Probability: Quick and (Hopefully) Gentle Intro

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Notes

Specializovaný předmět na pravděpodobnost a statistiku teprve přijde.

Outline

- Mathematics of uncertainty
- Random Experiment, Outcomes, Sample Space, Events, . . .
- Probability, Conditional Probability, Independence
- Random Variable, Expectation

Notes

Uncertainty is everywhere

- The probability of rain tomorrow is 70%.
- What are my chances to win in a lottery?
- I was tested positive for disease X, am I really sick?
- Given testimonies X, Y, and Z, is the suspect guilty?
- Unemployment changed by X, what will be the inflation?
- How will the stock prices evolve?
- We chose action X, how much will the robot move?
- What is the probability that the person on the photo is person X?
- How long will it take me to get to work if I take the tram?
- ...

We need a mathematical description ...

Notes

(Random) Experiment

Experiment :

- Vaguely: the act of observing certain feature of the world
- A procedure that
  - can be repeated many times under the same conditions and
  - has a well-defined set of possible outcomes.
- Deterministic experiment has only a single possible outcome.
- Random experiment has more than one possible outcomes.
- Before executing random experiment, we do not know the actual outcome. After execution this uncertainty vanishes.

Notes

Example 1: Three tosses of a coin (Head/Tails)

What is the probability of three heads?

Sample space (a set!) $\mathcal{S}$ of all elementary events (experiment outcomes) . How big is it?

A 3^2

B 2^3

C $2 \cdot 3$

D ∞

Events :

- A - $3\times$ head, $P(A) = ?$
- B - $3\times$ the same symbol, $P(B) = ?$
- C - at least one tail, $P(C) = ?$ ...

Notes

Pro výpočet velikosti prostoru všech elementárních jevů někdy poslouží vhodný model. Tady např. to může být $n-$bitové binární číslo.
Vyjděme z množiny elementárních jevů - $HHH, HHT, \ldots, TTT$.
Je pravděpodobnost některého elementárního jevu větší nebo menší než u ostatních? Je to tak vždy?
Jak definovat jevy A, B, C? Nešlo by jev C definovat snáze, pomocí množinových operací a jiných elementárních jevů?

Jaká je jejich pravděpodobnost? Jak by se dala spočítat $P(C)$ pomocí již známých pstí?

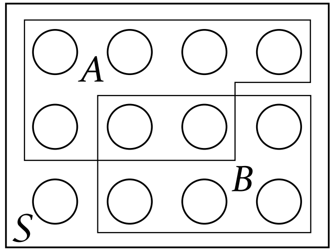
(Random) Events /(Náhodné) jevy/

Elementary events /elementární jevy/ are *all possible, mutually exclusive outcomes* of certain experiment.

The set of elementary events is called a sample space /množina elementárních jevů/, denoted as $\mathcal{S}$.

An event /jev/ is any subset of the sample space, $A \subseteq \mathcal{S}$.

- Event A occured if the experiment outcome belongs to A.
- An event is any statement about the experiment outcome for which we can decide if it occured or not.

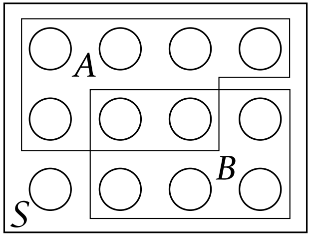


Notes

Naive probability (Bernoulli/Laplace)

$$P(A) = \frac{|A|}{|\mathcal{S}|} = \frac{\text{number of outcomes favorable to } A}{\text{total number of outcomes in } \mathcal{S}}$$

- Limited to *equally likely* outcomes/elementary events. (*Equally likely?*)
- It does not allow for infinite sample spaces, geometric probability, ...
- *Combinatorics!* Counting (variations, permutations, combinations, ...)



Notes

"Equally likely": we actually use an assumption on probability values in the definition of the probability.

