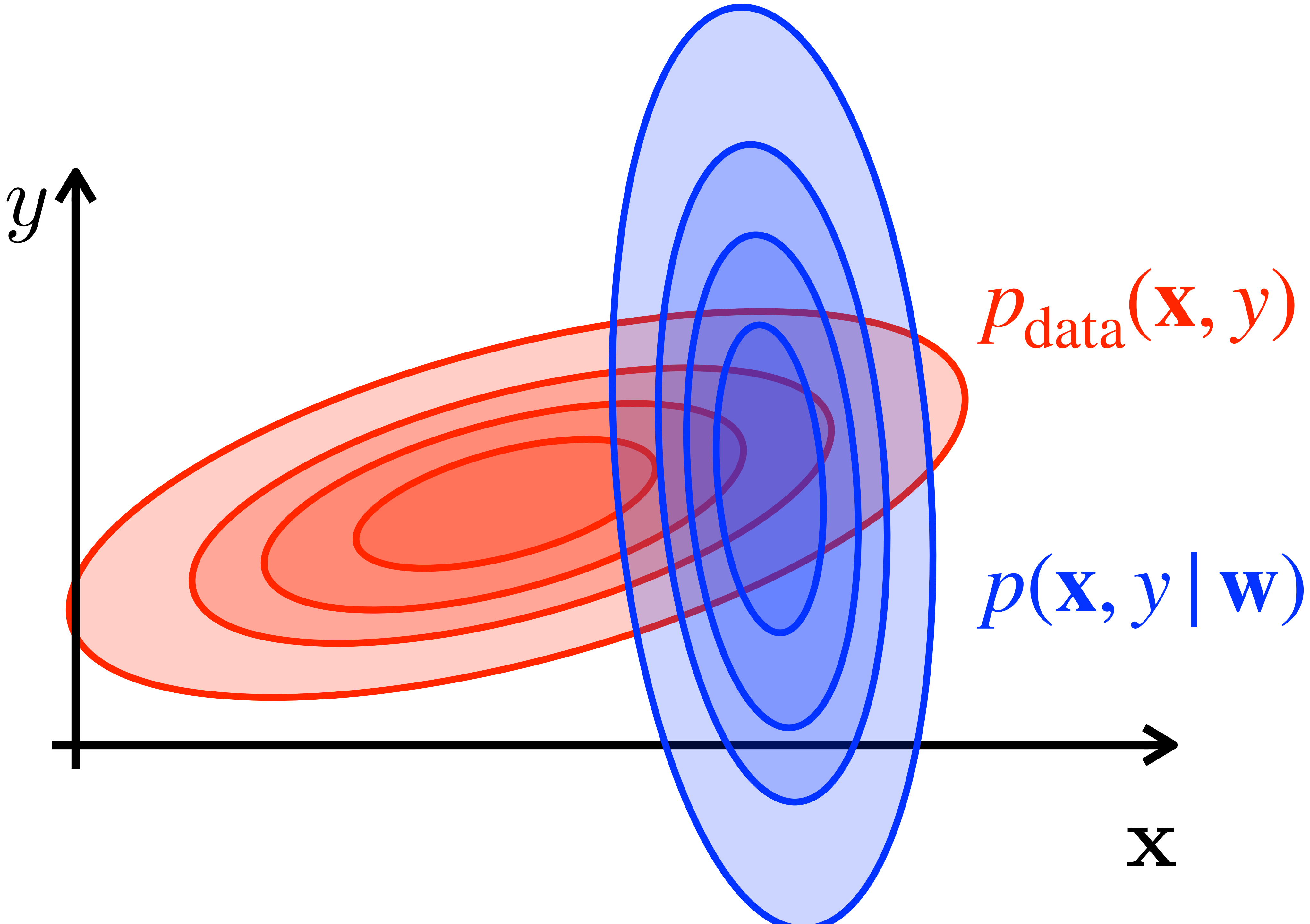

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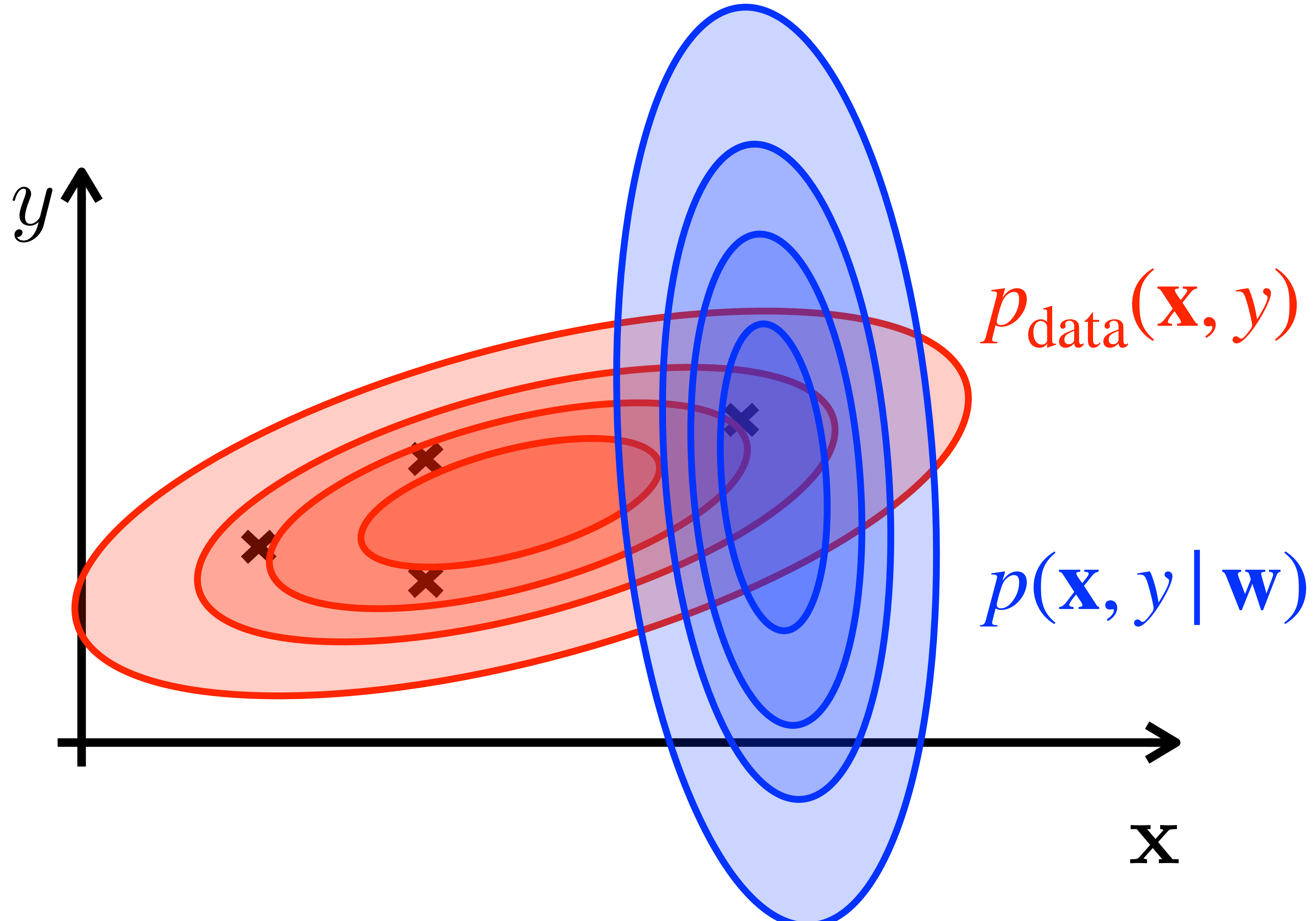
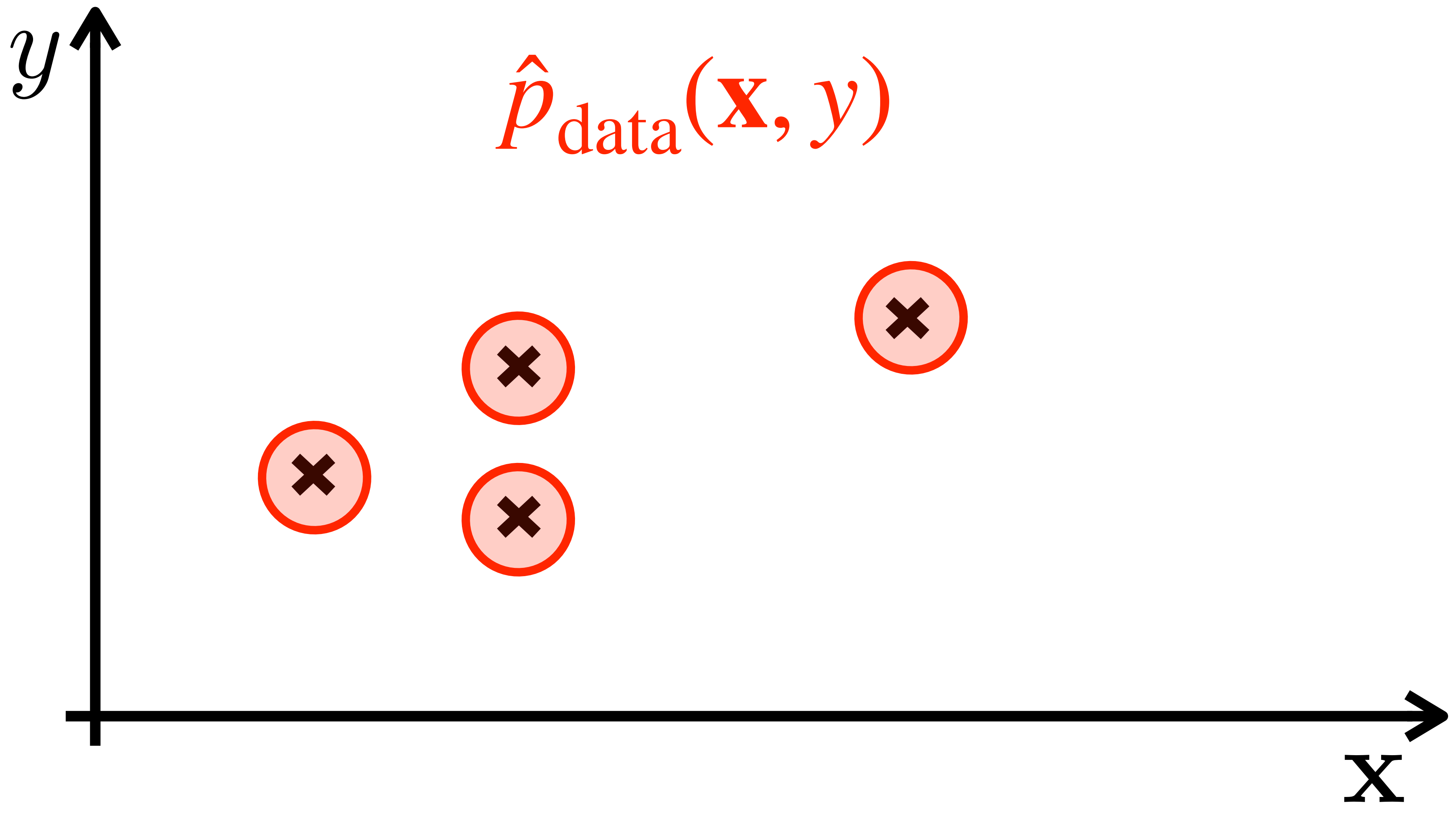
- $p_{\text{data}}(\mathbf{x}, y)$ is unknown => use samples (training set)
- Since the traning set is finite, we actually used different $\hat{p}_{\text{data}}(\mathbf{x}, y)$





