Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 8

Tomas Pajdla

pajdla@cvut.cz

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Let us now consider point

$$
\begin{aligned}
\vec{v}'_{\beta'} &= (\vec{x}'_{\beta'} \times \vec{y}'_{\beta'}) \times (\vec{z}'_{\beta'} \times \vec{w}'_{\beta'}) && (8.42)\\
&= \left(\frac{\mathtt{H}^{-\top}}{\lambda_1\,\lambda_2\,|\mathtt{H}^{-\top}|}(\vec{x}_\beta \times \vec{y}_\beta)\right) \times \left(\frac{\mathtt{H}^{-\top}}{\lambda_3\,\lambda_4\,|\mathtt{H}^{-\top}|}(\vec{z}_\beta \times \vec{w}_\beta)\right) && (8.43)\\
&= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}(\vec{x}_\beta \times \vec{y}_\beta) \times (\vec{z}_\beta \times \vec{w}_\beta) && (8.44)\\
&= \frac{\mathtt{H}\,|\mathtt{H}|}{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4}\vec{v}_\beta && (8.45)
\end{aligned}
$$

8.3.4 Note on homographies that are rotations

First notice that homogeneous coordinats of points and lines constructed as combinations of joins and meets indeed behave under a homography as homogeneous coordinates constructed from affine coordinates of points.

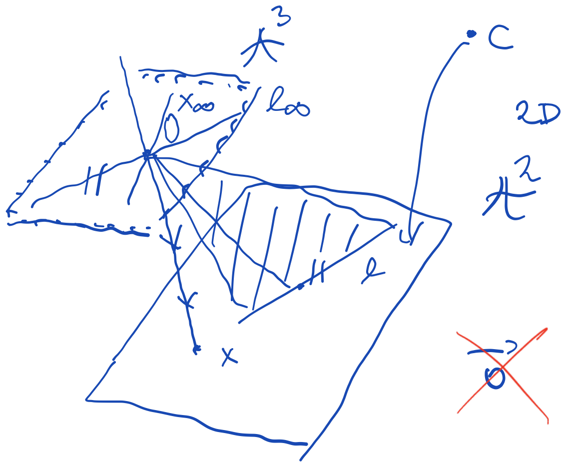
Secondly, when the homography is a rotation and homogeneous coordinates are unit vecors, all λ's become equal to one, the determinant of $\mathtt{H}$ is one and $\mathtt{H}^{-\top} = \mathtt{H}$. Therefore, all homogeneous coordinates in the previous formulas become related just by $\mathtt{H}$.

8.4 Vanishing points

When modeling perspective projection in the affine space with affine projection planes, we meet somewhat unpleasant situations. For instance, imagine a projection of two parallel lines K, L, which are in a plane τ in the space into the projection plane π through the center C, Figure 8.10.

The lines K, L project to image lines k, l. As we go with two points X, Y along the lines k, l away from the projection plane, their images x, y get closer and closer to the point v in the image but they do not reach point v. We shall call this point of convergence of lines K, L the *vanishing point*[7].

[7] Úběžník in Czech.



$A^2 \longrightarrow P^2$

$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix} \approx \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha w \end{bmatrix}$

points $\equiv$ 1D subspaces of $\mathbb{R}^3$

lines $\equiv$ 2D subspaces of $\mathbb{R}^3$

$\equiv$ 1D subspaces $\mathbb{R}^3$

$x \cdot l \qquad x^\top l = 0$

$l = \vec{x} \times \vec{y}\,, \quad \vec{x} = \vec{l} \times \vec{k}$

Vanishing points

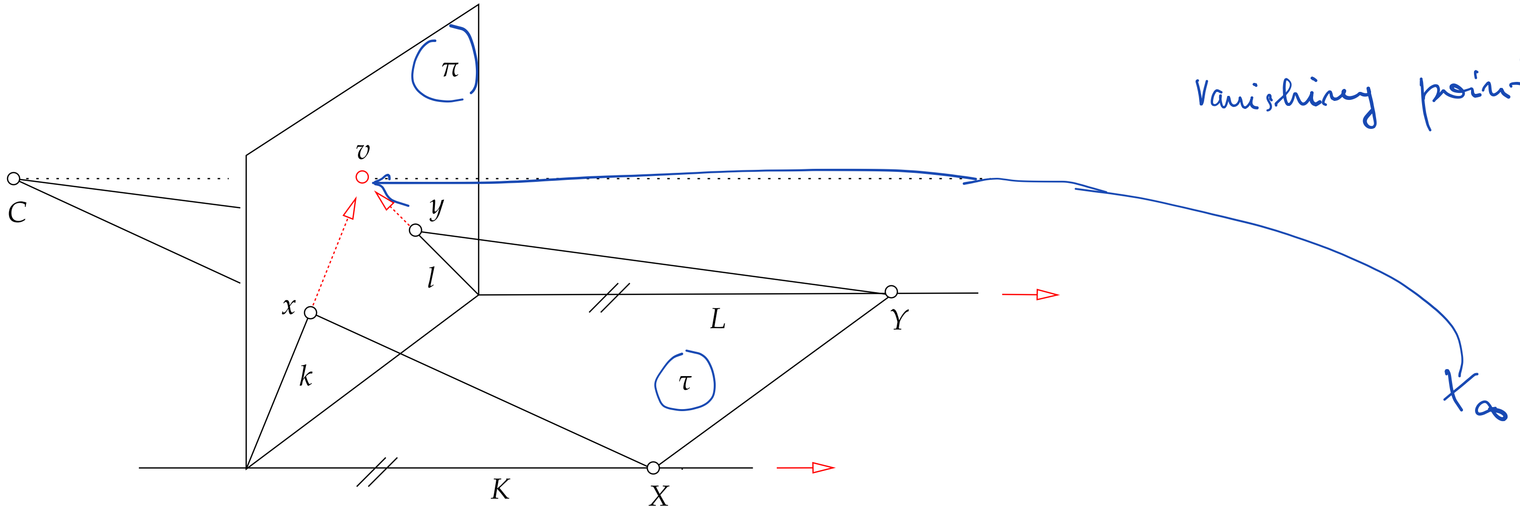


Figure 8.10: Vanishing point v is the point towards projections x an y tend as X and Y move away from π but which they never reach.

8.5 Vanishing line and horizon

If we take all sets of parallel lines in τ, each set with a different direction, then all the points of convergence in the image will fill a complete line h.

The line h is called the *vanishing line* or the *horizon*[8] when τ is the ground plane.

Now, imagine that we project all points from τ to π using the affine geometrical projection model. Then, no point from τ will project to h. Similarly, when projecting in the opposite direction, i.e. π to τ, line h has no image, i.e. it does not project anywhere to τ.