Events and their combinations

Important events:

- Certain event : $\mathcal{S}$, $\mathbf{1}$
- Impossible event : $\emptyset$, $\mathbf{0}$

Event combinations:

- Conjunction (A and B): $A \cap B$
- Disjuction (A or B): $A \cup B$
- Complementary event /jev opačný/ to A: $A^c = \mathcal{S} \setminus A$
- $A \Rightarrow B$: $A \subseteq B$
- Disjoint events /jevy neslučitelné/ : $A_1, \dots, A_n : \bigcap_{i \leq n} A_i = \emptyset$
- Mutually exclusive events /Jevy po dvou neslučitelné = vzájemně se vylučující/ : $A_1, \dots, A_n : \forall i, j \in \{1, \dots, n\},\ i \neq j : A_i \cap A_j = \emptyset$

Notes

Výrokovou logiku lze k popisu jevů použít místo množin, je to ekvivalentní popis. Je dobré ale obojí nemixovat.

Partition of sample space /Úplný systém jevů/

Partition of sample space $\mathcal{S}$ /Úplný systém jevů/ is composed of events $B_1, \ldots, B_n$ if they are *mutually exclusive* and $\bigcup_{i=1}^{n} B_i = \mathcal{S}$.

- The sample space $\mathcal{S}$ is its own partition by definition.
- Events $\{C, C^c\}$ form a partition: $C \cap C^c = \emptyset$ and $C \cup C^c = \mathcal{S}$.

Why is the partition of $\mathcal{S}$ an important concept?

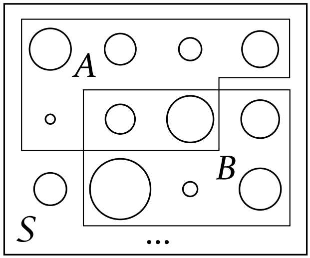
Notes

Proč je úplný systém jevů důležitý koncept?

Protože víme, že výsledkem experimentu je právě jeden z jevů v úplném systému. Proč?

Axiomatic probability (Kolmogorov)

- Sample space $\mathcal{S}$ may be infinite.
- Elementary events do not have to be equally likely.
- Axiomatic:
  1. state a set of constraints the probability function must obey
  2. find a function that fulfills them (next slides)

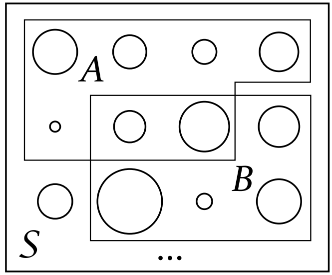


Notes

From discrete to continous, From Pebble world representation to Venn Diagrams.

Definition of probability

- Probability function /pravděpodobnostní funkce/ P is a function that assigns a real number between 0 and 1 to each event $A \subseteq \mathcal{S}$.
- P must satisfy the following axioms:



1. $P(\emptyset) = 0$, $P(\mathcal{S}) = 1$
2. For any mutually exclusive events $A_1, A_2, \ldots, A_n$:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

   (n may be infinite)

Notes

Interpretations of probability

Frequentist :

- Relative frequency of an event after many repetitions of random experiment.

Bayesian :

- Degree of belief that an event occurs.
- This allows us to assign probabilities to statements like "candidate A wins elections" or "suspect X is guilty", although we cannot repeat the same elections or the same crime over and over.

Notes

Example 2: Properties of P, rolling a die

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

Consider events:

- A - outcome is 6
- B - outcome is an even number

Using sets: $A \subset B$

Probability: $P(A) < P(B)$

Another event:

- C - outcome is 2 or 4

Using sets: $C = B \setminus A$

Probability: $P(C) = P(B) - P(A)$

Notes

Pro názornost nám opět dobře poslouží oblázkový svět.

Example 2: Properties of P (cont.)

Rolling a die, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, A - outcome is 6, B - outcome is an odd number. Obvioiusly $A \cap B = \emptyset$,

$$P(A \cup B) = P(A) + P(B)$$

A pump in a power plant is backed up by another, identical pump. Event A_i means that pump i is OK. What is the probability that at least one of them is OK?