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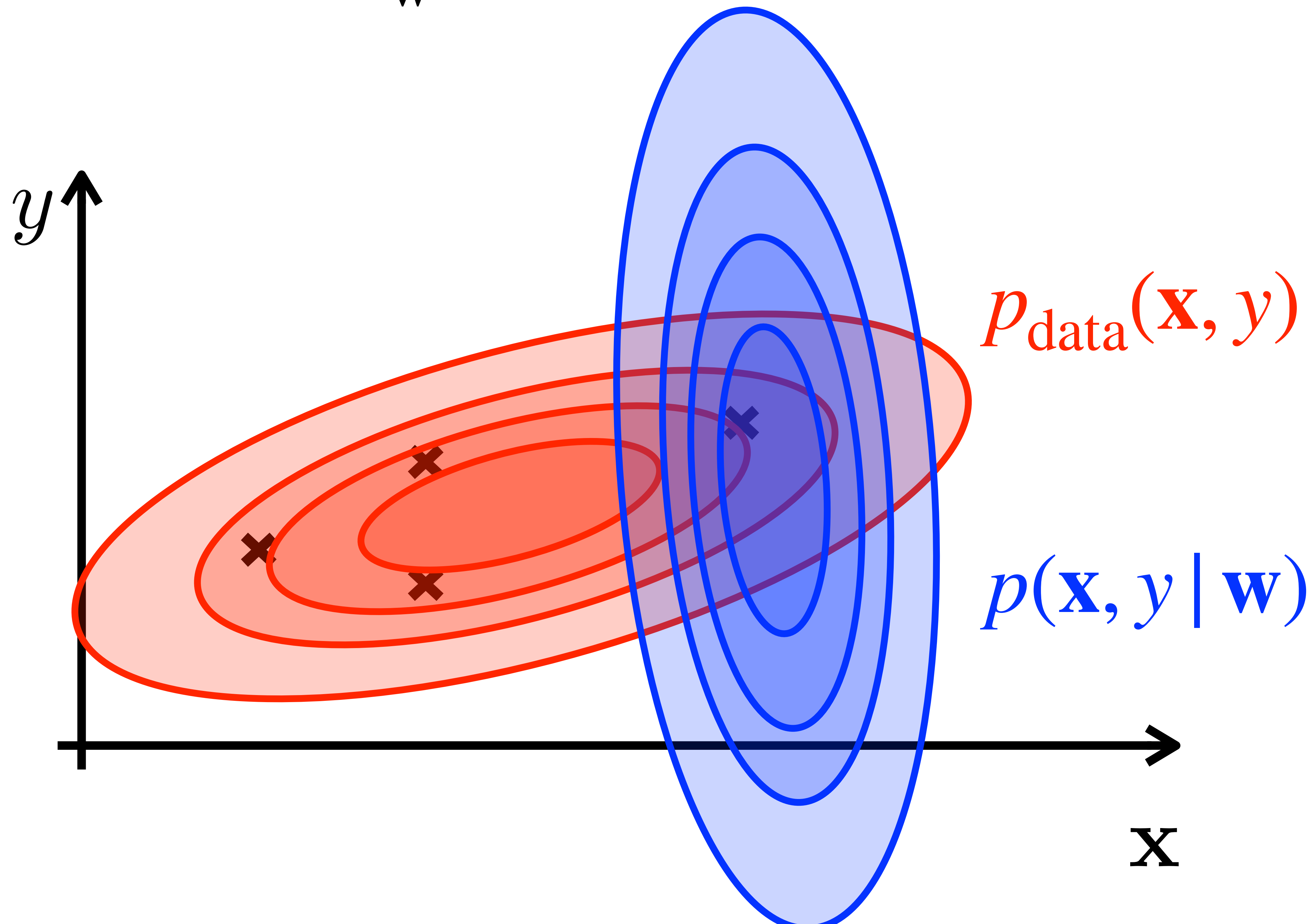

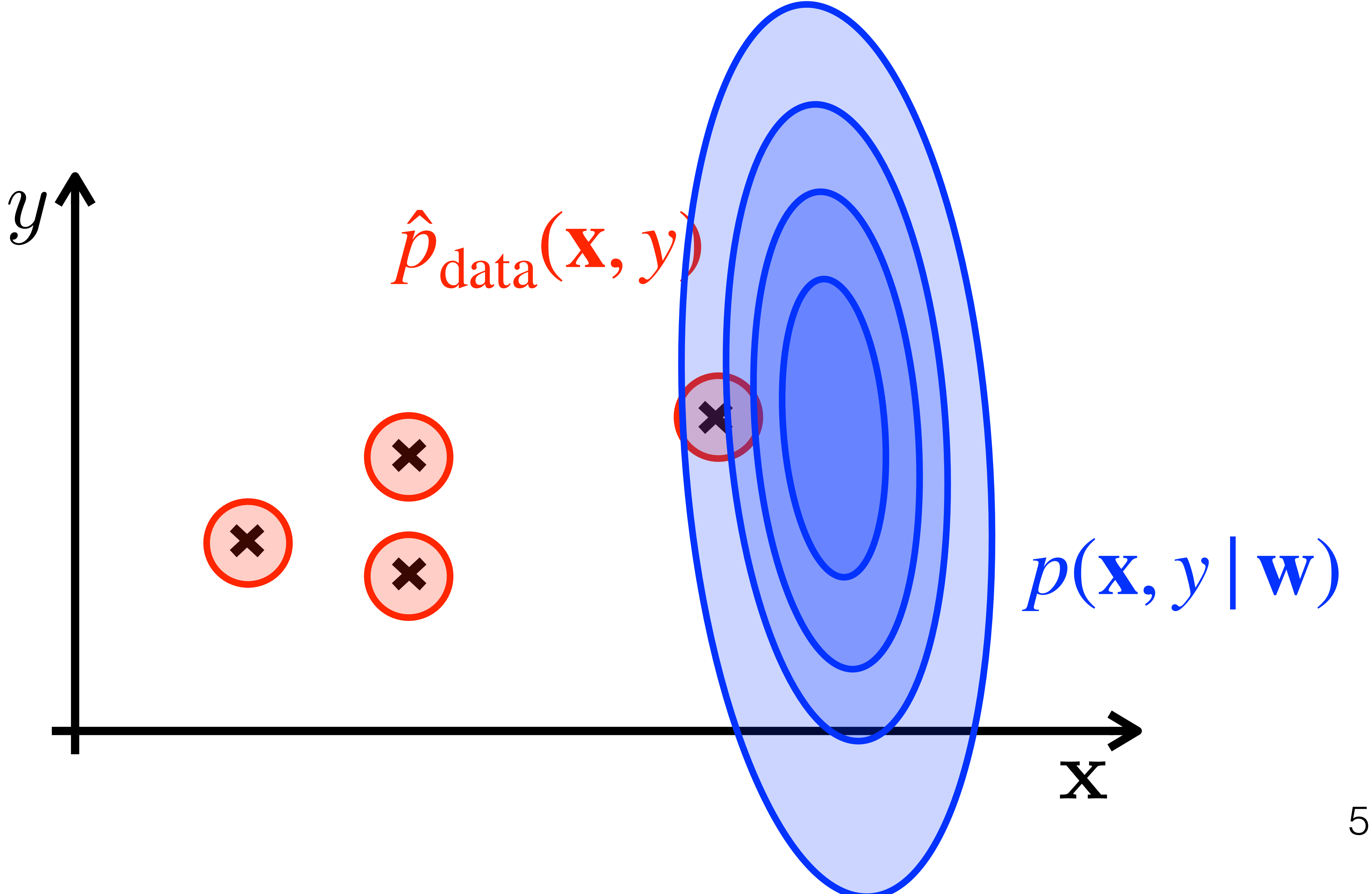

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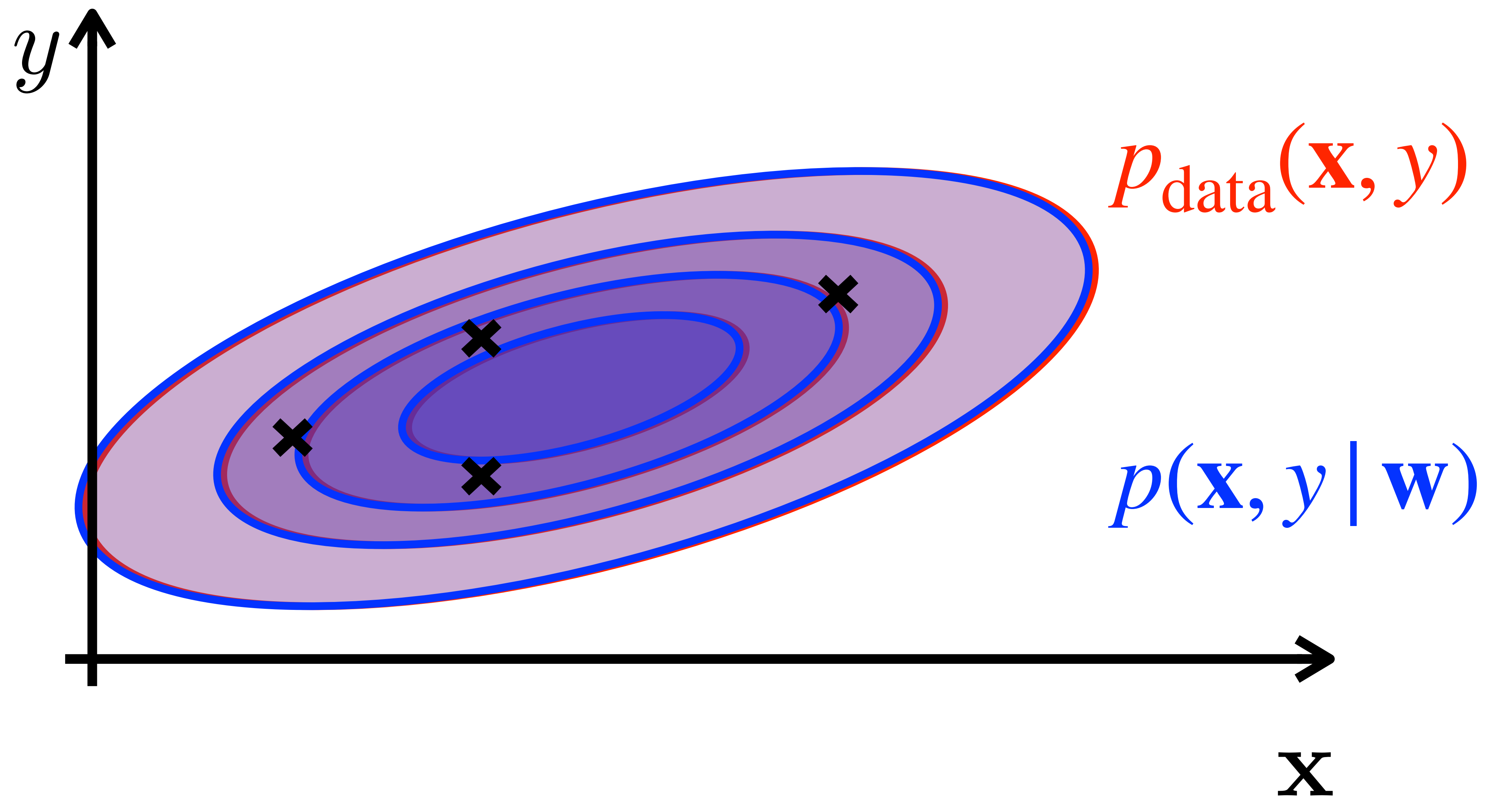