[8] Horizont in Czech

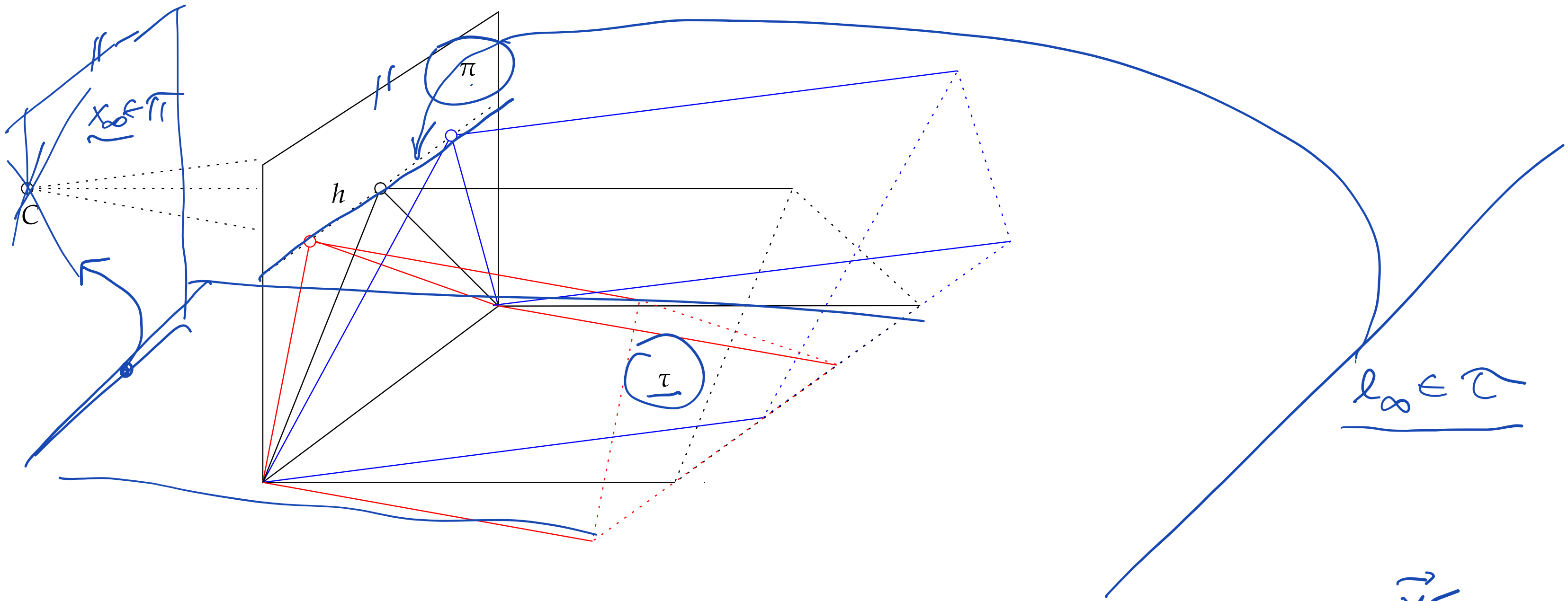


Figure 8.11: Vanishing line (horizon) h is the line of vanishing points.

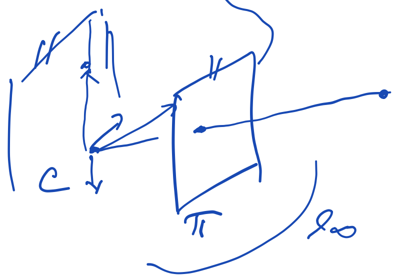
When using the affine geometrical projection model with the real projective plane to model the perspective projection (which is equivalent to the algebraic model in $\mathbb{R}^3$), all points of the projective plane τ (obtained as the projective completion of the affine plane τ) will have exactly one image in the projective plane π (obtained as the projective completion of the affine plane π) and vice versa. This total symmetry is useful and beautiful.

$\vec{X}_\delta$

$$\alpha \begin{bmatrix} u \\ v \\ w \end{bmatrix} = P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

0

1

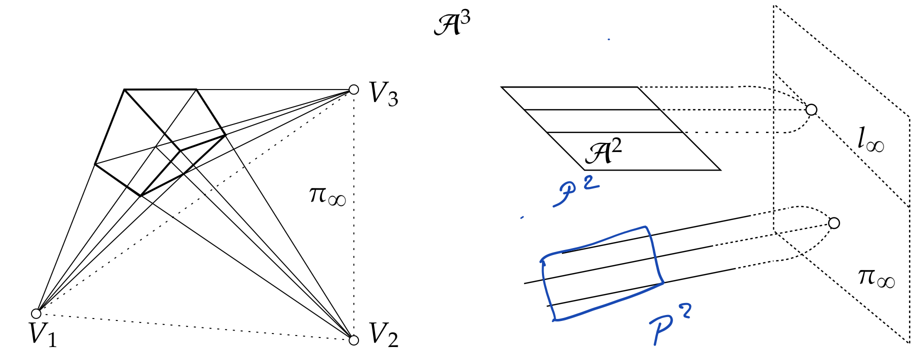


9 Projective space

9.1 Motivation – the union of ideal points of all affine planes

Figure 9.1(a) shows a perspective image of three sets of parallel lines generated by sides of a cube in the three-dimensional real affine space. The images of the three sets of parallel lines converge to vanishing points V_1, V_2 and V_3. The cube has six faces. Each face generates two pairs of parallel lines and hence two vanishing points. Each face generates an affine plane which can be extended into a projective plane by adding the line of ideal points of that plane. The projection of the three ideal lines are vanishing lines $l_{12} = V_1 \vee V_2$, $l_{23} = V_2 \vee V_3$ and $l_{31} = V_3 \vee V_1$. Imagine now all possible affine planes of the three-dimensional affine space and their corresponding ideal points. Let us take the union V of the sets of ideal points of all such planes. There is exactly one ideal point for every set of parallel lines in V, i.e. there is a one-to-one correspondence between elements of V (ideal points) and directions in the three-dimensional affine space. Notice also that every plane π generates one ideal line l_∞ of its ideal points and that all other planes parallel with π generate the same l_∞, Figure 9.1.

It suggests itself to extend the three-dimensional affine space by adding the set V to it, analogically to how we have extended the affine plane. In this new space, all parallel lines will intersect. We will call this space the *three-dimensional real projective space* and denote it $\mathbb{P}^3$. Let us develop an algebraic model of $\mathbb{P}^3$. It is practical to require this model to encompass the model of the real projective plane. The real projective plane is modeled



(a) (b)

Figure 9.1: (a) A perspective image of a cube generates three vanishing points V_1, V_2 and V_3 and hence also three vanishing lines l_{12}, l_{23} and l_{31}. (b) Every plane adds one line of ideal points to the three-dimensional affine space. Every ideal point corresponds to one direction, i.e. to a set of parallel lines. Each ideal line corresponds to a set of parallel planes.

algebraically by subspaces of $\mathbb{R}^3$. Let us observe that subspaces of $\mathbb{R}^4$ will be a convenient algebraic model of $\mathbb{P}^3$.

We start with the three-dimensional real affine space $\mathbb{A}^3$ and fix a coordinate system (O, δ) with $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$. An affine plane π is a set of points of $\mathbb{A}^3$ represented in (O, δ) by the set of vectors

$$\pi = \{[x, y, z]^\top \mid a\,x + b\,y + c\,z + d = 0,\ a, b, c, d \in \mathbb{R},\ a^2 + b^2 + c^2 \neq 0\} \quad (9.1)$$

We see that the point of π represented by vector $[x, y, z]^\top$ can also be represented by one-dimensional subspace $\{\lambda\,[x, y, z, 1]^\top \mid \lambda \in \mathbb{R}\}$ of $\mathbb{R}^4$ and

Projective plane → Projective space

$A^2 \longrightarrow P^2$

$\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ w \end{bmatrix}$

$A^3 \longrightarrow P^3$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$

1D subspaces = points

3D subspaces = planes

= 1D subspaces

2D subspaces

points ⟺ planes

lines

hence π can be seen as the set

$$\pi = \{\{\lambda\,[x,y,z,1]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, [a,b,c,d]\,[x,y,z,1]^\top = 0,\; a,b,c,d \in \mathbb{R},\; a^2+b^2+c^2 \neq 0\} \tag{9.2}$$

of one-dimensional subspaces of $\mathbb{R}^4$.

Notice that we did not require $\lambda \neq 0$ in the above definition. This is because we establish the correspondence between a vector $[x,y,z]$ and the corresponding complete one-dimensional subspace $\{\lambda\,[x,y,z,1]^\top, \lambda \in \mathbb{R}\}$ of $\mathbb{R}^4$ and since every linear space contains zero vector, we admit zero λ.

