GVG Test-α Solution

Task 1. *Complete vectors $\vec{b}_2$ and $\vec{b}_3$ to form a basis in $\mathbb{R}^3$:* $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} . \\ . \\ . \end{bmatrix}$ $\vec{b}_3 = \begin{bmatrix} . \\ . \\ . \end{bmatrix}$

Solution: One of the infinitely many ways to define $\vec{b}_2$ and $\vec{b}_3$ is

$$\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to see that $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are linearly independent, since the last row in the row echelon form of

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is nonzero. □

Task 2. *Use LD, resp. LI, to mark linearly dependent, resp. linearly independent, sets of vectors in ( )*

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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Solution:

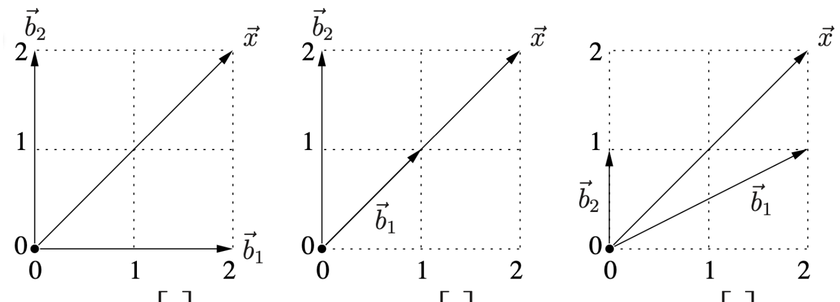
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ are LI}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\} \text{ are LD}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ are LD}$$

□

Task 3. *Write the coordinates of vector $\vec{x}$ in ordered basis $\beta = (\vec{b}_1, \vec{b}_2)$:*



$\vec{x}_\beta = \begin{bmatrix} . \\ . \end{bmatrix}$ $\vec{x}_\beta = \begin{bmatrix} . \\ . \end{bmatrix}$ $\vec{x}_\beta = \begin{bmatrix} . \\ . \end{bmatrix}$

Solution:

1. $\vec{x} = \vec{b}_1 + \vec{b}_2 \Rightarrow \vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. $\vec{x} = 2\vec{b}_1 \Rightarrow \vec{x}_\beta = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

3. $\vec{x} = \vec{b}_1 + \vec{b}_2 \Rightarrow \vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\square$

Task 4. *Vector $\vec{x}$ has coordinates $(1, -1)$ in ordered basis $\vec{b}_1$, $\vec{b}_2$. What are its coordinates in ordered basis $\vec{b}'_1 = 2\,\vec{b}_1$, $\vec{b}'_2 = \vec{b}_1 - \vec{b}_2$?*

Solution: Let $\beta = (\vec{b}_1, \vec{b}_2)$ and $\beta' = (\vec{b}'_1, \vec{b}'_2)$. By the task,

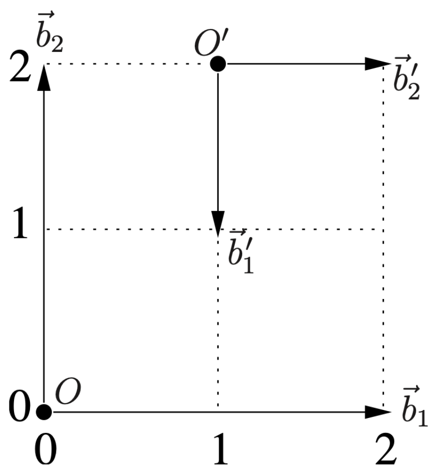
$$\vec{x}_\beta = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence

$$\vec{x} = \vec{b}_1 - \vec{b}_2 = 0 \cdot \vec{b}'_1 + 1 \cdot \vec{b}'_2 \Rightarrow \vec{x}_{\beta'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\square$

Task 5. *The following figure shows two coordinate systems (O, β) and (O', β'), with bases $\beta = (\vec{b}_1, \vec{b}_2)$ and $\beta' = (\vec{b}'_1, \vec{b}'_2)$*