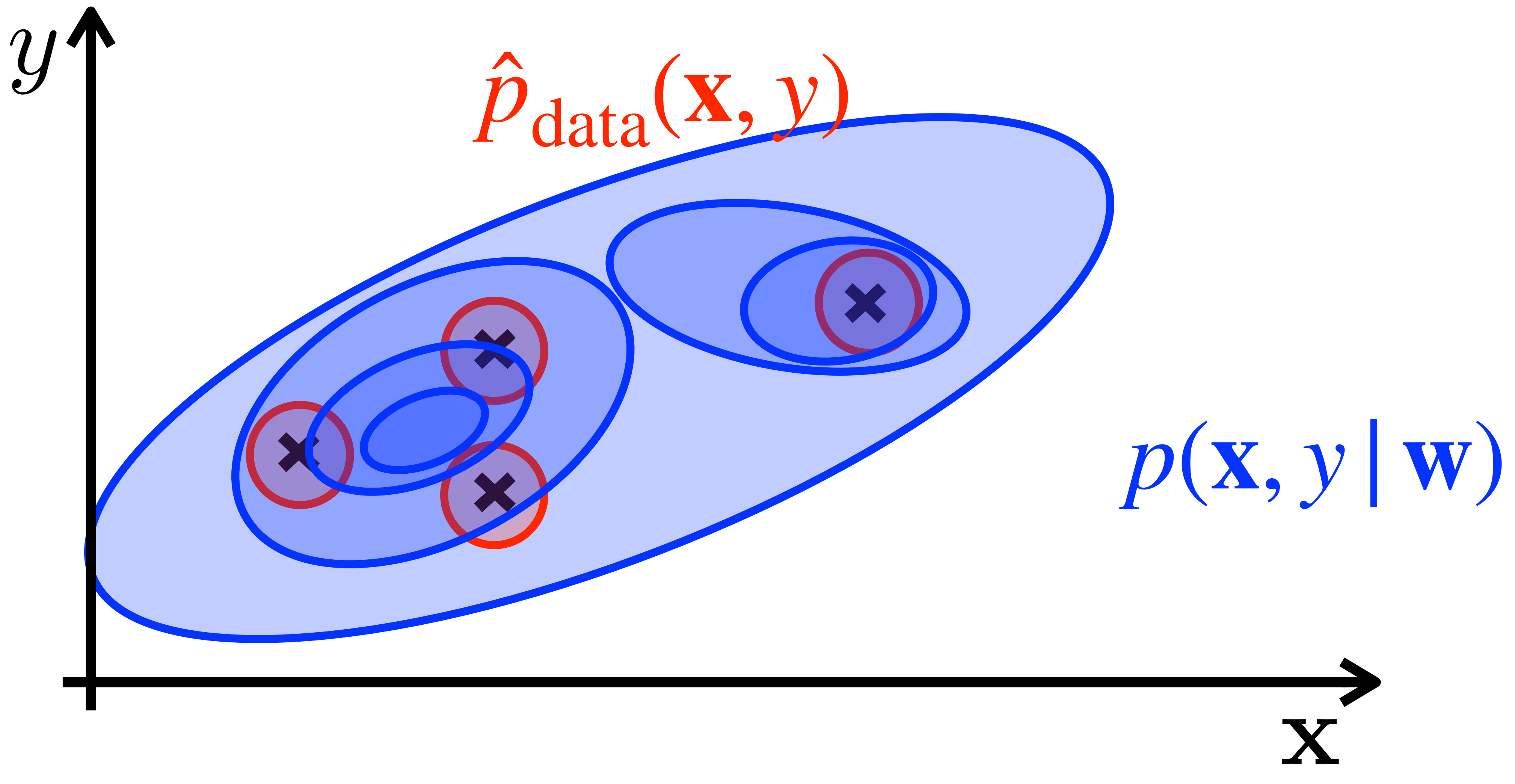

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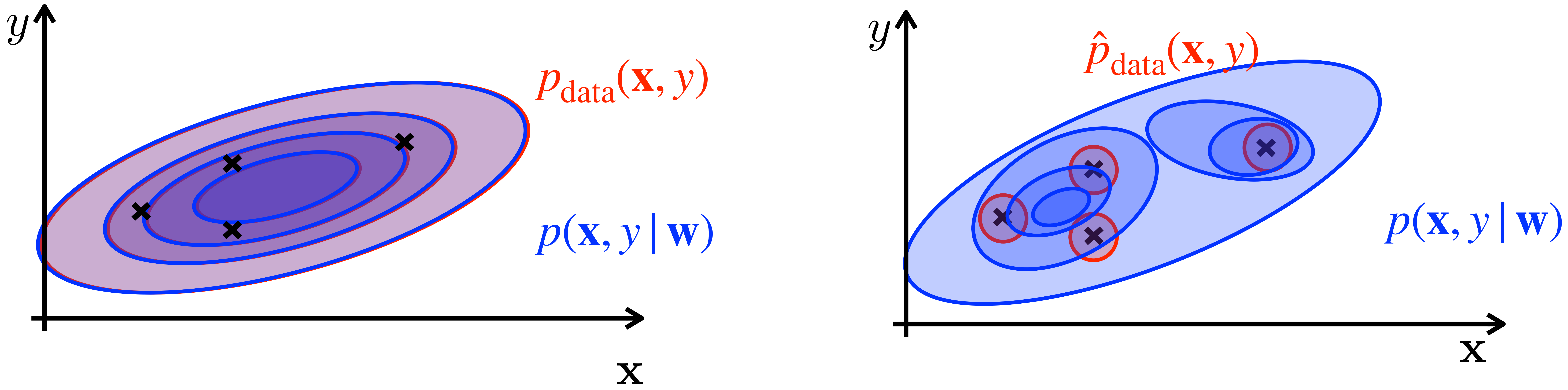
Take home message: **Optimization ≠ Machine learning**

Machine learning is optimization of the criterion, we do not have access to.

Therefore approximated criterion is optimized instead

Suppress overfitting = 1) Use the right $p(\mathbf{x}, y \,|\, \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$



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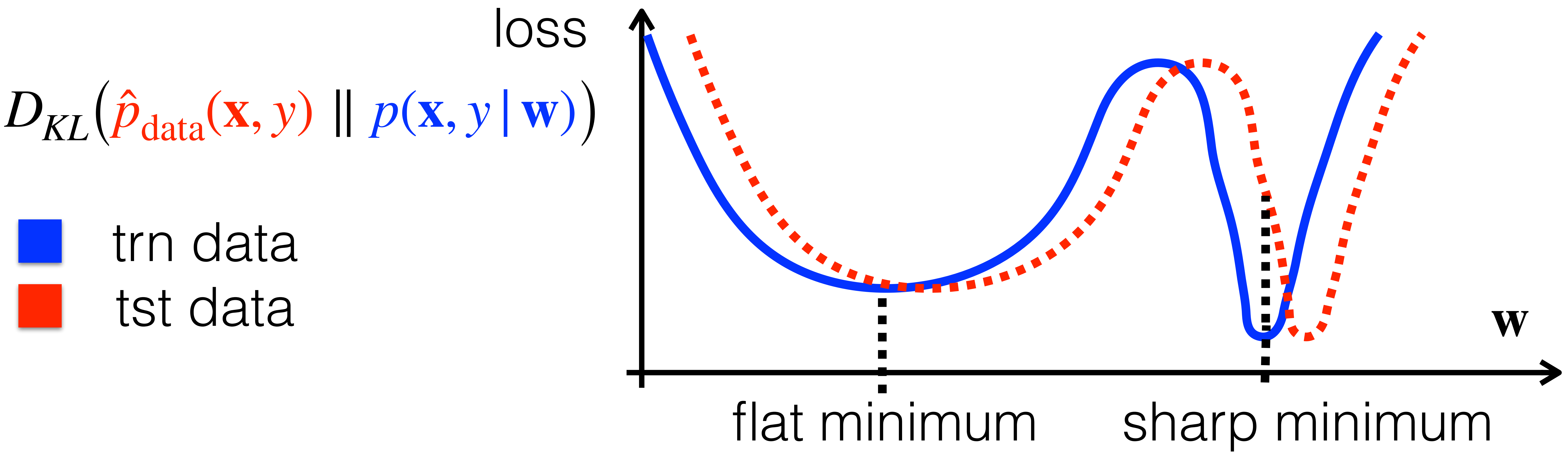
Suppress overfitting =
1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$
2) **Which one is better???**



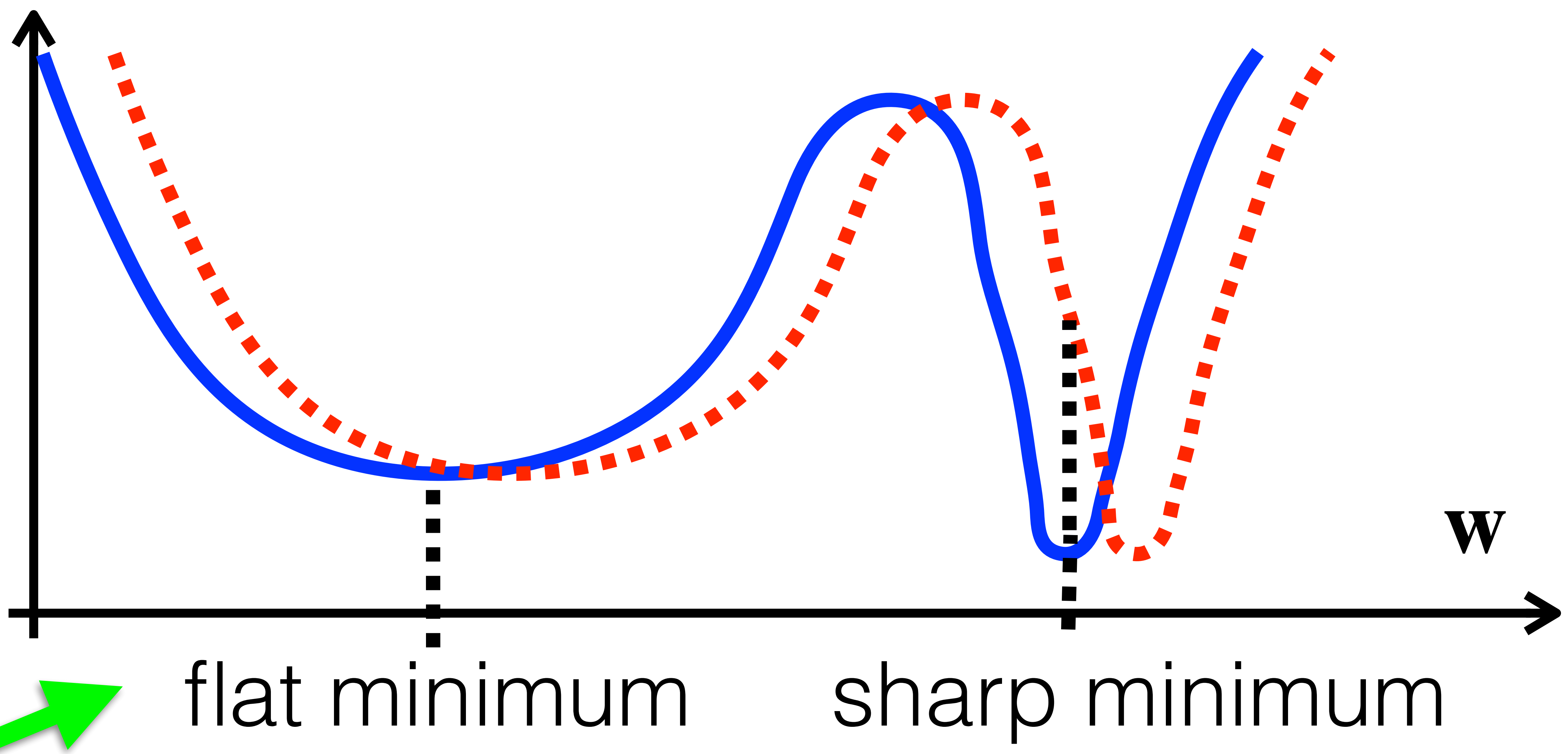
Where does the **overfitting** come from?

2) **Which one is better???**

$$D_{KL}\left(\hat{p}_{\text{data}}(\mathbf{x}, y) \,\|\, p(\mathbf{x}, y \mid \mathbf{w})\right)$$

🟦 trn data

🟥 tst data



Good generalization

Testing error remains small

Weak generalization => optimum prone to overfitting

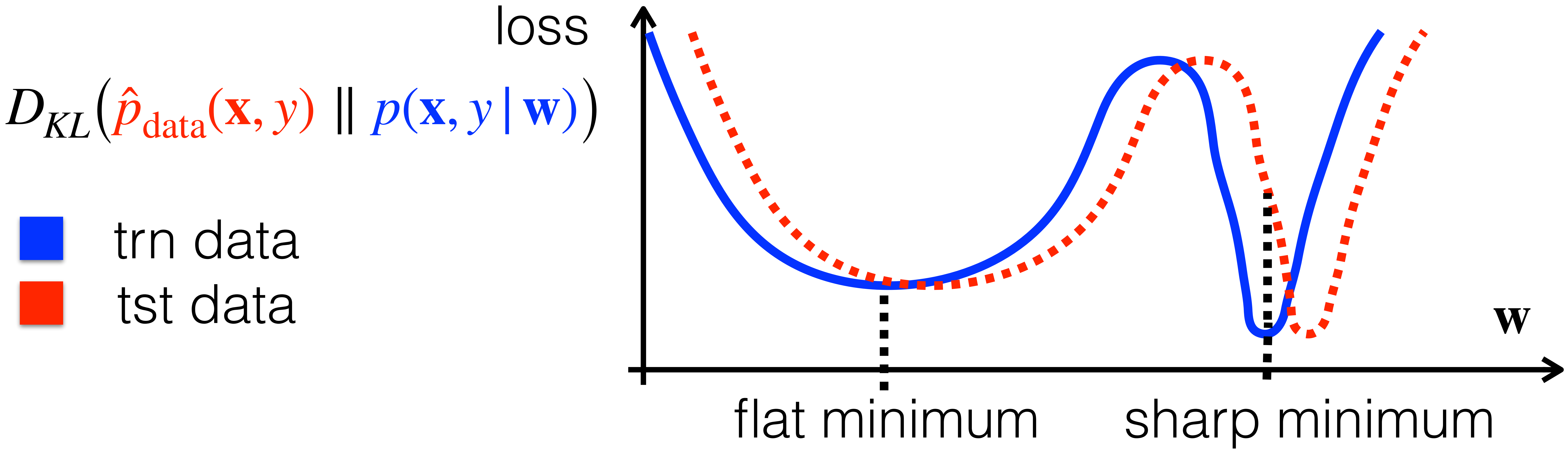
Testing error grows fast with a small trn/tst shift

Weaker learning methods are surprisingly better in generalization

[Dai, NIPS, 2018]   https://arxiv.org/pdf/1812.00542.pdf

Where does the **overfitting** come from?