Every $[x,y,z]^\top \in \mathbb{R}^3$ represents in (O,δ) a point of $\mathbb{A}^3$ and hence the subset

$$\mathbb{A}^3 = \{\{\lambda\,[x,y,z,1]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, x,y,z \in \mathbb{R}\} \tag{9.3}$$

of one-dimensional subspaces of $\mathbb{R}^4$ represents $\mathbb{A}^3$.

We observe that we have not used all one-dimensional subspaces of $\mathbb{R}^4$ to represent $\mathbb{A}^3$. The subset

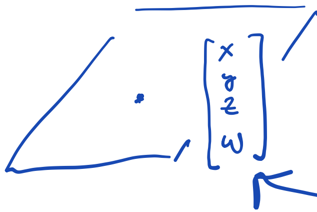
$$\pi_\infty = \{\{\lambda\,[x,y,z,0]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, x,y,z \in \mathbb{R},\; x^2+y^2+z^2 \neq 0\} \tag{9.4}$$

of one-dimensional subspaces of $\mathbb{R}^4$ is in one-to-one correspondence with all non-zero vectors of $\mathbb{R}^3$, i.e. in one-to-one correspondence with the set of directions in $\mathbb{A}^3$. This is the set of ideal points which we add to $\mathbb{A}^3$ to get the three-dimensional real projective space

$$\mathbb{P}^3 = \{\{\lambda\,[x,y,z,w]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, x,y,z,w \in \mathbb{R},\; x^2+y^2+z^2+w^2 \neq 0\} \tag{9.5}$$

which is the set of all one-dimensional subspaces of $\mathbb{R}^4$. Notice that $\mathbb{P}^3 = \mathbb{A}^3 \cup \pi_\infty$.

§1 Points Every non-zero vector of $\mathbb{R}^4$ generates a one-dimensional subspace and thus represents a point of $\mathbb{P}^3$. The zero vector $[0,0,0,0]^\top$ does not represent any point.



$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} \in \mathbb{R}^4$

$[a\ b\ c\ d]$

$ax + by + cz + dw = 0$

$[a\ b\ c\ d] \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$

$\mathbb{P}^3 \equiv \mathbb{R}^4$

lines $\equiv$ 1D subspaces of 6D space

$\equiv$ 2D subspaces of 4D space

§ 2 Planes Affine planes $\pi_{\mathbb{A}^3}$, Equation 9.2, are in one-to-one correspondence to the subset

$$\pi_{\mathbb{A}^3} = \{\{\lambda\,[a,b,c,d]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, a,b,c,d \in \mathbb{R},\ a^2+b^2+c^2 \neq 0\} \tag{9.6}$$

$$\alpha \begin{bmatrix} u \\ v \\ w \end{bmatrix} = P \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$\in \mathbb{R}^3 \setminus \{\vec{0}\}$ plane $\in \mathbb{R}^4 \setminus \{\vec{0}\}$ space

of the set of one-dimensional subspaces of $\mathbb{R}^4$. There is only one one-dimensional subspace of $\mathbb{R}^4$, $\{\lambda\,[0,0,0,1]^\top \,|\, \lambda \in \mathbb{R}\}$ missing in $\pi_{\mathbb{A}^3}$. It is exactly the one-dimensional subspace corresponding to the set π_∞ of ideal points of $\mathcal{P}^3$

$$\pi_\infty = \{\{\lambda\,[x,y,z,w]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, x,y,z,w \in \mathbb{R},\ x^2+y^2+z^2 \neq 0,\ [0,0,0,1]\,[x,y,z,w]^\top = 0\} \tag{9.7}$$

$$\pi_\infty = \left\{ \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} \;\middle|\; x,y,z \in \mathbb{R}^3 \right\}$$

not all $x,y,z = 0$

We can take another view upon planes and observe that affine planes are in one-to-one correspondence with the three-dimensional subspaces of $\mathbb{R}^4$. The set π_∞ also corresponds to a three-dimensional subspace of $\mathbb{R}^4$. Hence π_∞ can be considered another plane, *the ideal plane* of $\mathbb{P}^3$.

The set of planes of $\mathbb{P}^3$ can be hence represented by the set of one-dimensional subspaces of $\mathbb{R}^4$

$$\pi_{\mathbb{P}^3} = \{\{\lambda\,[a,b,c,d]^\top \,|\, \lambda \in \mathbb{R}\} \,|\, a,b,c,d \in \mathbb{R},\ a^2+b^2+c^2+d^2 \neq 0\} \tag{9.8}$$

but can also be viewed as the set of three-dimensional subspaces of $\mathbb{R}^4$.

We see that there is a duality between points and planes of $\mathbb{P}^3$. They both are represented by one-dimensional subspaces of $\mathbb{R}^4$ and we see that point X represented by vector $\vec{X} = [x,y,x,w]^\top$ is incident to plane π represented by vector $\vec{\pi} = [a,b,c,d]^\top$, i.e. $X \circ \pi$, when

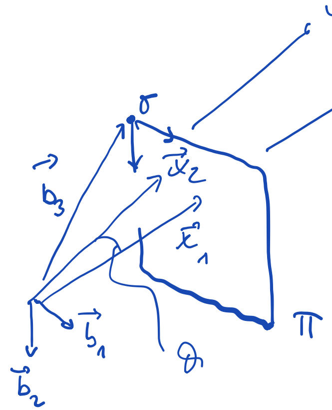
$$\vec{\pi}^\top \vec{X} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = a\,x + b\,y + c\,z + d\,w = 0 \tag{9.9}$$

§ 3 Lines Lines in $\mathbb{P}^3$ are represented by two-dimensional subspaces of $\mathbb{R}^4$. Unlike in $\mathbb{P}^2$, lines are not dual to points.

Get K

$\vec{x}_\beta = K \vec{x}_\gamma$

10 Camera auto-calibration



$\vec{x}_\delta \leftrightarrow \vec{x}_\beta$

$P \downarrow K$

$\omega \downarrow K$

Camera auto-calibration is a process when the parameters of image formation are determined from properties of the observed scene or knowledge of camera motions. We will study camera auto-calibration methods and tasks related to metrology in images. We have seen in Chapter 6 that to measure the angle between projection rays we needed only matrix K. Actually, it is enough to know matrix[1]

$$\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$$

symmetric PSD

$\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ $\quad \vec{x}_\beta = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$

not orthogonal basis
normal
in general

to measure the angle between the rays corresponding to image points $\vec{x}_{1\beta}$, $\vec{x}_{2\beta}$ as

$$\boxed{\cos\angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{2\beta}}{\|\mathtt{K}^{-1}\vec{x}_{1\beta}\|\|\mathtt{K}^{-1}\vec{x}_{2\beta}\|} = \frac{\vec{x}_{1\beta}^\top\,\omega\,\vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^\top\,\omega\,\vec{x}_{1\beta}}\sqrt{\vec{x}_{2\beta}^\top\,\omega\,\vec{x}_{2\beta}}}} \tag{10.1}$$

Knowing ω is however (almost) equivalent to knowing K since K can be recovered from ω up to two signs as follows.