1. *Write down the coordinates of vectors of basis β in basis β'.*

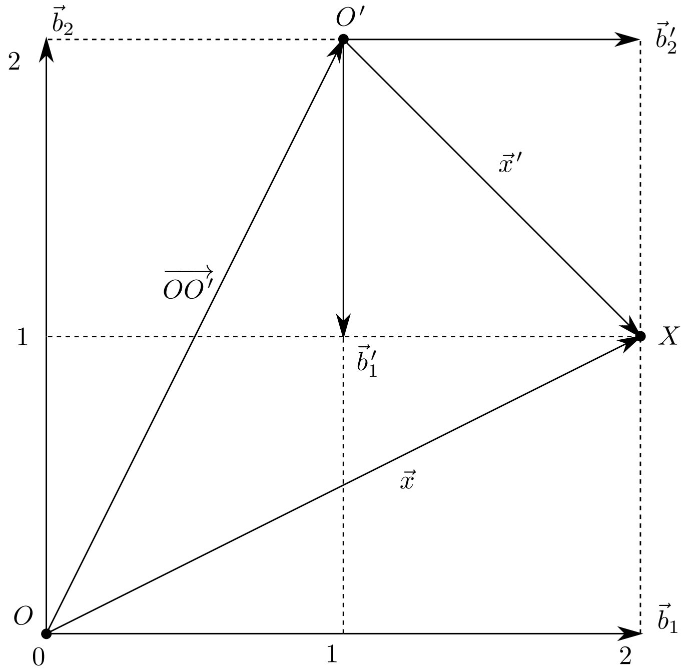
2. *Write down the coordinates of vectors of basis β' in basis β.*

3. *Write down the general formula for transforming the coordinates of $\vec{x}_\beta$ representing a general point X in (O, β) into the coordinates of $\vec{x}'_{\beta'}$ representing X in (O', β') and fill in the concrete numerical values for the situation in the figure.*

Solution:

Remark. *In this task we talk about vectors as "free vectors" [1, Section 3.3]. This means that, figuratively speaking, arrows are not bounded to concrete points and we can freely move them through the whole space of points and it will still be the same vector. We call this class of arrows which differ by translation a "free vector". In order to add two free vectors we must first move the arrows which represent them to the same point and then apply the parallelogram rule to get an arrow representing the sum of the given free vectors.*

1. In order to compute $\vec{b}_{1\beta'}$ we move the arrow representing $\vec{b}_1$ so that it points from O' and then we see that
$$\vec{b}_1 = 0 \cdot \vec{b}_1' + 2 \cdot \vec{b}_2' \Rightarrow \vec{b}_{1\beta'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
Similarly,
$$\vec{b}_2 = -2 \cdot \vec{b}_1' + 0 \cdot \vec{b}_2' \Rightarrow \vec{b}_{2\beta'} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
2. Similarly,
$$\vec{b}_1' = 0 \cdot \vec{b}_1 - \frac{1}{2} \cdot \vec{b}_2 \Rightarrow \vec{b}_{1\beta}' = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$$
$$\vec{b}_2' = \frac{1}{2} \cdot \vec{b}_1 + 0 \cdot \vec{b}_2 \Rightarrow \vec{b}_{2\beta}' = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$
3. Let's, for example, pick a point X as it is shown in the picture below:



When we say that vector $\vec{x}_\beta$ represents point X in coordinate system (O, β) it means that $\vec{x}_\beta = \overrightarrow{OX}_\beta$ (similarly, $\vec{x}'_{\beta'} = \overrightarrow{O'X}_{\beta'}$). By looking at the triangle $\triangle OO'X$ we can write
$$\vec{x} = \overrightarrow{OO'} + \vec{x}'$$

Remark. *It can be proved that the map φ_β from the linear space of free vectors to their coordinates in some basis β (which is itself a linear space) is linear, and thus $(\vec{x}+\vec{y})_\beta = \varphi_\beta(\vec{x}+\vec{y}) = \varphi_\beta(\vec{x}) + \varphi_\beta(\vec{y}) = \vec{x}_\beta + \vec{y}_\beta$.*