2) Avoid sharp minima of $D_{KL}\left(\hat{p}_{\text{data}}(\mathbf{x}, y) \parallel p(\mathbf{x}, y \mid \mathbf{w})\right)$



Can you guess how to enforce flat minimum???

$$\min_{w} \max_{\|\epsilon\|_2 \leq \rho} L_{train}(w + \epsilon)$$

[Foret 2021]
https://arxiv.org/pdf/2106.01548.pdf

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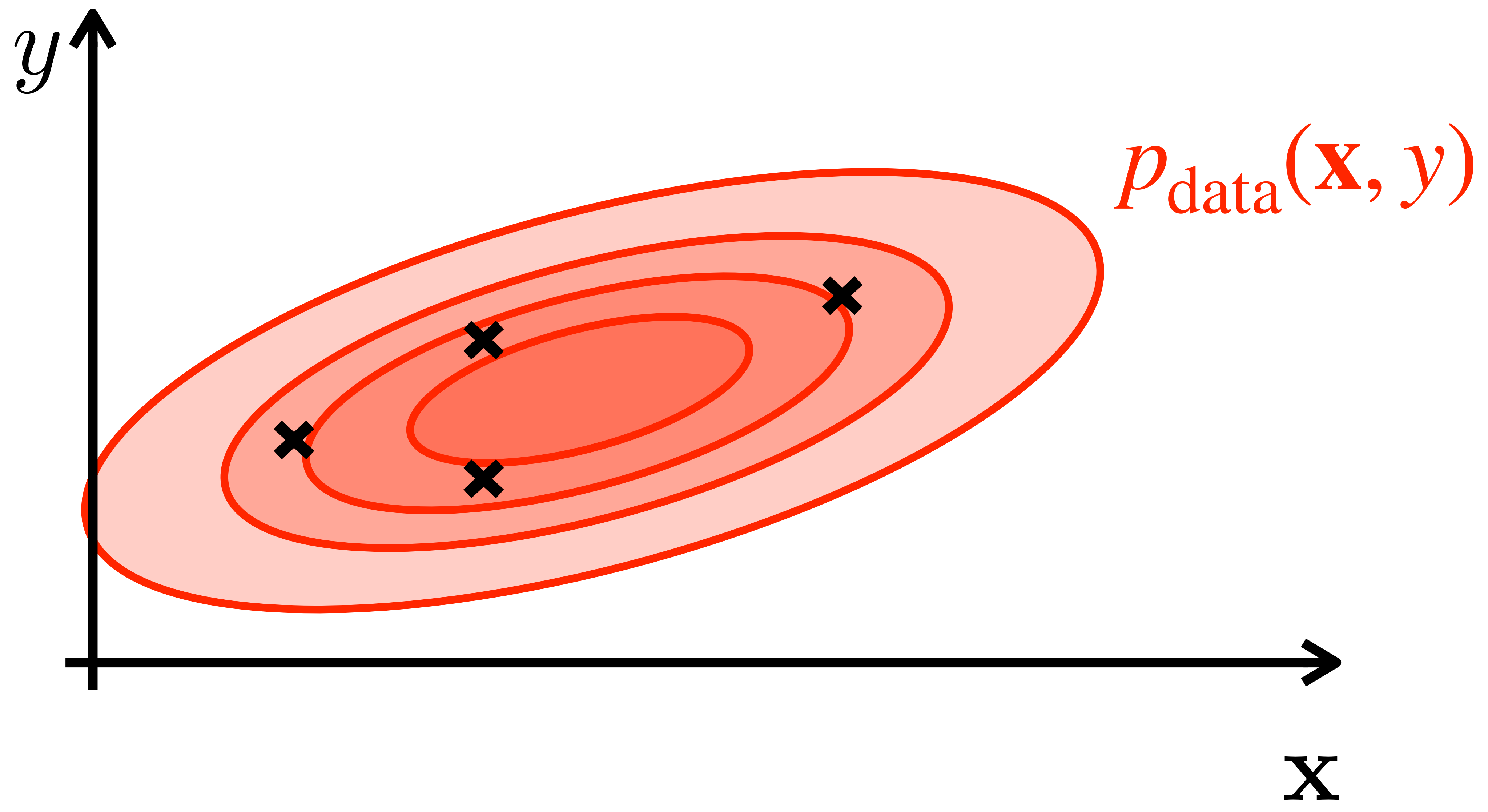
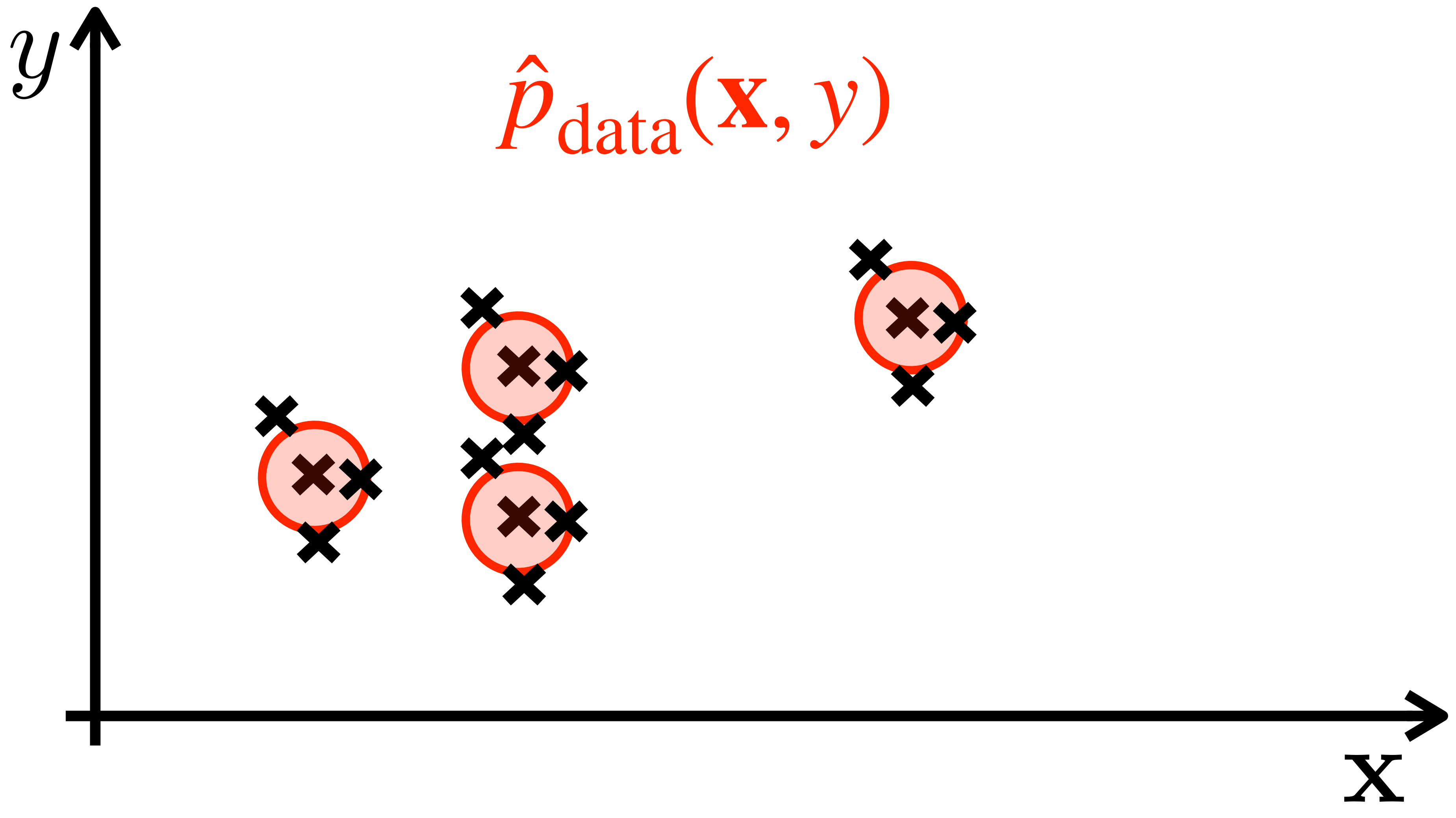
3) Use close-to-infinite dataset

Where does the **overfitting** come from?

3) Use close-to-infinite dataset

Data augmentation





Where does the **overfitting** come from?

3) Use close-to-infinite dataset

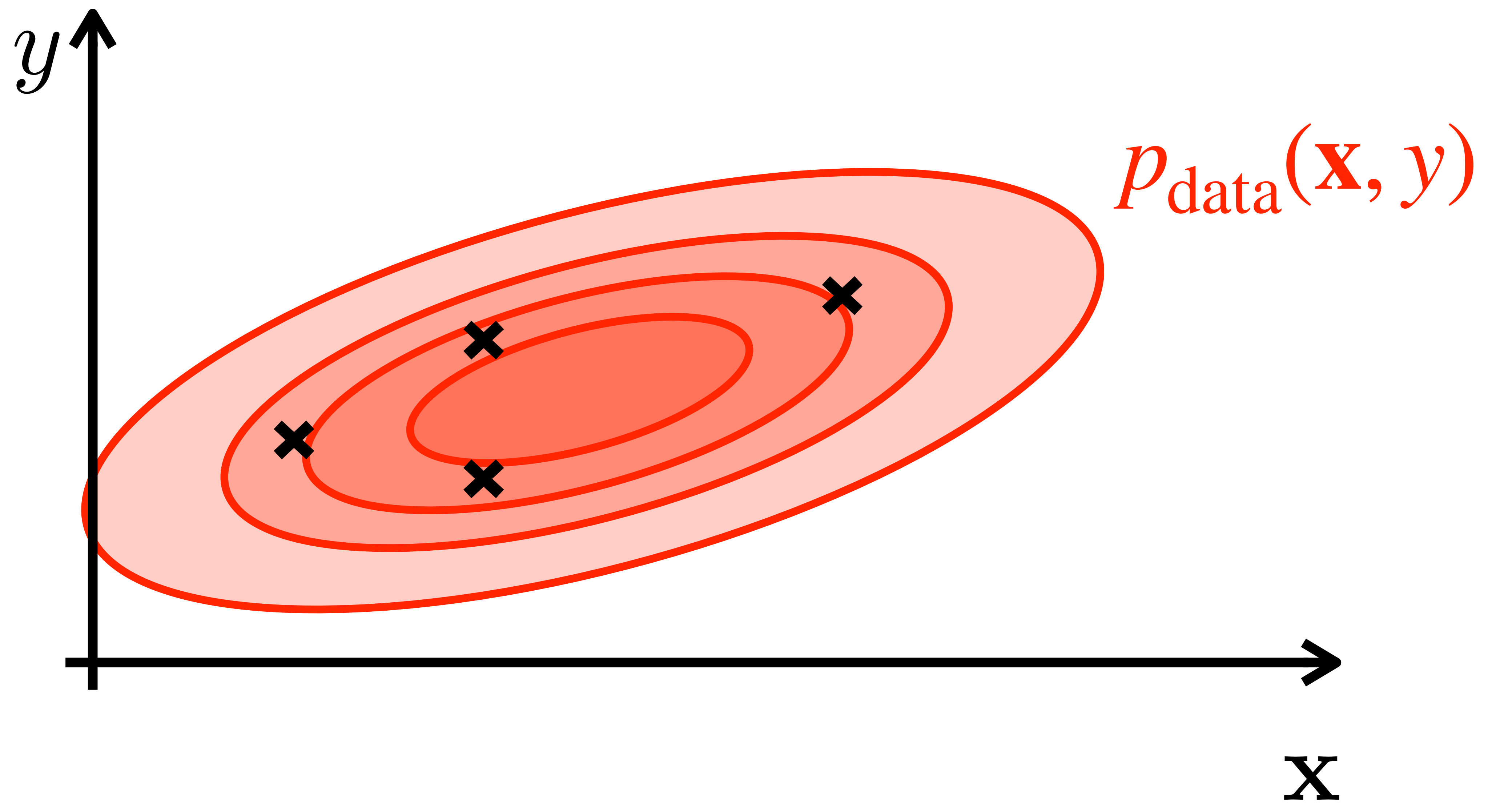
Data augmentation

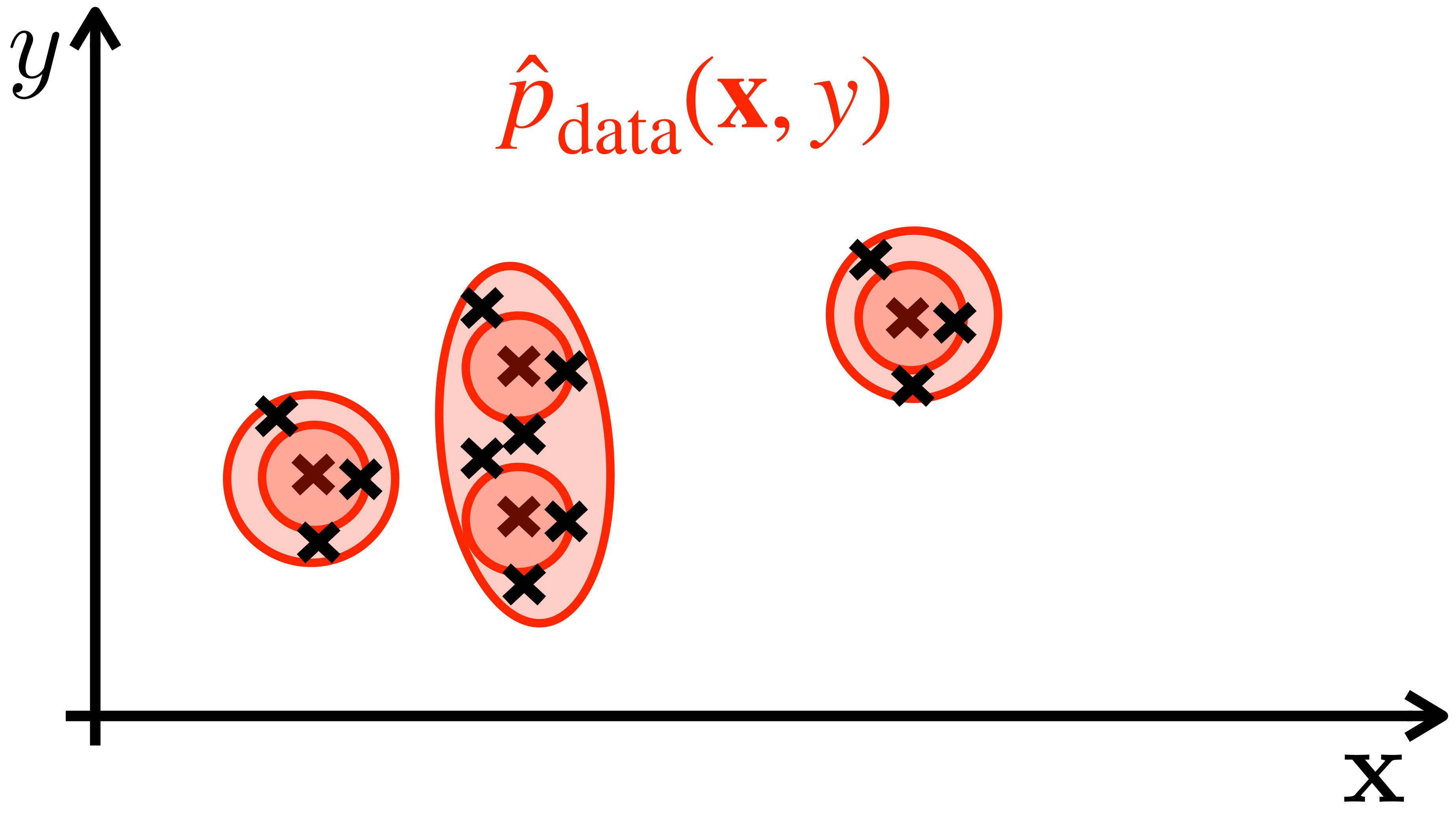
Which fake data can you generate???