$P = K[R \mid -R\vec{C}_\delta]$

internal calibration

K

$\beta \leftarrow \gamma$

§1 Recovering K from ω Let us give a procedure for recovering K from ω. Assuming

$$\boxed{\mathtt{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}} \tag{10.2}$$

[1] In [13], ω is called the image of the absolute conic.

we get

$$\mathtt{K}^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k_{11}} & \frac{-k_{12}}{k_{11}k_{22}} & \frac{k_{12}\,k_{23}-k_{13}\,k_{22}}{k_{11}\,k_{22}} \\ 0 & \frac{1}{k_{22}} & \frac{-k_{23}}{k_{22}} \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix}}^{M} \tag{10.3}$$

$\omega \longrightarrow K^{-1} \longrightarrow K$

$\omega = K^{-T} K^{-1}$

known

unknown

deg 2 poly. eqns

$(K^{-1})^{T} (K^{-1})$

for some real $m_{11}, m_{12}, m_{13}, m_{22}$ and m_{23}. Equivalently, we get

$$\mathtt{K} = \begin{bmatrix} \frac{1}{m_{11}} & \frac{-m_{12}}{m_{11}m_{22}} & \frac{m_{12}\,m_{23}-m_{13}\,m_{22}}{m_{11}\,m_{22}\,m_{23}} \\ 0 & \frac{1}{m_{22}} & \frac{-m_{23}}{m_{22}} \\ 0 & 0 & 1 \end{bmatrix} \tag{10.4}$$

Introducing the following notation

$$\underline{\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} \tag{10.5}$$

yields

$$\begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}\,m_{12} & m_{11}\,m_{13} \\ m_{11}\,m_{12} & m_{12}^2 + m_{22}^2 & m_{12}\,m_{13} + m_{22}\,m_{23} \\ m_{11}\,m_{13} & m_{12}\,m_{13} + m_{22}\,m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix} = M^T M \tag{10.6}$$

$\omega = M^T M$

which can be solved for $\mathtt{K}^{-1}$ up to the sign of the rows of $\mathtt{K}^{-1}$ as follows. Equation 10.6 provides equations

$$
\begin{aligned}
\omega_{11} = m_{11}^2 \quad &\Rightarrow \quad m_{11} = s_1\,\sqrt{\omega_{11}} \\
\omega_{12} = m_{11}\,m_{12} \quad &\Rightarrow \quad m_{12} = \omega_{12}/(s_1\,\sqrt{\omega_{11}}) = s_1\,\omega_{12}/\sqrt{\omega_{11}} \\
\omega_{13} = m_{11}\,m_{13} \quad &\Rightarrow \quad m_{13} = \omega_{13}/(s_1\,\sqrt{\omega_{11}}) = s_1\,\omega_{13}/\sqrt{\omega_{11}} \\
\omega_{22} = m_{12}^2 + m_{22}^2 \quad &\Rightarrow \quad m_{22} = s_2\,\sqrt{\omega_{22} - m_{12}^2} = s_2\,\sqrt{\omega_{22} - \omega_{12}^2/\omega_{11}} \\
\omega_{23} = m_{12}\,m_{13} + m_{22}\,m_{23} \quad &\Rightarrow \quad m_{23} = s_2\,(\omega_{23} - \omega_{12}\,\omega_{13}/\omega_{11})/\sqrt{\omega_{22} - \omega_{12}^2/\omega_{11}} \\
&\qquad\qquad = s_2\,(\omega_{11}\,\omega_{23} - \omega_{12}\,\omega_{13})/\sqrt{\omega_{11}^2\omega_{22} - \omega_{11}\,\omega_{12}^2}
\end{aligned}
$$

$\omega \to K$

Choleski decomposition

which can be solved for m_{ij} with $s_1 = \pm 1$ and $s_2 = \pm 1$. Hence

$$
\mathtt{K} = \begin{bmatrix} s_1\,\sqrt{\omega_{11}} & s_1\,\omega_{12}/\sqrt{\omega_{11}} & s_1\,\omega_{13}/\sqrt{\omega_{11}} \\ 0 & s_2\,\sqrt{\omega_{22} - \omega_{12}^2/\omega_{11}} & s_2\,(\omega_{23} - \omega_{12}\,\omega_{13}/\omega_{11})/\sqrt{\omega_{22} - \omega_{12}^2/\omega_{11}} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \tag{10.7}
$$

$s_1, s_2 > 0 = 1$

$K = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & 1 \end{bmatrix}$

Signs s_1, s_2 are determined by the choice of the image coordinate system. The standard choice is $s_1 = s_2 = 1$, which corresponds to $k_{11} > 0$ and $k_{22} > 0$.

Notice that $\sqrt{\omega_{11}}$ is never zero for a real camera since $m_{11} = \frac{1}{k_{11}} \neq 0$. There also holds true

$$
\sqrt{\omega_{22} - \omega_{12}^2/\omega_{11}} = \sqrt{m_{11}^2 - m_{12}^2} = \sqrt{\frac{1}{k_{11}^2} - \frac{k_{12}^2}{k_{11}^2\,k_{22}^2}} = \frac{1}{k_{11}\,k_{22}}\sqrt{k_{22}^2 - k_{12}^2} \neq 0 \tag{10.8}
$$

since $|k_{12}|$ is much smaller than $|k_{22}|$ for all real cameras.

10.1 Constraints on ω

Matrix ω is a 3×3 symmetric matrix and by this it has only six independent elements $\omega_{11}, \omega_{12}, \omega_{13}, \omega_{22}, \omega_{23}$ and ω_{33}. Let us next investigate additional constratints on ω, which follow from different choices of $\mathtt{K}$.

5 vs 6

↓

1 constraint on ω

§ 1 Constraints on ω for a general $\mathtt{K}$ Even a general $\mathtt{K}$ yields a constraint on ω. Equation 10.6 relates the six parameters of ω to only five parameters $m_{11}, m_{12}, m_{13}, m_{22}$ and m_{23} and hence the six parameters of ω can't be independent. Indeed, let us see that the following identity holds true

$\omega_{11} \neq 0$

$$
\begin{aligned}
&(\omega_{23}^2 - \frac{\omega_{13}^2\,\omega_{12}^2}{\omega_{11}^2} - (\omega_{22} - \frac{\omega_{12}^2}{\omega_{11}})\,(\omega_{33} - \frac{\omega_{13}^2}{\omega_{11}} - 1))^2 - 4\,\frac{\omega_{13}^2\,\omega_{12}^2}{\omega_{11}^2}\,(\omega_{22} - \frac{\omega_{12}^2}{\omega_{11}})\,(\omega_{33} - \frac{\omega_{13}^2}{\omega_{11}} - 1) \\
= &\Bigg((m_{12}m_{13} + m_{22}m_{23})^2 - \frac{(m_{11}m_{13})^2(m_{11}m_{12})^2}{m_{11}^4} \\
&-(m_{12}^2 + m_{22}^2 - \frac{(m_{11}m_{12})^2}{m_{11}^2})(m_{13}^2 + m_{23}^2 + 1 - \frac{(m_{11}m_{13})^2}{m_{11}^4} - 1) \Bigg)^2 \\
- &\;4\,\frac{(m_{11}m_{13})^2(m_{11}m_{12})^2}{m_{11}^4}(m_{12}^2 + m_{22}^2 - \frac{(m_{11}m_{12})^2}{m_{11}^2})(m_{13}^2 + m_{23}^2 + 1 - \frac{(m_{11}m_{13})^2}{m_{11}^4} - 1) \\
= &\;((m_{12}m_{13} + m_{22}m_{23})^2 - (m_{12}m_{13})^2 - (m_{22}m_{23})^2)^2 - 4\,(m_{12}m_{13})^2(m_{22}m_{23})^2 \\
= &\;(2\,(m_{12}m_{13})(m_{22}m_{23}))^2 - 4\,(m_{12}m_{13})^2(m_{22}m_{23})^2 \\
= &\;0
\end{aligned}
\tag{10.9}
$$

Since $\omega_{11} \neq 0$, we get the following equivalent identity

$$
\begin{aligned}
&(\omega_{11}^2\omega_{23}^2 - \omega_{13}^2\,\omega_{12}^2 - (\omega_{11}\omega_{22} - \omega_{12}^2)\,(\omega_{11}\omega_{33} - \omega_{13}^2 - \omega_{11}))^2 \\
&\qquad - 4\,\omega_{13}^2\,\omega_{12}^2\,(\omega_{11}\omega_{22} - \omega_{12}^2)\,(\omega_{11}\omega_{33} - \omega_{13}^2 - \omega_{11}) = 0
\end{aligned}
\tag{10.10}
$$

deg 8 constraint in ω

which is a polynomial equation of degree eight in elements of ω.