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Both pumps are OK:

$$P(A_1 \cap A_2) = ?$$

Notes

Základní kontrola pravděpodobnosti libovolně složité události A: $0 \leq P(A) \leq 1$.

Jak spočítat $P(A_1 \cap A_2)$? Existuje nějaký případ, kdy to lze spočítat snadno?

Properties of probability

For any valid probability function:

- $P(A) \in \langle 0, 1 \rangle$ (definition)
- $P(\emptyset) = 0, \qquad P(\mathcal{S}) = 1$ (axioms)
- $P(A^c) = 1 - P(A)$
- If $A \subseteq B$, then $P(A) \leq P(B)$
- If $A \subseteq B$, then $P(B \setminus A) = P(B) - P(A)$
- If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ (*aditivity*)
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Notes

Example 3: Probability of parts

If we choose a person form the population at random,

- he/she suffers from disease X and is younger than 18 years with probability 0.01,
- he/she suffers from disease X and is between 18 and 65 years with probability 0.05, and
- he/she suffers from disease X and is older than 65 years with probability 0.09.

What is the probability a randomly chosen person suffers from disease X?

Notes

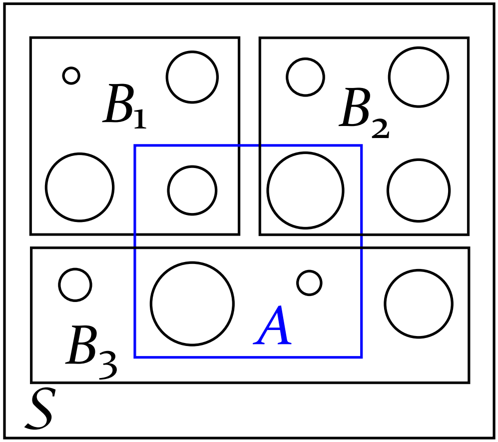
Properties of probability (cont.)

If $\{B_1, \ldots, B_n\}$ is a partition of sample space then for any event $A \subseteq \mathcal{S}$

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) \, .$$

In particular, for partition $\{C, C^c\}$

$$P(A) = P(A \cap C) + P(A \cap C^c) \, .$$



Notes

Independent events /Nezávislé jevy/

Events A and B are independent if and only iff

$$P(A \cap B) = P(A) \cdot P(B).$$

If A, B are independent, then

- $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$,
- and pairs A, B^c and A^c, B and A^c, B^c are independent too.

Notes

Independence of events: tossing two coins

- A - head on the first coin
- B - head on the second coin
- C - different symbols on the coins

Which groups of events are independent?

A no group of events

B pairs (A, B), (B, C), (A, C)

C pairs (A, B), (B, C), (A, C) and triple (A, B, C)

D only triple (A, B, C)

Notes

Mohou být dvojice (A, A) a (A, A^c) nezávislé?

Kdy pro (A, A) platí, že $P(A \cap A) = P(A)P(A) = P(A)$?

Kdy pro (A, A^c) platí, že $P(A \cap A^c) = P(A)P(A^c) = P(A)(1 - P(A))$?

Example: Soldier or technician?

Tom likes order, is decisive, and has a good sense of justice. When he was a kid, he liked to play strategic games and shooting RPGs. He has always been interested in weapons and military equipment.

What do you think is Tom's occupation now?

A Soldier

B Technician

Notes

Nechť je identifikace povolání jev V a T. Můžeme zobecnit na hypotézu V, buď platí $H = V$ nebo $H = T$. Daný popis osobnosti nechť je E jako evidence.

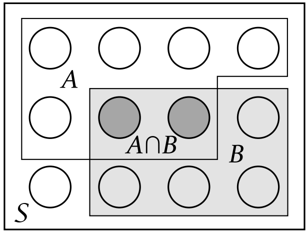
Podle Sčítání 2021 je v ČR cca 22 tis. zaměstnanců v ozbrojených silách a cca 860 tis. technických a odborných pracovníků.