Reasonable geometrical and histogram transformations:

Mirroring, scaling, rotation, squeezing, contrast, brightness, …

Or any generative model





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Take home message: **Optimization ≠ Machine learning**

Machine learning is optimization of the criterion, we do not have access to.

Therefore approximated criterion is optimized instead

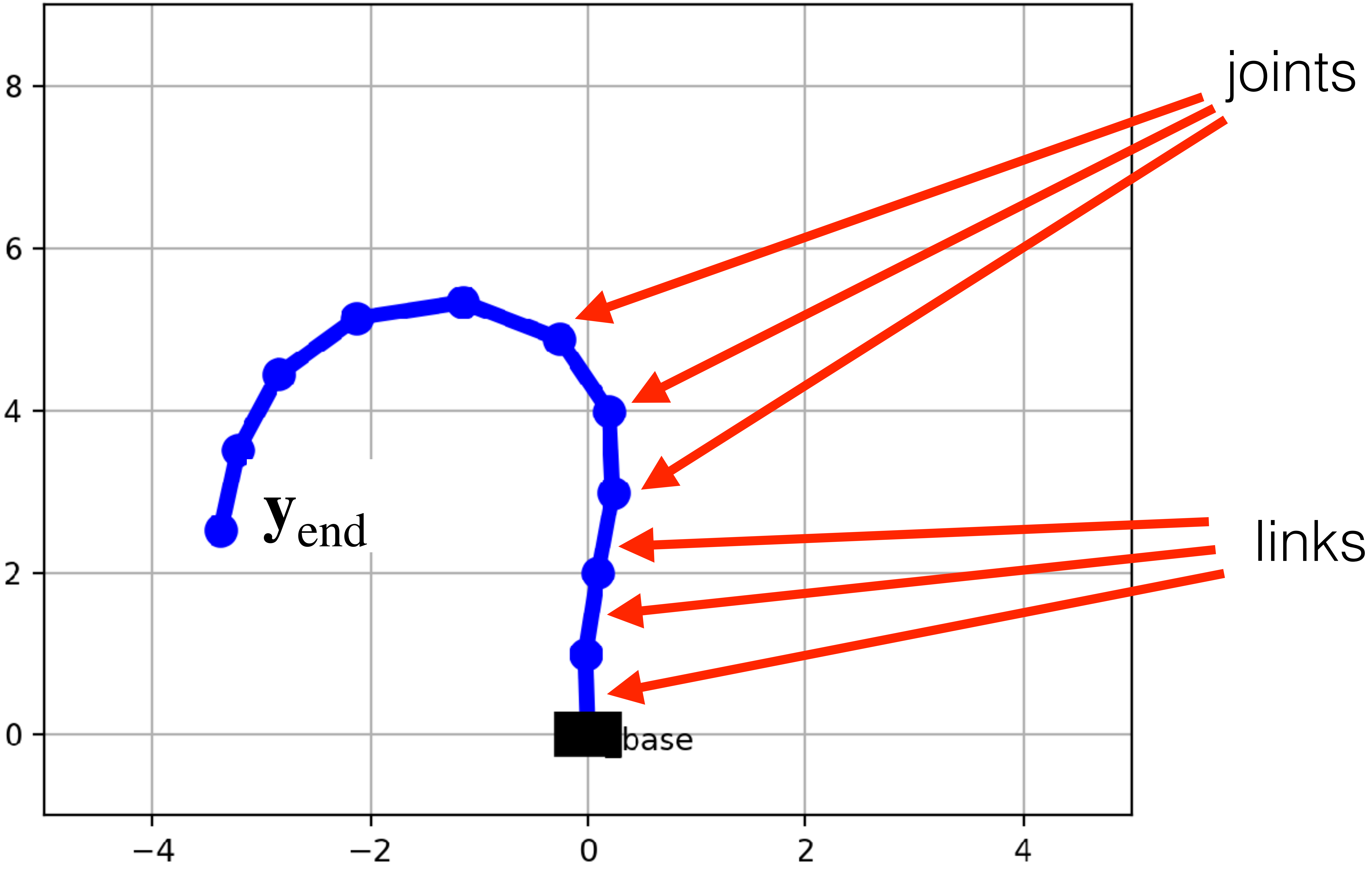
Suppress overfitting =

1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$
2) Avoid sharp minima of $D_{KL}\big(\hat{p}_{\text{data}}(\mathbf{x}, y) \,\|\, p(\mathbf{x}, y \mid \mathbf{w})\big)$
3) Use close-to-infinite dataset

1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

2D manipulator

HW1 IKT: (links + end_effector) => joints



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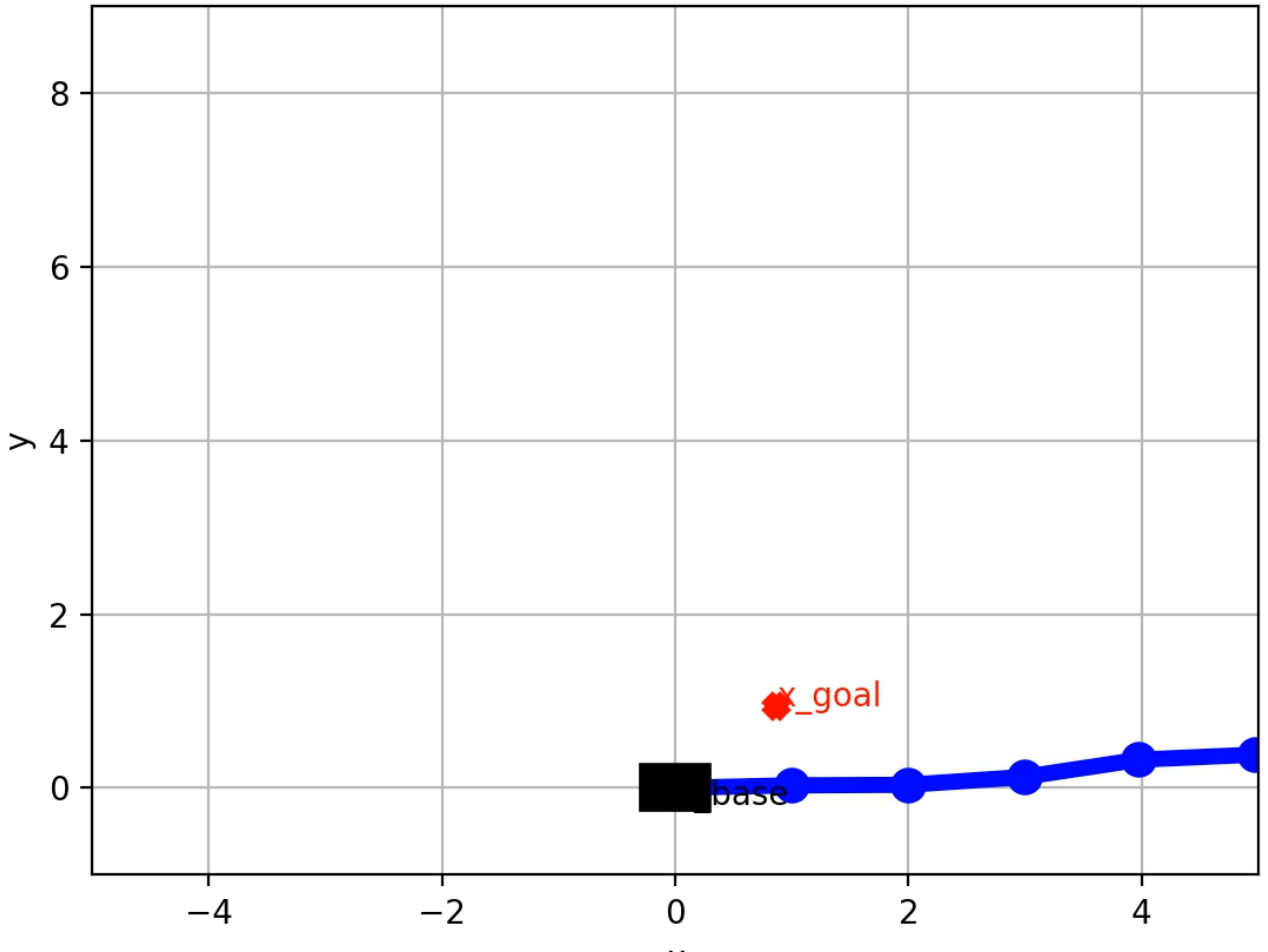
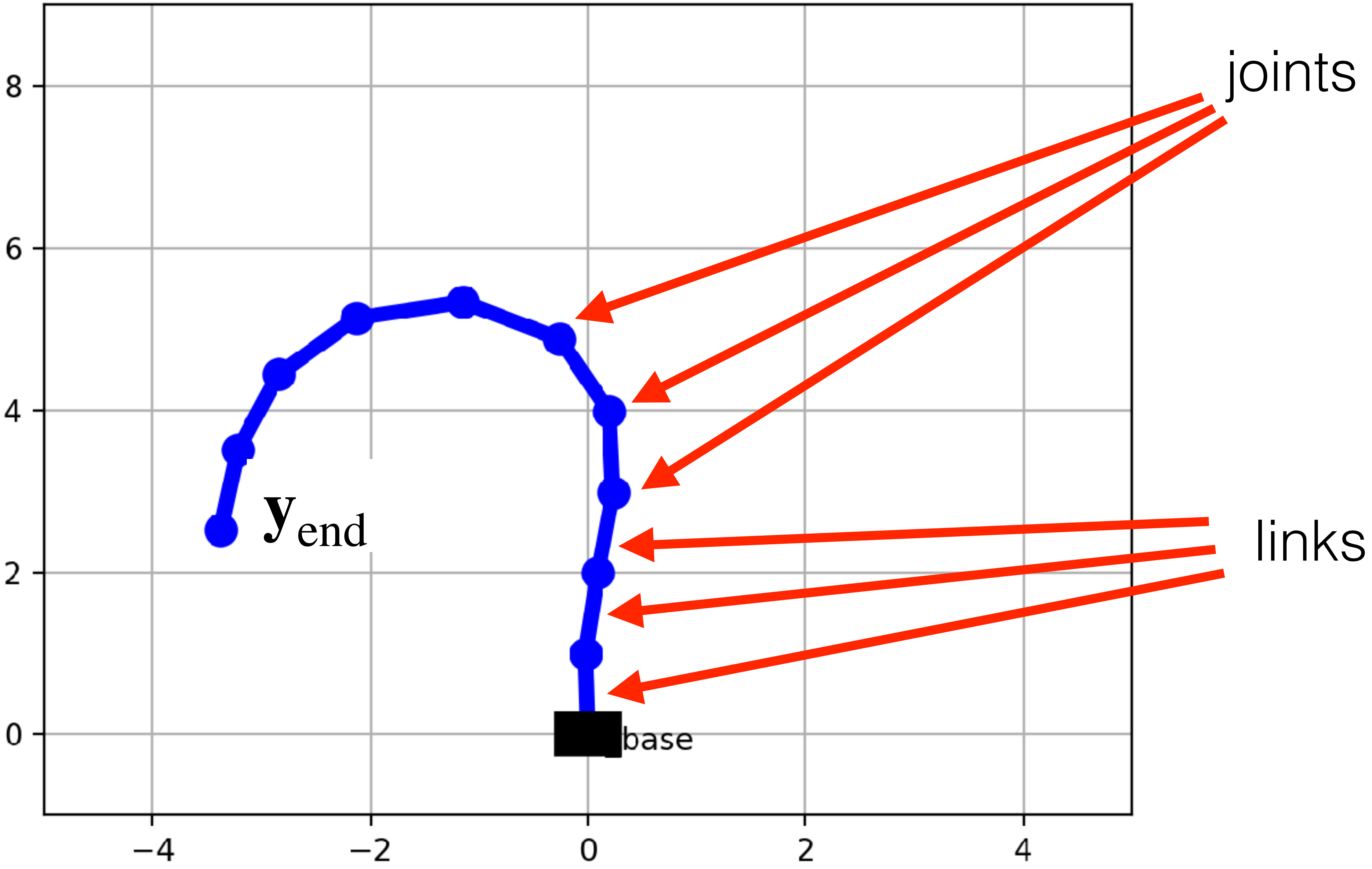