We shall see next that it makes sense to introduce a new matrix

$$\mathbf{\Omega} = \begin{bmatrix} 1 & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\omega_{12}}{\omega_{11}} & \frac{\omega_{13}}{\omega_{11}} \\ \frac{\omega_{12}}{\omega_{11}} & \frac{\omega_{22}}{\omega_{11}} & \frac{\omega_{23}}{\omega_{11}} \\ \frac{\omega_{13}}{\omega_{11}} & \frac{\omega_{23}}{\omega_{11}} & \frac{\omega_{33}}{\omega_{11}} \end{bmatrix} \tag{10.11}$$

$\omega_{11} \neq 0$ for practical K

ω / ω_{11}

which contains only five unknowns, and use Equation 10.10 to get the positive ω_{11} from $\mathbf{\Omega}$ by solving the following quadratic equation

$\omega \rightarrow \Omega$ $\quad \Omega_{11} = 1$

$$a_2\, \omega_{11}^2 + a_1\, \omega_{11} + a_0 = 0 \tag{10.12}$$

$\Omega = \frac{1}{\omega_{11}} \omega$

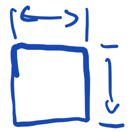
with

$$\begin{aligned} a_2 &= -4\, {o_{23}}^2 {o_{13}}^2 {o_{12}}^2 + {o_{23}}^4 - 2\, {o_{23}}^2 o_{22}\, o_{33} + 2\, {o_{13}}^2 {o_{12}}^2 o_{22}\, o_{33} \\ &\quad -2\, {o_{22}}^2 o_{33} {o_{13}}^2 + {o_{12}}^4 {o_{33}}^2 + 2\, {o_{23}}^2 o_{22}\, {o_{13}}^2 + 2\, {o_{23}}^2 {o_{12}}^2 o_{33} \\ &\quad +{o_{22}}^2\, {o_{13}}^4 + {o_{22}}^2 {o_{33}}^2 - 2\, o_{22}\, {o_{33}}^2 {o_{12}}^2 \end{aligned} \tag{10.13}$$

$$\begin{aligned} a_1 &= 2\, {o_{13}}^2 {o_{12}}^2\, o_{22} + 2\, {o_{23}}^2 o_{22} - 2\, {o_{22}}^2\, o_{33} - 2\, {o_{12}}^4\, o_{33} \\ &\quad +4\, o_{22}\, o_{33} {o_{12}}^2 - 2\, {o_{23}}^2 {o_{12}}^2 + 2\, {o_{22}}^2 {o_{13}}^2 \end{aligned} \tag{10.14}$$

$$a_0 = -2\, o_{22}\, {o_{12}}^2 + {o_{22}}^2 + {o_{12}}^4 \tag{10.15}$$

Square pixels



§ 2 Constraints on ω for K from square pixels Cameras have often square pixels, i.e. $\|\vec{b}_1\| = |\vec{b}_2\| = 1$ and $\angle(\vec{b}_1, \vec{b}_2) = \pi/2$, which implies, Equations 6.13, 6.15, 6.16, a simplified

$$\mathtt{K} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{10.16}$$

$\omega = K^{-T} K^{-1}$

This gives also simpler

$$\omega = \frac{1}{k_{11}^2} \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix} \tag{10.17}$$

We see that we get the following three identities

$$\omega_{12} = 0 \tag{10.18}$$

$$\omega_{22} - \omega_{11} = 0 \tag{10.19}$$

$$\omega_{13}^2 + \omega_{23}^2 - \omega_{11}\omega_{33} + \omega_{11} = 0 \tag{10.20}$$

We also get simpler

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & 0 & o_{13} \\ 0 & 1 & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = k_{11}^2\,\omega = \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix} \tag{10.21}$$

and use Equation 10.21 to get

$$k_{11}^2 = o_{33} - o_{13}^2 - o_{23}^2 \tag{10.22}$$

$$k_{13} = -o_{13} \tag{10.23}$$

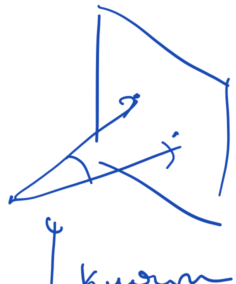
$$k_{23} = -o_{23} \tag{10.24}$$

$\left.\begin{matrix}\Omega \\ \omega\end{matrix}\right\} \longrightarrow K$

10.2 Camera calibration from angles between projection rays

We will now show how to calibrate a camera by finding the matrix $\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$.

In general, matrix ω is constrained by knowing angles contained between pairs of projection rays. Consider two projection rays with direction vectors $\vec{x}_1$, $\vec{x}_2$. Then the angle between them is related to ω and $\boldsymbol{\Omega}$ by



$$\cos\angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\beta}^\top\,\omega\,\vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^\top\,\omega\,\vec{x}_{1\beta}}\,\sqrt{\vec{x}_{2\beta}^\top\,\omega\,\vec{x}_{2\beta}}} = \frac{\vec{x}_{1\beta}^\top\,\boldsymbol{\Omega}\,\vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^\top\,\boldsymbol{\Omega}\,\vec{x}_{1\beta}}\,\sqrt{\vec{x}_{2\beta}^\top\,\boldsymbol{\Omega}\,\vec{x}_{2\beta}}} \tag{10.25}$$

constant known

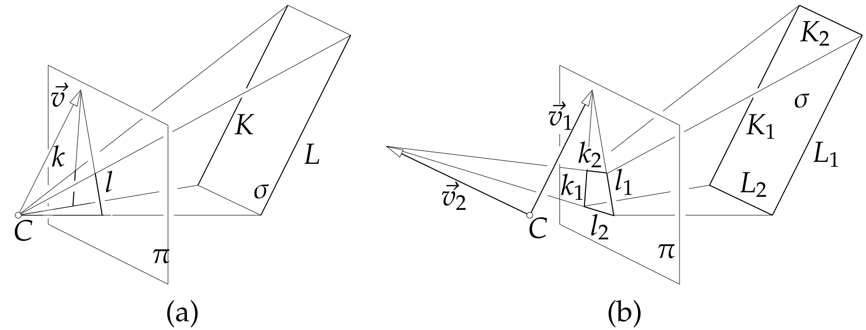
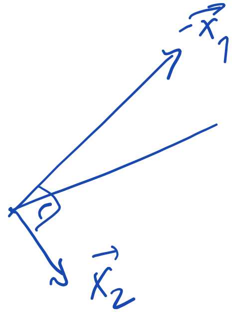


Figure 10.1: (a) Parallel lines K, L are projected to lines k, l with vanishing point represented by $\vec{v}$. Vector $\vec{v}$ is parallel to k, l. (b) Vectors $\vec{v}_1, \vec{v}_2$ contain the same angle as pairs of lines K_1, K_2 or L_1, L_2.