Diskutujme, podle čeho jsme se rozhodovali. Nakresleme diagram s jevy V a T, znázorníme v obou jevech části, kdy platí i E.

Conditional probability

Conditional probability of event A given event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) \neq 0.$$



- All probabilities are conditional: $P(A) = P(A|\mathcal{S})$.
- Interpretation:
  1. $P(A)$ is our current belief that event A occurs.
  2. We get a new information that a different event B occured.
  3. $P(A|B)$ is now our updated belief about A.
- Conditional probability is still a probability: it maps any event $A \subseteq \mathcal{S}$ to $\langle 0, 1 \rangle$.

Notes

Properties of Conditional Probability

- $P(\mathcal{S}|B) = 1$, $P(\emptyset|B) = 0$.
- $P(A|A) = 1$, $P(A^c|A) = 0$.
- If $B \subseteq A$, then $P(A|B) = 1$.
- If $P(A \cap B) = 0$, then $P(A|B) = 0$.
- If $A_1, \ldots, A_n$ are mutually exclusive events, then $P\left(\bigcup_{i=1}^{n} A_i \middle| B\right) = \sum_{i=1}^{n} P(A_i|B)$.
- Events A, B are independent iff $P(A|B) = P(A)$ (if $P(A|B)$ is defined).

Notes

Belief update

1. Probability $P(A)$ is our initial (prior) belief that event A occurs.
2. We learn that another event, B, occured.
3. Probability $P(A|B)$ is our updated (posterior) belief that event A occurs.

No other info about events A and B is available. Which of the following options is correct?

A $P(A|B) < P(A)$

B $P(A|B) = P(A)$

C $P(A|B) > P(A)$

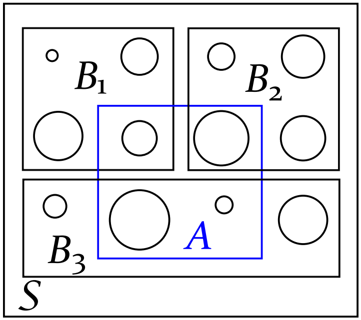
D Any of the above options can happen.

Notes

The Law of Total Probability

Let $B_1, \ldots, B_n$ be a partition of the sample space $\mathcal{S}$ (i.e., the B_i are disjoint events and their union is $\mathcal{S}$), with $P(B_i) > 0$ for all i.
Then

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$$



Notes

Bayes rule

Probability of the intersection of two events A and B, $P(A \cap B)$, can be expressed in 2 ways:

- $P(A \cap B) = P(A|B) \cdot P(B)$
- $P(A \cap B) = P(B|A) \cdot P(A)$

From that it follows that

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

Applying the law of total probability from previous slide:

$$P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{P(A)} = \frac{P(A|B_i) \cdot P(B_i)}{\sum_{j \in I} P(A|B_j) \cdot P(B_j)}.$$

Notes

Working with random events becomes cumbersome ...

Experiment: 3 tosses of a coin. Outcomes $s \in \mathcal{S}$. Events $A_i \subseteq \mathcal{S}$:

- three heads – $X(s) = 3$
- at least one head – $X(s) \geq 1$
- three equal symbols – $X(s) \in \{3, 0\}$
- ...

We can define each event as a set (often quite large) of outcomes s.
Or we can define a random variable :

$$X(s) = \text{number of heads in } s$$

Before the experiment, how many heads do I expect to be tossed?

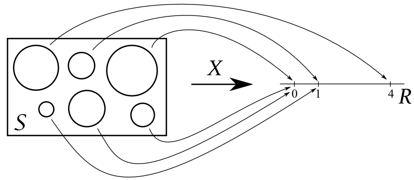
Notes

Lze uvažovat např. i o hodu 3 kostkami. Kolik existuje různých výsledků experimentu?

Jak je asi složité pracovat s jevy zahrnujícími stovky elementárních jevů?