1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

model: $p(y \mid \mathbf{x}, \mathbf{w})$= **???**



1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

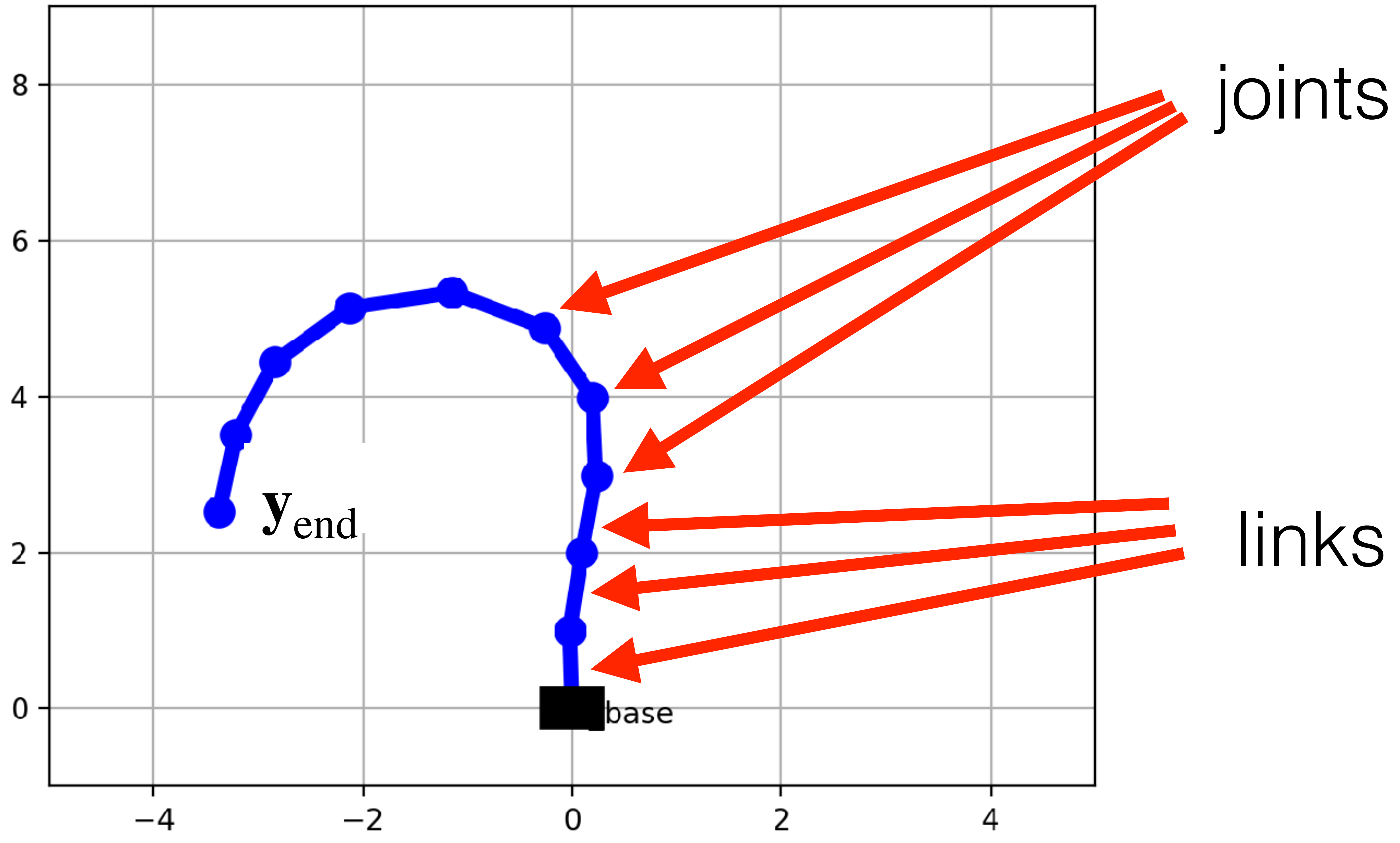
model: $p(y \mid \mathbf{x}, \mathbf{w}) = \mathcal{N}(y_{\text{end}};\ f(\mathbf{x}, \mathbf{w}), \sigma)$

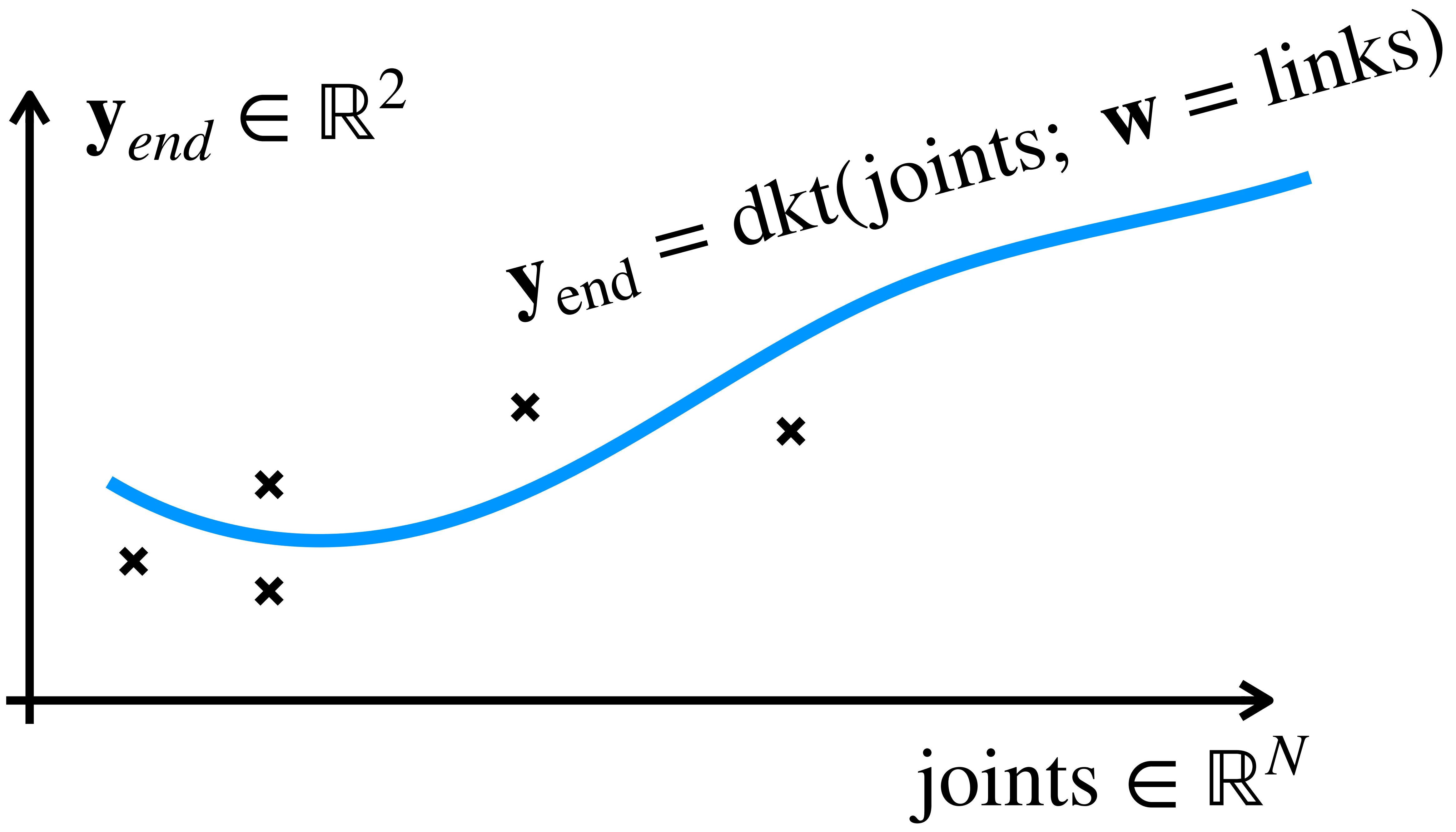
What is the right $f(\mathbf{x}, \mathbf{w})$?

(a) linear function $\mathbf{y}_{\text{end}} = f(\text{joints}; \mathbf{W}) = \mathbf{W} \cdot \text{joints}$ (underfit)

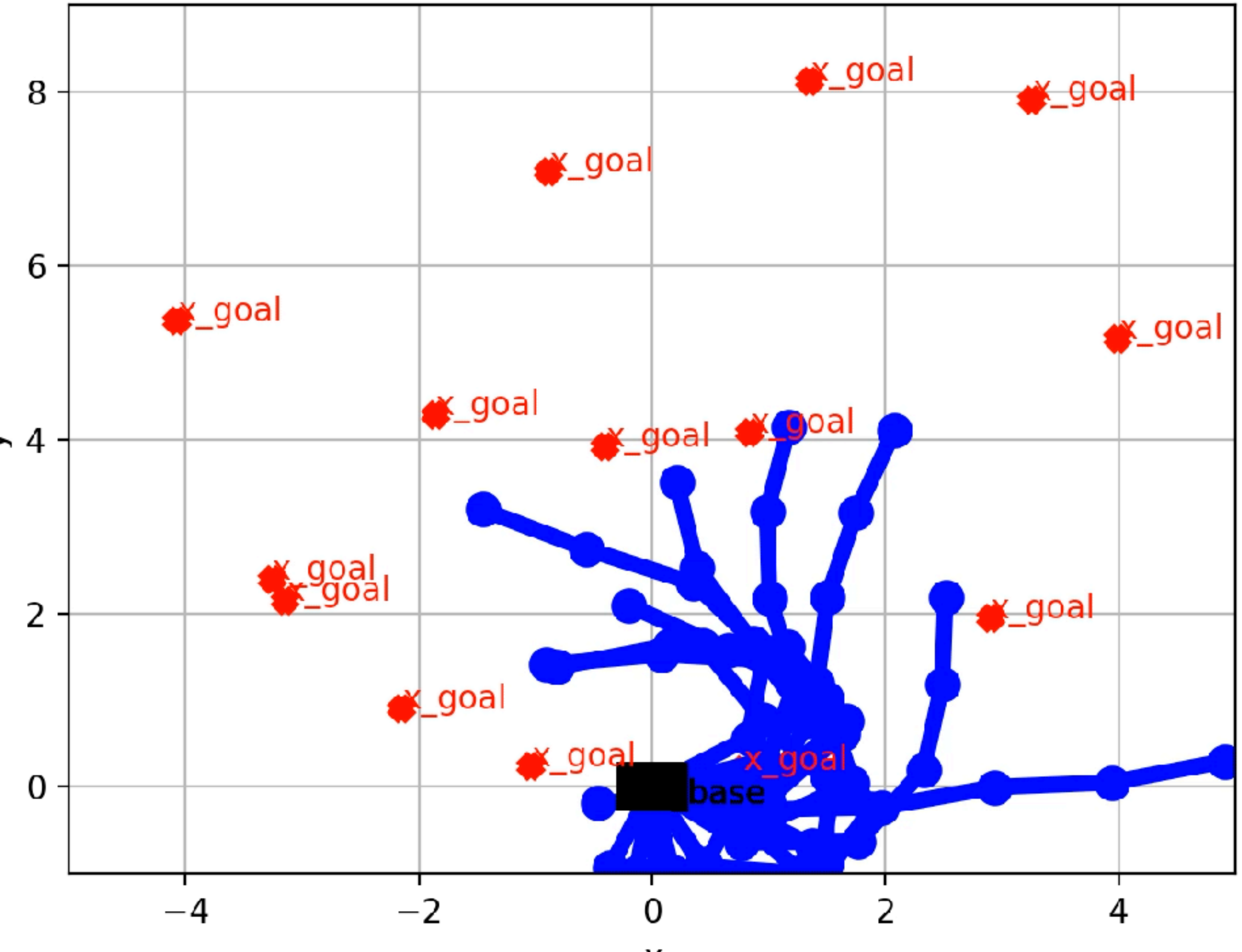
(b) deep ConvNet $\mathbf{y}_{\text{end}} = f(\text{joints}; \mathbf{w})$ (overfit)