Squaring the above and clearing the denominators gives

$$\boxed{(\cos\angle(\vec{x}_1,\vec{x}_2))^2(\vec{x}_{1\beta}^\top\,\Omega\,\vec{x}_{1\beta})\,(\vec{x}_{2\beta}^\top\,\Omega\,\vec{x}_{2\beta}) = (\vec{x}_{1\beta}^\top\,\Omega\,\vec{x}_{2\beta})^2} \qquad (10.26)$$

Polynomial deg 2 in elements of ω, Ω

which is a second order equation in elements of Ω. To find Ω, which has five independent parameters for a general K, we need to be able to establish five pairs of rays with known angles and solve a system of five quadratic equations 10.26 above.

in general not linear

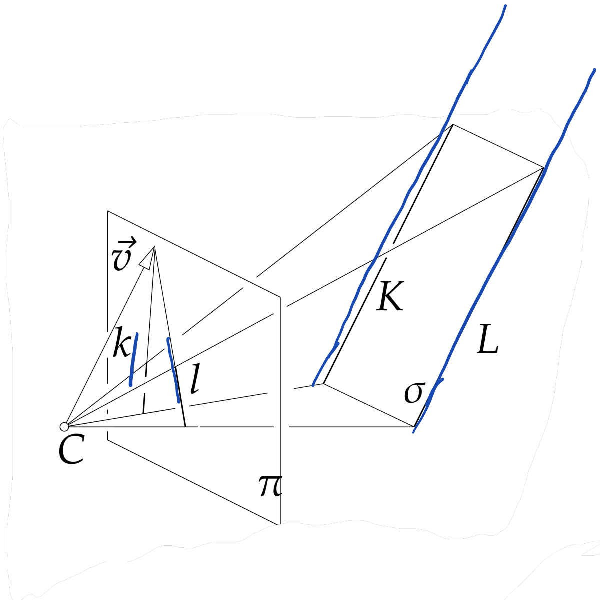
$$\underbrace{\cos\angle(x_1,x_2)}_{= 0} = \frac{\vec{x}_{1\beta}^\top\,\Omega\,\vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^\top\,\Omega\,\vec{x}_{1\beta}}\sqrt{\vec{x}_{2\beta}^\top\,\Omega\,\vec{x}_{2\beta}}}$$

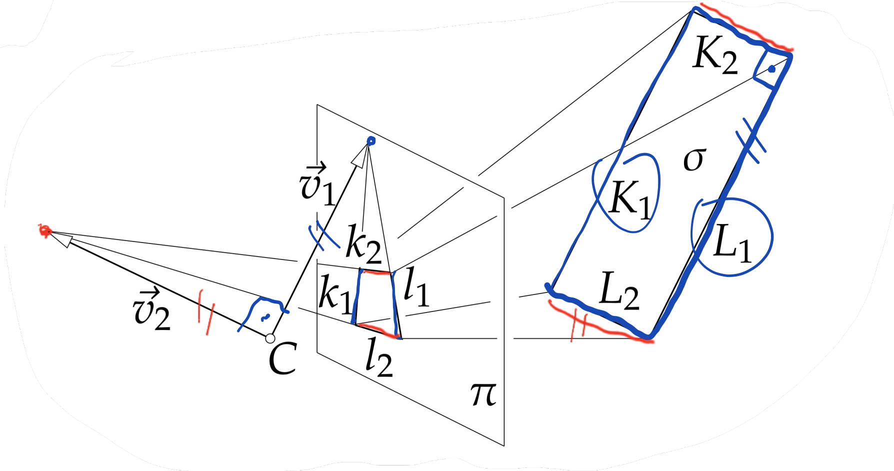
§ 1 **Camera with square pixels** A simpler situation arises when the camera has square pixels. Then, we can use constraints from § 2 to recover ω and K from three pairs of rays containing known angles. That amounts to solving three second order equations 10.26 in o_{13}, o_{23}, o_{33}.

$$0 = \vec{x}_{1\beta}^\top\,\Omega\,\vec{x}_{2\beta}$$

linear constraint

However, this is actually exactly the same problem as we have already solved in Section 6.3. Figure 10.2 shows an image plane π with a coordinate system (o, δ') with $\delta' = (\vec{b}_1, \vec{b}_2, \vec{b}_3')$ derived from the image coordinate


$\vec{v}$
k
l
K
L
σ
C
π


K_2
σ
$\vec{v}_1$
K_1
L_1
k_2
k_1
l_1
L_2
$\vec{v}_2$
C
l_2
π

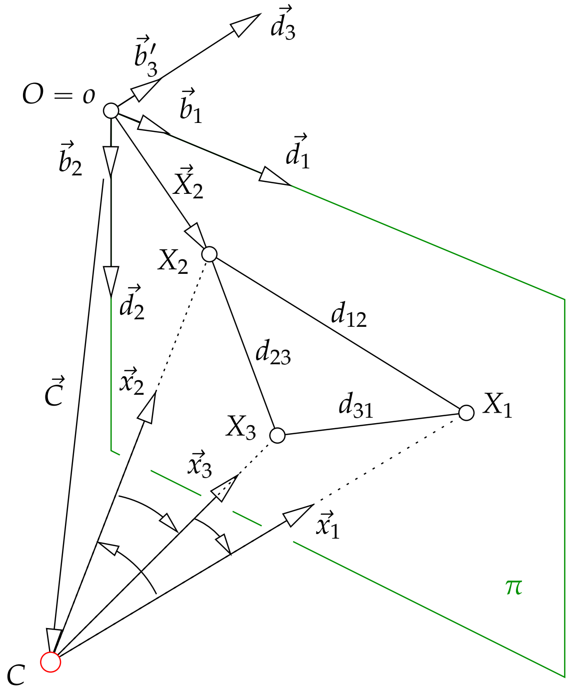


Figure 10.2: Images of three points with known angles between their rays can be used to calibrate cameras with square pixels. The position of image center $\vec{C}_{\delta'}$ can be computed in the ortogonal coordinate system (o, δ') using the absolute pose problem from Chapter 6.3. Matrix $\mathtt{K}$ is composed from coordinates of $\vec{C}_{\delta'}$.

system (o, α). Having square pixels, vectors $\vec{b}_1$, $\vec{b}_2$ can be complemented with $\vec{b}'_3$ to form an orthogonal coordinates system $(O = o, \delta')$. Next, we choose the global orthonormal coordinate system, $(O = o, \delta)$, $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$, such that

$$\vec{d}_1 = \frac{\vec{b}_1}{||\vec{b}_1||}, \quad \vec{d}_2 = \frac{\vec{b}_2}{||\vec{b}_1||}, \quad \text{and} \quad \vec{d}_3 = \frac{\vec{b}'_3}{||\vec{b}_1||} \tag{10.27}$$

and hence

$$\vec{x}_\delta = \begin{bmatrix} ||\vec{b}_1|| & 0 & 0 \\ 0 & ||\vec{b}_1|| & 0 \\ 0 & 0 & ||\vec{b}_1|| \end{bmatrix} \vec{x}_{\delta'} \tag{10.28}$$

We know angles $\angle(\vec{x}_1, \vec{x}_2)$, $\angle(\vec{x}_2, \vec{x}_3)$ and $\angle(\vec{x}_3, \vec{x}_1)$. We also know image points $\vec{u}_{1\alpha} = \vec{X}_{1\delta'}$, $\vec{u}_{2\alpha} = \vec{X}_{2\delta'}$, $\vec{u}_{3\alpha} = \vec{X}_{3\delta'}$ and thus we can compute distances $d_{12} = ||\vec{X}_{2\delta'} - \vec{X}_{1\delta'}||$, $d_{23} = ||\vec{X}_{3\delta'} - \vec{X}_{2\delta'}||$ and $d_{31} = ||\vec{X}_{3\delta'} - \vec{X}_{1\delta'}||$. Having that, we can find the pose $\vec{C}_{\delta'} = [c_1, c_2, c_3]^\top$ of the camera center C in (O, δ') by solving the absolute pose problem from Chapter 6.3. We will select a solution with $c_3 < 0$ and, if necessary, use a fourth point in π to choose the right solution among them. To find $\mathtt{K}$, we can form the following equation