Random Variable

Random variable (náhodná proměnná/veličina) on a probability space $(\mathcal{S}, P)$ is a function X mapping elementary events $s \in \mathcal{S}$ to real numbers $\mathbb{R}$, i.e., $X : \mathcal{S} \to \mathbb{R}$.



"Random variable is a numerical 'summary' of an aspect of the experiment."

- R.v. X assigns a numerical value $X(s)$ to each possible outcome $s \in \mathcal{S}$.
- The mapping is *deterministic*; the randomness comes from outcomes of random experiment (with outcome probabilities described by probability function P).
- Before the experiment, we know neither the value of s, nor the value of $X(s)$. But we can compute the probability that X will take on a given value, or a range of values.
- After the experiment, s was realized, and the r.v. crystalizes into value $X(s)$.

Notes

Events vs Values of Random Variable

Let X be a random variable, i.e., $X : \mathcal{S} \to \mathbb{R}$.

- $X = x$ denotes the event $\{s \in \mathcal{S} : X(s) = x\}$, i.e., the event consisting of all outcomes s such that $X(s) = x$.
- $X \in \langle a, b)$ denotes the event $\{s \in \mathcal{S} : a \leq X(s) < b\}$, i.e., the event consisting of all outcomes s such that $a \leq X(s) < b$.

Notes

Discrete Random Variable

Random variable X is called discrete if the values of $X(s)$ for all $s \in \mathcal{S}$ form either

- a finite set of values $a_1, a_2, \ldots, a_n$, or
- an infinite set of countably many values $a_1, a_2, \ldots$

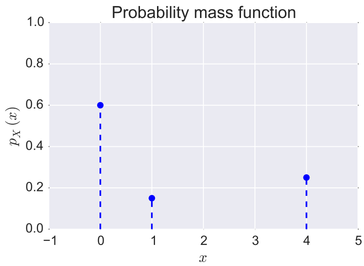
Support of X:
$\mathcal{S}_X = \{x \in \mathbb{R} : P(X = x) > 0\} = \{a_1, a_2, \ldots\}$

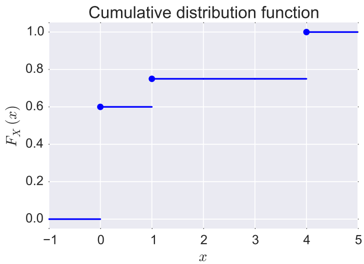
Probability Mass Function (PMF) /pstní fce/ of a discrete r.v. X is the function p_X given by

$$p_X(x) = P(X = x) = P(\{s \in \mathcal{S} : X(s) = x\}).$$

Cumulative Distribution Function (CDF) /distribuční fce/ of a discrete r.v. X is the function F_X defined as

$$F_X(x) = P(X \leq x) = \sum_{t \leq x} p_X(t).$$





Notes

Expected value

Expected value (střední hodnota) of a discrete r.v. X is denoted as $\operatorname{E} X$ and is defined as

$$\operatorname{E} X = \sum_{t \in \mathbb{R}} t \cdot p_X(t) = \sum_{t \in \mathcal{S}_X} t \cdot p_X(t).$$

For equally probable outcomes $s \in \mathcal{S}$ also $\operatorname{E} X = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} X(s)$.

Characteristics of $\operatorname{E} X$:

- $\operatorname{E} r = r$, $\operatorname{E}(\operatorname{E} X) = \operatorname{E} X$
- $\operatorname{E}(X + Y) = \operatorname{E} X + \operatorname{E} Y$, $\operatorname{E}(X + r) = \operatorname{E} X + r$, $\operatorname{E}(X - Y) = \operatorname{E} X - \operatorname{E} Y$
- $\operatorname{E}(rX + sY) = r \operatorname{E} X + s \operatorname{E} Y$
- For *independent* r.v.s: $\operatorname{E}(X \cdot Y) = \operatorname{E} X \cdot \operatorname{E} Y$.

Notes

References, further reading

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Notes