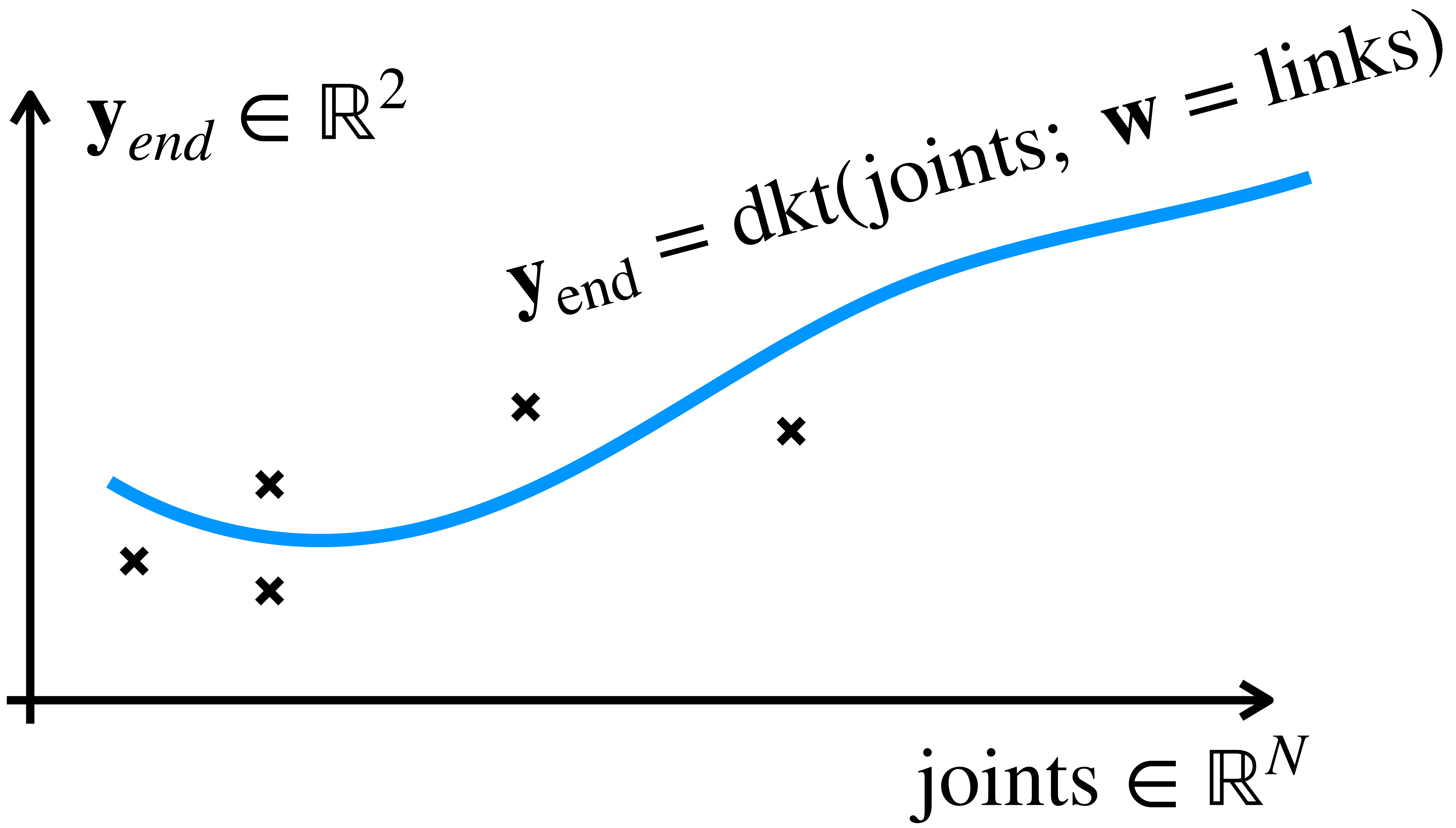
(c) DKT: $\mathbf{y}_{\text{end}} = \text{dkt}(\text{joints};\ \ \mathbf{w} = \text{links})$ (well-justified model)





1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$





1) Use the right $p(\mathbf{x}, y \,|\, \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

Take home message: **Always use the right tool/model**

Embed prior knowledge (physics, geometry, biology) about the problem into the network architecture

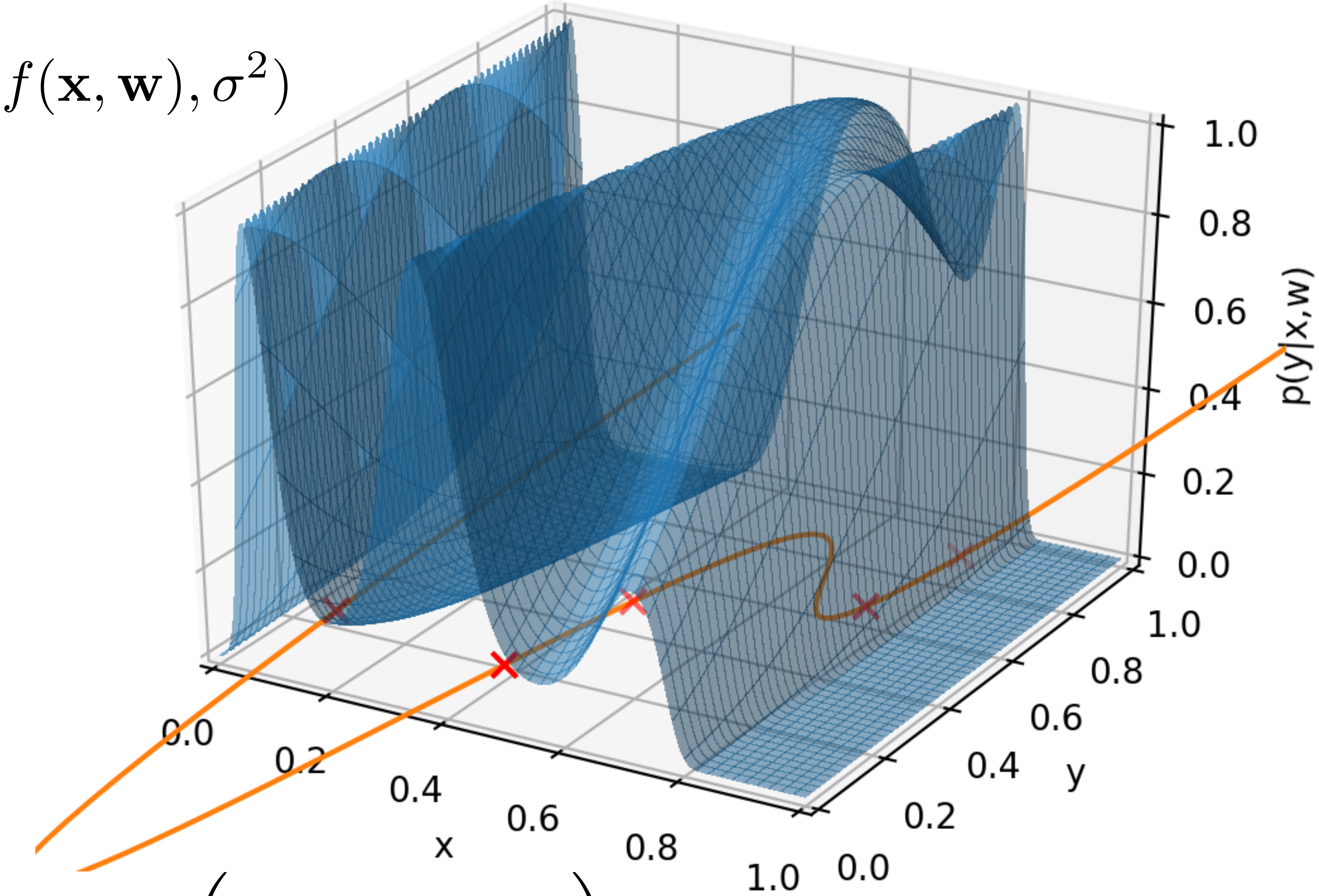
Examples:

- Projective transformation of pinhole cameras (for camera calibration or stereo)
- Geometry of Euclidean motion (for point cloud alignment, direct kinematic tasks)
- Motion model of robots such Dubins car, flight, pendulum, ballistic trajectory
- Structure of animal cortex (for ConvNets)

If you cannot do it, at least penalize wild solutions

1) Use the right $p(\mathbf{x}, y \,|\, \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

$$p(y|\mathbf{x}, \mathbf{w}) \sim \mathcal{N}_y(f(\mathbf{x}, \mathbf{w}), \sigma^2)$$



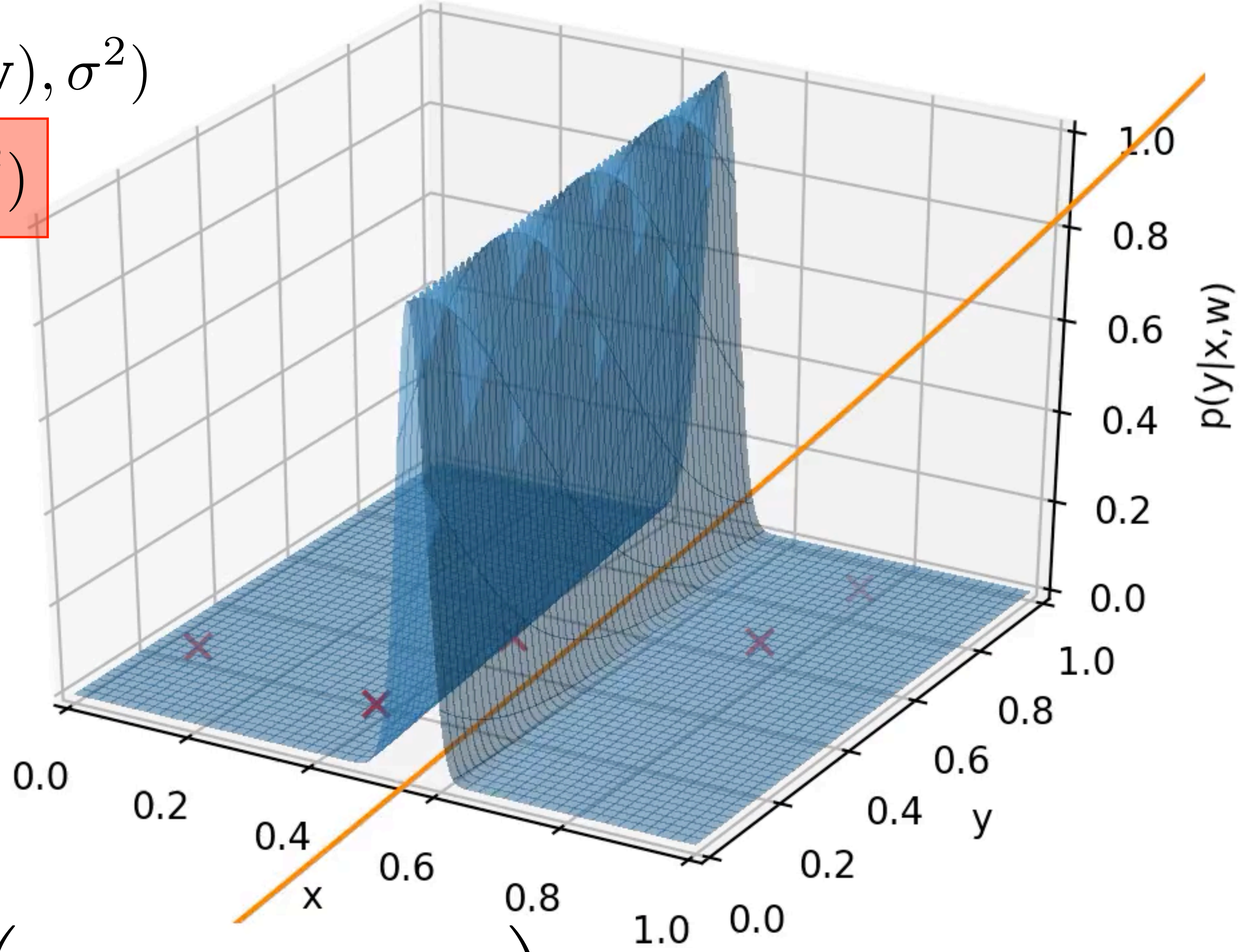
$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \left(\prod_i p(y_i|\mathbf{x}_i, \mathbf{w}) \right) = \arg\min_{\mathbf{w}} \sum_i (f(\mathbf{x}_i, \mathbf{w}) - y_i)^2$$

1) Use the right $p(\mathbf{x}, y \,|\, \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

What prior/regularizer should I use?

$$p(y|\mathbf{x}, \mathbf{w}) \sim \mathcal{N}_y(f(\mathbf{x}, \mathbf{w}), \sigma^2)$$