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{f} \left[\mathtt{K}\,\mathtt{R} \,|\, -\mathtt{K}\,\mathtt{R}\,\vec{C}_\delta \right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{10.29}$$

since point o is represented by $[0, 0, 1]^\top$ in β and by $[0, 0, 0]^\top$ in δ. Coordinate system (O, δ) is chosen such that $\mathtt{R} = \mathtt{I}$ and $\vec{C}_\delta = ||\vec{b}_1||\,\vec{C}_{\delta'}$ and thus we get

$$\mathtt{K}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{||\vec{b}_1||}{f} \vec{C}_{\delta'} \tag{10.30}$$

Now, let us consider matrix $\mathtt{K}$ as in Equation 10.16 and use the intepretation of elements of $\mathtt{K}$ from Chapter 6, Equations 6.16, 6.17. We can write

$$\mathtt{K} = \begin{bmatrix} \frac{f}{\|\vec{b}_1\|} & 0 & k_{13} \\ 0 & \frac{f}{\|\vec{b}_1\|} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{an thus} \quad \mathtt{K}^{-1} = \begin{bmatrix} \frac{\|\vec{b}_1\|}{f} & 0 & -\frac{\|\vec{b}_1\|}{f}k_{13} \\ 0 & \frac{\|\vec{b}_1\|}{f} & -\frac{\|\vec{b}_1\|}{f}k_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{10.31}$$

and use it in Equation 10.30 to get

$$\begin{bmatrix} k_{13} \\ k_{23} \\ -\frac{f}{\|\vec{b}_1\|} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{10.32}$$

and thus

$$\mathtt{K} = \begin{bmatrix} -c_3 & 0 & c_1 \\ 0 & -c_3 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{10.33}$$

10.3 Camera calibration from vanishing points

Let us first make an interesting observation about parallel lines in space an its corresponding vanishing point in an image. Let us consider a pair of parallel lines K, L in space as shown in Figure 10.1(a). There is an affine plane σ containing the lines. The lines K, L are projected to image plane π into lines k, l, respectively.

Now, first extend affine plane σ to a projective plane Σ using the camera center C. Then, define a coordinate system (C, δ) with orthonormal basis $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$ such that vectors $\vec{d}_1, \vec{d}_2$ span affine plane σ.

Let $\vec{K}_{\bar{\delta}}, \vec{L}_{\bar{\delta}}$ be homogeneous coordinates of lines K, L w.r.t. $\bar{\delta}$. Then

$$\vec{w}_\delta = \vec{K}_{\bar{\delta}} \times \vec{L}_{\bar{\delta}} \tag{10.34}$$

are homogeneous coordinates of the intersection of lines K, L in Σ.

Next, extend the affine plane π to a projective plane Π using the camera center C with the (camera) coordinate system (C, β).

Let $\vec{k}_{\bar\beta}$, $\vec{l}_{\bar\beta}$ be homogeneous coordinates of lines k, l w.r.t. $\bar\beta$. Then

$$\boxed{\vec{v}_{\beta} = \vec{k}_{\bar\beta} \times \vec{l}_{\bar\beta}} \tag{10.35}$$

are homogeneous coordinates of the intersection of lines k, l in Π.

Now, consider Equation 7.14 for planes Σ and Π. Since δ is orthonormal, we have $\mathtt{K}' = \mathtt{I}$ and thus that there is a homoghraphy

$$\mathtt{H} = \mathtt{K}\,\mathtt{R} \tag{10.36}$$

which maps plane Σ to plane Π. Matrices $\mathtt{K}$ and $\mathtt{R}$ of the camera are here w.r.t. the world coordinate system (C, δ).

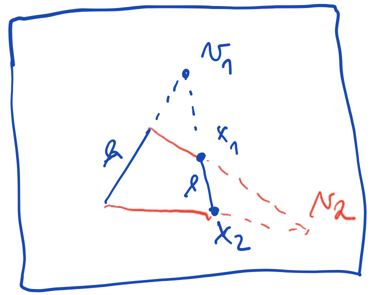
We see that there is a real λ such that there holds

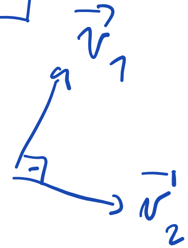
$$\lambda\, \vec{v}_{\beta} = \mathtt{K}\,\mathtt{R}\, \vec{w}_{\delta} \tag{10.37}$$

true.

§ 1 Pairs of "orthogonal" vanishing points and camera with square pixels Let us have two pairs of parallel lines in space, Figure 10.1(b), such that they are also orthogonal, i.e. let K_1 be parallel with L_1 and K_2 be parallel with L_2 and at the same time let K_1 be orthogonal to K_2 and L_1 be orthogonal to L_2. This, for instance, happens when lines K_1, L_1, K_2, L_2 form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

Let lines k_1, l_1, k_2, l_2 be the projections of K_1, L_1, K_2, L_2, respectively, represented by the corresponding vectors $\vec{k}_{1\bar\beta}, \vec{l}_{1\bar\beta}, \vec{k}_{2\bar\beta}, \vec{l}_{2\bar\beta}$ in the camera coordinates system with (in general non-orthogonal) basis β. Lines k_1 and l_1,





$\vec{v}_{1\beta}$, $\vec{v}_{2\beta}$

$\vec{l} = \vec{x}_1 \times \vec{x}_2$

$v_1 = \vec{k} \times \vec{l}$

$\vec{v}_{1\beta}\ \Omega\ \vec{v}_{2\beta} = 0$

linear

resp. k_2 and l_2, generate vanishing points

$$\boxed{\begin{aligned} \vec{v}_{1\beta} &= \vec{k}_{1\bar{\beta}} \times \vec{l}_{1\bar{\beta}} \\ \vec{v}_{2\beta} &= \vec{k}_{2\bar{\beta}} \times \vec{l}_{2\bar{\beta}} \end{aligned}}$$

The perpendicularity of $\vec{w}_1$ to $\vec{w}_2$ is, in the camera orthogonal basis δ, modeled by

$$\vec{w}_{1\delta}^\top \vec{w}_{2\delta} = 0 \tag{10.38}$$

We therefore get from Equation 10.37

$$\vec{v}_{1\beta}^\top \mathtt{K}^{-\top}\mathtt{R}^{-\top}\mathtt{R}^{-1}\mathtt{K}^{-1}\vec{v}_{2\beta} = 0 \tag{10.39}$$

$$\vec{v}_{1\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{v}_{2\beta} = 0 \tag{10.40}$$

$$\vec{v}_{1\beta}^\top \,\omega\, \vec{v}_{2\beta} = 0 \tag{10.41}$$

which is a linear homogeneous equation in ω. Assuming further square pixels, we get, § 2,

$$\begin{aligned} \vec{v}_{1\beta}^\top \,\omega\, \vec{v}_{2\beta} &= 0 \\ \vec{v}_{1\beta}^\top \,\Omega\, \vec{v}_{2\beta} &= 0 \end{aligned}$$

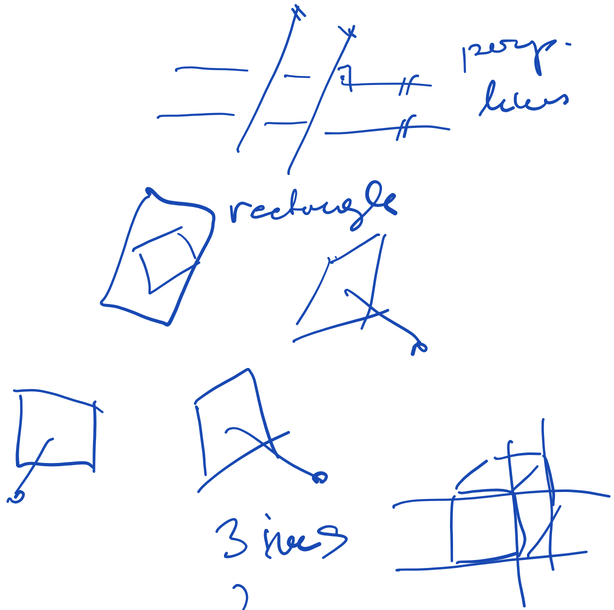
$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & o_{13} \\ 0 & 1 & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