Hence, by the previous remark, after passing to the coordinates of the above free vectors in basis β' we obtain
$$\vec{x}_{\beta'} = \left(\overrightarrow{OO'} + \vec{x}'\right)_{\beta'} = \overrightarrow{OO'}_{\beta'} + \vec{x}'_{\beta'}$$
$$\vec{x}'_{\beta'} = \vec{x}_{\beta'} - \overrightarrow{OO'}_{\beta'}$$
$$\vec{x}'_{\beta'} = \mathtt{A}_{\beta\to\beta'} \left(\vec{x}_\beta - \overrightarrow{OO'}_\beta\right)$$
$$\vec{x}'_{\beta'} = \begin{bmatrix} \vec{b}_{1\beta'} & \vec{b}_{2\beta'} \end{bmatrix} \left(\vec{x}_\beta - \overrightarrow{OO'}_\beta\right)$$

$$\vec{x}'_{\beta'} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \left(\vec{x}_\beta - \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

$$\vec{x}'_{\beta'} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \vec{x}_\beta + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

□

Task 6. *Write down a basis of the one-dimensional subspace of $\mathbb{R}^3$, which results as the intersection of two two-dimensional subspaces of $\mathbb{R}^3$, which are determined by their bases*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution: Finding the intersection of these two planes means solving

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{1}$$

Rewriting it in a matrix form (but differently) we obtain

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve this system of linear equations by Gaussian Elimination:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Looking at the last equation $-c + 2d = 0$ we let d to be any real number and $c = 2d$. From the second equation $b - d = 0$ we get $b = d$. From the first equation $a - d = 0$ we get $a = d$. Thus, the set of solution is described by

$$S = \left\{ \begin{bmatrix} d \\ d \\ 2d \\ d \end{bmatrix} \;\middle|\; d \in \mathbb{R} \right\}$$

To obtain a basis of the intersection of planes we substitute $a = d$ and $b = d$ to Equation (1):

$$L = \left\{ d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; d \in \mathbb{R} \right\} = \left\{ d \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \;\middle|\; d \in \mathbb{R} \right\}$$

Thus, a basis of the linear subspace L is

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

□

Task 7. *Change one element of the following matrix so it becomes a rank one matrix.*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

Solution: There are two ways:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0.5 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

□

Task 8. *Find all solutions to the systems*

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: We solve the first system by Gaussian Elimination:

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second equation $y + z = 0$ we let z to be any real number and express $y = -z$. From the first equation $-x + z = 0$ we get $x = z$. Thus, the set of solutions is

$$S = \left\{ \begin{bmatrix} z \\ -z \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

From linear algebra we know that if S is the set of solutions to $\mathbf{Ax} = \mathbf{0}$, then the set of solutions to $\mathbf{Ax} = \mathbf{b}$ can be written as $\mathbf{x}_0 + S$, where $\mathbf{Ax}_0 = \mathbf{b}$. We can guess that

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence the set of solutions to the second system can be described as

$$\mathbf{x}_0 + S = \left\{ \begin{bmatrix} z+1 \\ -z+1 \\ z \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

□

Task 9. *Find the eigenvalues and eigenvectors of matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: To find the eigenvalues we compute the characteristic polynomial of $\mathbf{A}$:

$$p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix} \right) = (\lambda - 1)^3$$

The eigenvalues are the roots of $p(\lambda)$:

$$p(\lambda) = 0 \Rightarrow \lambda_{1,2,3} = 1.$$

The eigenspace corresponding to the eigenvalue $\lambda = 1$ of multiplicity 3 can be computed as the kernel of $\lambda\mathbf{I} - \mathbf{A}$:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The set of solutions is

$$S = \left\{ \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} \;\middle|\; x_1, x_3 \in \mathbb{R} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; x_1, x_3 \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

A basis of the eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

□

Task 10. *How many roots, including multiplicities, does the equation $x^6 + x^4 - x^2 - 1 = 0$ have in complex space? Find as many of its roots as possible.*

Solution: *The given polynomial $f = x^6 + x^4 - x^2 - 1$ can be factorized as follows:*

$$f = x^4(x^2+1) - (x^2+1) = (x^2+1)(x^4-1) = (x^2+1)(x^2+1)(x^2-1) = (x^2+1)^2(x^2-1)$$

The solutions to $f = 0$ are

$$x_{1,2} = i, \quad x_{3,4} = -i, \quad x_5 = 1, \quad x_6 = -1.$$

In total (including multiplicities) there are 6 *complex roots, since* $\deg f = 6$. □

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf`.