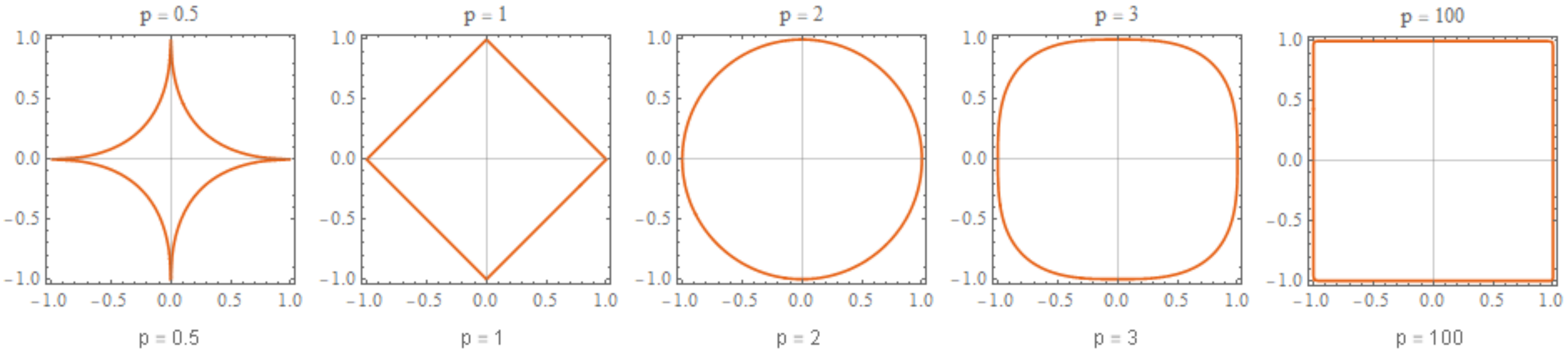
$$\boxed{p(\mathbf{w}) \sim \mathcal{N}_w(\mathbf{0},\ \sigma^2)}$$

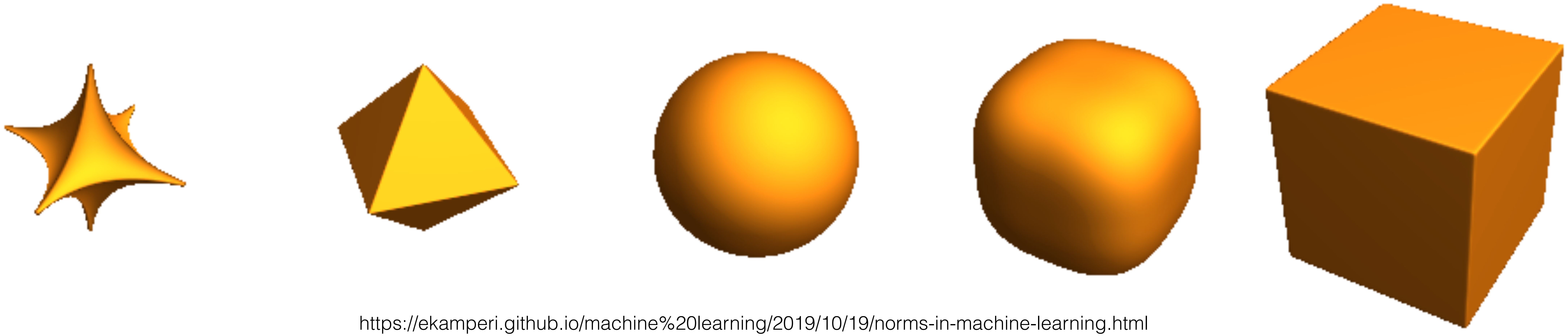


$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \left(\prod_i p(y_i|\mathbf{x}_i, \mathbf{w}) \boxed{p(\mathbf{w})} \right) = \arg\min_{\mathbf{w}} \sum_i (f(\mathbf{x}_i, \mathbf{w}) - y_i)^2 + \boxed{\|\mathbf{w}\|}$$

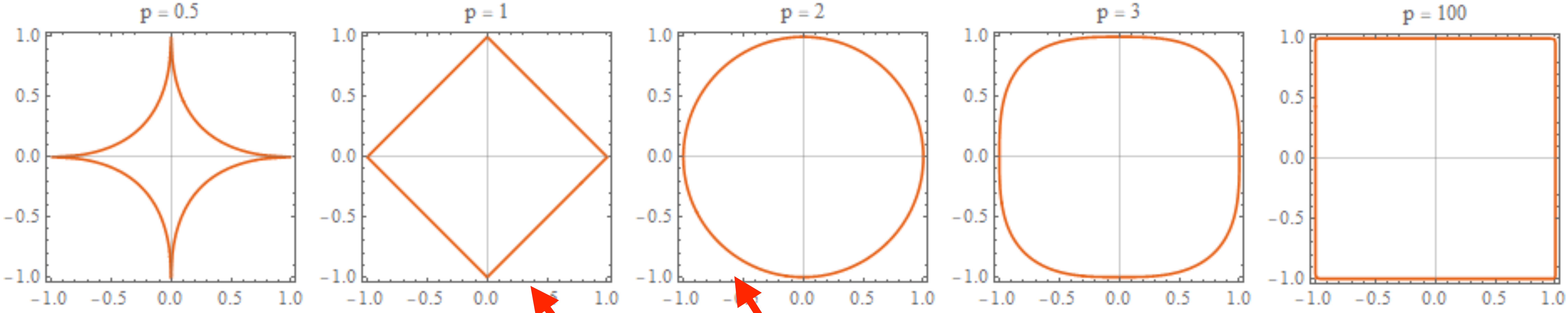
1) Use the right $p(\mathbf{x}, y \mid \mathbf{w})$ that generates only shapes similar to $p_{\text{data}}(\mathbf{x}, y)$

Lp-norm: $\|\mathbf{w}\|_p = \left(\sum_i |w_i|^p \right)^{\frac{1}{p}}$





https://ekamperi.github.io/machine%20learning/2019/10/19/norms-in-machine-learning.html



- Gaussian prior $p(\mathbf{w}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\|\mathbf{w}\|_2^2}{2\sigma^2}}$ => L2-regularization: $\|\mathbf{w}\|_2$

  It says: the smaller the better (the more probable)

- Laplace prior $p(\mathbf{w}) = \frac{1}{2b} e^{(-\frac{|\mathbf{w}|}{b})}$ => L1-regularization: $\|\mathbf{w}\|_1$

  It says: the sparser the better (the more probable)

- L2-regression with L1-regularization is known as Lasso

Summary

- Machine learning = optimization of the criterion, we do not have access to (KL divergence between true distribution and model)

Optimization $\neq$ Machine learning

- Avoid any "*not-well justified leprechauns*" in the model, => avoid overfitting

Always use the right ("leprechauns-free") architecture

  - Projective transformation of pinhole cameras (for camera calibration or stereo)
  - Geometry of Euclidean motion (point cloud alignment, direct kinematic tasks)
  - Motion model of robots such Dubins car, pendulum, ODE …
  - Structure of animal cortex (for ConvNets)

Prefer simplier solutions (flat minima, lower weights, …)

  - Less is sometimes more

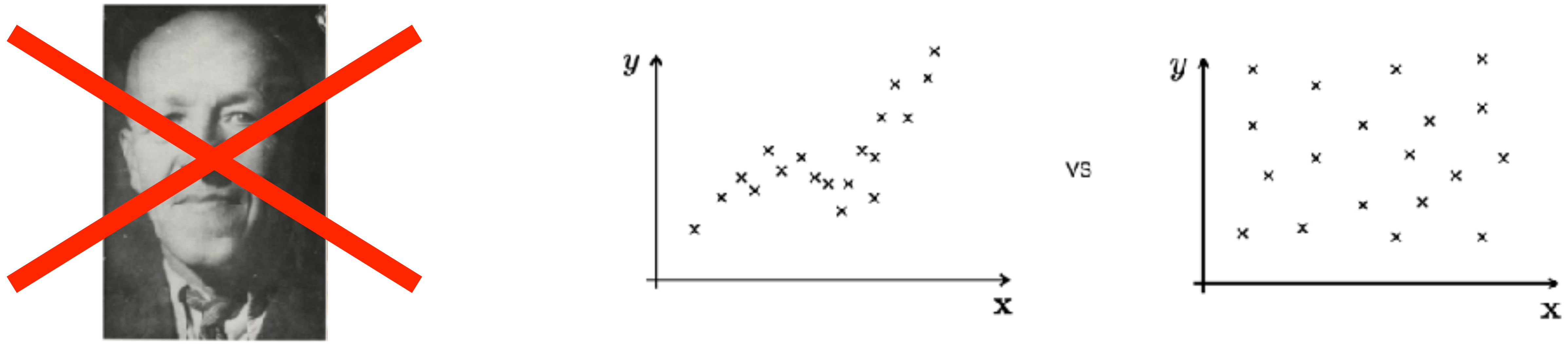
Use close-to-infinite training data

  - More is sometimes more ;-)

- Avoid oversimplifications of the model, => avoid underfitting

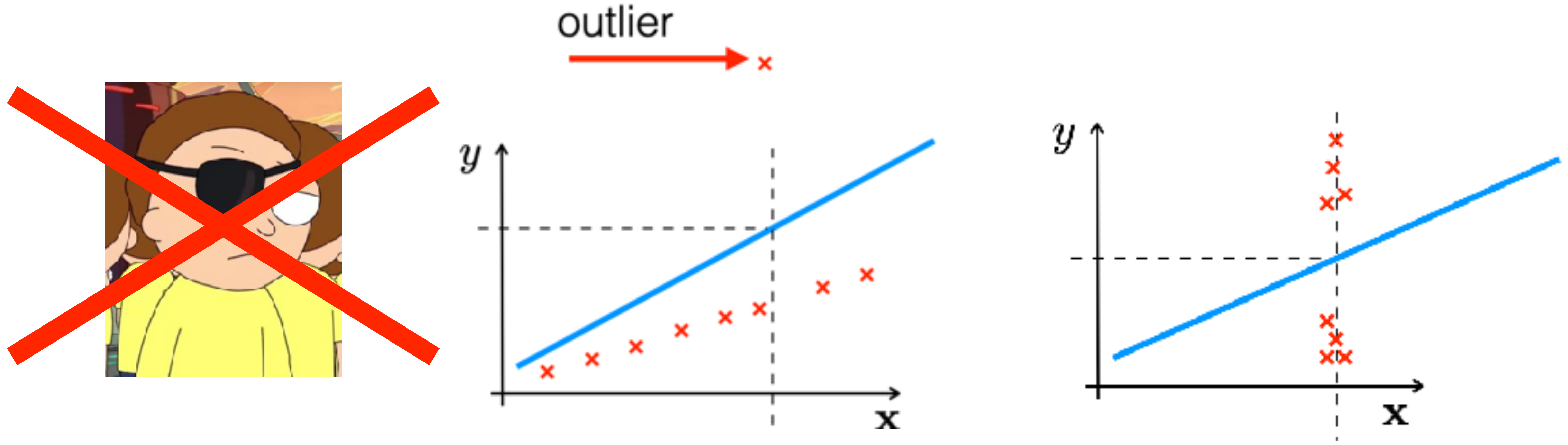
Golden grale:

- Solve only “Pilcik-free” problems



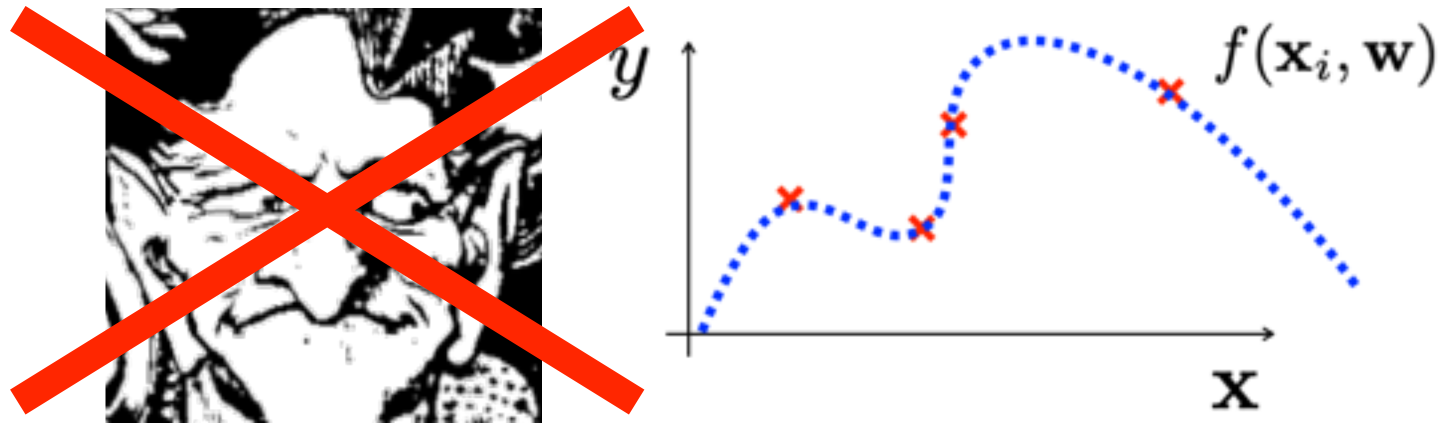
Lecture 1

- Use “Morty-free” data (or at least correct noise model)



Lecture 5

- Provide “sufficiently rich” + “lepricon-free” model.



Lecture 4,8,10
ConvNets
∇ODE, ∇argmin

- Avoid traps in learning

Lecture 7
Optimizers

Competencies required for the test T1

- Derive MLE estimate for regression and classification for different noise models
- Derive L2/L1/cross-entropy/logistic losses,
- Understand connection between KL divergence, loss, optimization, machine learning, underfitting, overfitting and model architecture.