LI

$$\begin{bmatrix} v_{23}\,v_{11} + v_{21}\,v_{13} & v_{23}\,v_{12} + v_{22}\,v_{13} & v_{23}\,v_{13} \end{bmatrix} \begin{bmatrix} o_{13} \\ o_{23} \\ o_{33} \end{bmatrix} = -(v_{21}\,v_{11} + v_{22}\,v_{12})$$

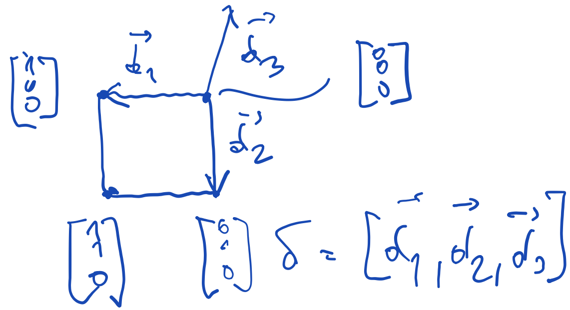
Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute o_{13}, o_{23}, o_{33} and then

$$\begin{aligned} k_{13} &= -o_{13} \\ k_{23} &= -o_{23} \\ k_{11} &= \sqrt{o_{33} - k_{13}^2 - k_{23}^2} \end{aligned}$$



$$\begin{bmatrix} = \end{bmatrix} \begin{bmatrix} o_{13} \\ o_{23} \\ o_{33} \end{bmatrix} = \begin{bmatrix} \vdots \end{bmatrix}$$

10.4 Camera calibration from images of squares



Let us exploit the relationship between the coordinates of points X, which all lie in a plane σ and are measured in a coordinate system $(O, \vec{d}_1, \vec{d}_2)$ in σ, Figure 7.2. The points X are projected by a perspective camera with the camera coordinate system is $(C, \beta), \beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and projection matrix $\mathtt{P}$ into image coordinates $\begin{bmatrix} u & v \end{bmatrix}^\top$, w.r.t. an image coordinate system $(o, \vec{b}_1, \vec{b}_2)$, Equation 7.16. See paragraph § 1 to recall that the columns of $\mathtt{P}$ can be writen as

$$\mathtt{P} = \begin{bmatrix} \mathtt{K}\,\mathtt{R} \,|\, -\mathtt{K}\,\mathtt{R}\,\vec{C}_\delta \end{bmatrix} = \begin{bmatrix} \vec{d}_{1\nu} & \vec{d}_{2\nu} & \vec{d}_{3\nu} & -\vec{C}_\nu \end{bmatrix} \tag{10.42}$$

and therefore we get the columns

$$\boxed{\begin{aligned} \mathbf{h}_1 = \mathbf{p}_1 &= \vec{d}_{1\nu} && (10.43)\\ \mathbf{h}_2 = \mathbf{p}_2 &= \vec{d}_{2\nu} && (10.44)\\ \mathbf{h}_3 = \mathbf{p}_4 &= -\vec{C}_\nu && (10.45) \end{aligned}}$$

of the homography $\mathtt{H}$ mapping σ to π as defined in Equation 7.17.

ν … orthonormal coordinate system

$$\mathtt{H}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{P}\begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$

Now imagine that we are observing a square with 4 corner points X_1, X_2, X_3 and X_4 in the plane σ and we construct the coordinate system in σ by assigning coordinates to the corners as

$$\begin{aligned} \vec{X}_{1\delta} &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} && (10.46)\\ \vec{d}_{1\delta} = \vec{X}_{2\delta} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} && (10.47)\\ \vec{d}_{2\delta} = \vec{X}_{3\delta} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} && (10.48)\\ \vec{X}_{4\delta} &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} && (10.49) \end{aligned}$$

We see that we get two constraints on $\vec{d}_{1\delta}, \vec{d}_{2\delta}$

$$\boxed{\begin{aligned} \vec{d}_{1\delta}^{\top}\,\vec{d}_{2\delta} &= 0 && (10.50)\\ \vec{d}_{1\delta}^{\top}\,\vec{d}_{1\delta} - \vec{d}_{2\delta}^{\top}\,\vec{d}_{2\delta} &= 0 && (10.51) \end{aligned}}$$

which lead to

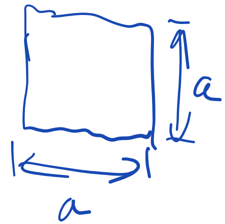
ω

$$
\begin{aligned}
\vec{d}_{1\nu}^{\top}\,\mathtt{K}^{-\top}\,\mathtt{K}^{-1}\,\vec{d}_{2\nu} &= 0 && (10.52)\\
\vec{d}_{1\beta}^{\top}\,\mathtt{K}^{-\top}\,\mathtt{K}^{-1}\,\vec{d}_{1\beta} - \vec{d}_{2\nu}^{\top}\,\mathtt{K}^{-\top}\,\mathtt{K}^{-1}\,\vec{d}_{2\nu} &= 0 && (10.53)
\end{aligned}
$$

by using $\vec{d}_{i\nu} = \mathtt{K}\,\mathtt{R}\,\vec{d}_{i\delta}$ for $i = 1, 2$, and $\mathtt{R}^{\top}\,\mathtt{R} = \mathtt{I}$.

These are two linear equations on ω and hence also, see § 1, on Ω

$$
\begin{aligned}
\boxed{\vec{d}_{1\nu}^{\top}\,\Omega\,\vec{d}_{2\nu} = 0} && (10.54)\\
\vec{d}_{1\nu}^{\top}\,\Omega\,\vec{d}_{1\nu} - \vec{d}_{2\nu}^{\top}\,\Omega\,\vec{d}_{2\nu} = 0 && (10.55)
\end{aligned}
$$



on ω in terms of estimated $\lambda\,\mathtt{H}$

$$
\begin{aligned}
\mathtt{h}_1^{\top}\,\Omega\,\mathtt{h}_2 &= 0 && (10.56)\\
\mathtt{h}_1^{\top}\,\Omega\,\mathtt{h}_1 - \mathtt{h}_2^{\top}\,\Omega\,\mathtt{h}_2 &= 0 && (10.57)
\end{aligned}
$$

} 2 linear eqns on Ω

$H = [h_1\ h_2\ h_3]$

One square provides two equations and therefore three squares in two planes in a general position suffice to calibrate full $\mathtt{K}$. Actually, such three squares provide one more equations than necessary since Ω has only five parameters. Hence, it is enough observe two squares and one rectangle to get five constraints. Similarly, one square and one rectangle in a plane then suffice to calibrate $\mathtt{K}$ when pixels are square.

Notice also that we have never used the special choice of coordinates of $\vec{X}_\delta$. Indeed, point X_4 could be anywhere provided that we know how to assign it coordinates in $(O, \vec{d}_1, \vec{d}_2)$.

To calibrate the camera, we first assign coordinates to the corners of the square as above, then find the homography $\mathtt{H}$ from the plane to the image

$$
\lambda_i\,\vec{x}_{i\beta} = \mathtt{H}\,\vec{X}_{i\delta} \qquad (10.58)
$$

for $\alpha_i = 1, \ldots, 4$ and finally use columns of $\mathtt{H}$ the find Ω.