Linear Models for Regression and Classification, Learning

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thanks to Matěj Hoffmann, Daniel Novák, Filip Železný, Ondřej Drbohlav

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Supervised Learning

A training multi-set of examples is available. Correct answers (hidden state, class, the quantity we want to predict) are *known* for all training examples.

Classification :

- Nominal dependent variable
- Examples: predict spam/ham based on email contents, predict 0/1/.../9 based on the image of a number, etc.

Regression :

- Quantitative/continuous dependent variable
- Examples: predict temperature in Prague based on date and time, predict height of a person based on weight and gender, etc.

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Learning by minimization of empirical risk

- Given the set of parametrized strategies $\delta\colon \mathcal{X} \to \mathcal{D}$, penalty/loss function $\ell\colon \mathcal{S} \times \mathcal{D} \to \mathbb{R}$, the quality of each strategy δ could be described by the risk

$$R(\delta) = \sum_{s \in \mathcal{S}} \sum_{x \in \mathcal{X}} P(x, s)\ell(s, \delta(x)),$$

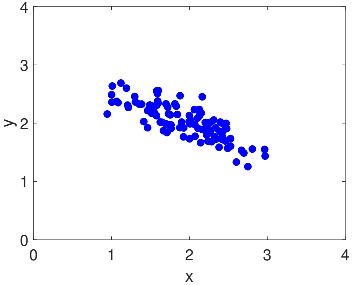
  but P is unknown.
- We thus use the empirical risk R_{emp} error on training (multi)set $\mathcal{T} = \{(x^{(i)}, s^{(i)})\}_{i=1}^{N}$, $x \in \mathcal{X}$, $s \in \mathcal{S}$:

$$R_{\mathrm{emp}}(\delta) = \frac{1}{N} \sum_{(x^{(i)}, s^{(i)}) \in \mathcal{T}} \ell(s^{(i)}, \delta(x^{(i)})).$$

- Optimal strategy $\delta^* = \operatorname{argmin}_{\delta} R_{\mathrm{emp}}(\delta)$.
- We expect the data are from the right distribution.

Quiz: Line fitting

We would like to fit a line of the form $\widehat{y} = w_0 + w_1 x$ to the following data:



The parameters of a line with a good fit will likely be

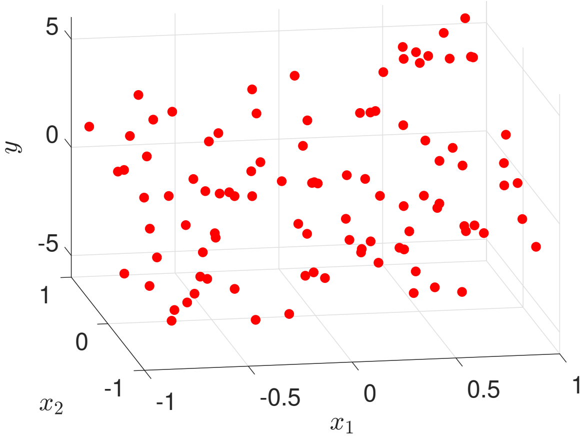
A $w_0 = -1,\ w_1 = -2$

B $w_0 = -\frac{1}{2},\ w_1 = 1$

C $w_0 = 3,\ w_1 = -\frac{1}{2}$

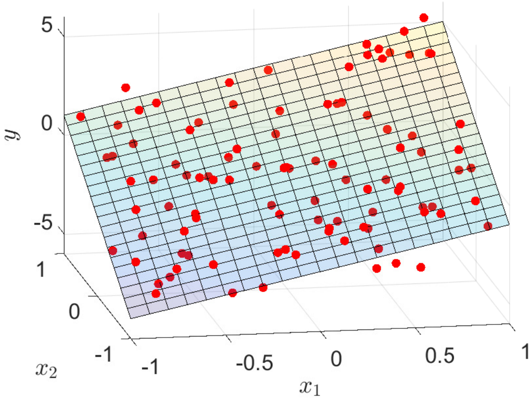
D $w_0 = 2,\ w_1 = \frac{1}{3}$

Linear regression: Illustration



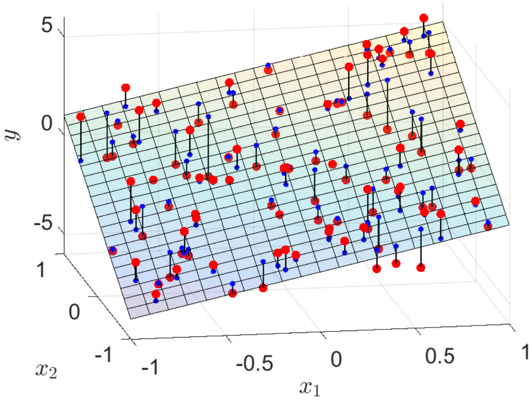
Given a dataset of input vectors $\boldsymbol{x}^{(i)}$ and the respective values of output variable $y^{(i)}$ ...

Linear regression: Illustration



...we would like to find a linear model of this dataset ...

Linear regression: Illustration



...minimizing the errors between target values and the model predictions.

Regression

Reformulating Linear algebra in a machine learning language.

Regression task is a supervised learning task, i.e.

- a training (multi)set $\mathcal{T} = \{(\boldsymbol{x}^{(1)}, y^{(1)}), \dots, (\boldsymbol{x}^{(N)}, y^{(N)})\}$ is available, where
- the labels $y^{(i)}$ are *quantitative*, often *continuous* (as opposed to classification tasks where $y^{(i)}$ are nominal).
- Its purpose is to model the relationship between independent variables (inputs) $\boldsymbol{x} = (x_1, \dots, x_D)$ and the dependent variable (output) y.

Linear Regression

Linear regression is a particular regression model which assumes (and learns) linear relationship between the inputs and the output:

$$\widehat{y} = \delta(\boldsymbol{x}) = w_0 + w_1 x_1 + \ldots + w_D x_D = w_0 + \langle \boldsymbol{w}, \boldsymbol{x} \rangle = w_0 + \boldsymbol{w}^\top \boldsymbol{x},$$

where

- $\widehat{y}$ is the model *prediction* (*estimate* of the true value y),
- $\delta(\boldsymbol{x})$ is the decision strategy (a linear model in this case),
- $w_0, \ldots, w_D$ are the coefficients of the linear function (weights), w_0 is the *bias*,
- $\langle \boldsymbol{w}, \boldsymbol{x} \rangle$ is a *dot product* of vectors $\boldsymbol{w}$ and $\boldsymbol{x}$ (scalar product),
- which can be also computed as a matrix product $\boldsymbol{w}^\top \boldsymbol{x}$ if $\boldsymbol{w}$ and $\boldsymbol{x}$ are *column vectors*, i.e. matrices of size $[D \times 1]$.

Notation remarks

Homogeneous coordinates :

- If we add "1" as the first element of $\boldsymbol{x}$ so that $\boldsymbol{x} = (1, x_1, \ldots, x_D)$, and
- include the bias term w_0 in the vector $\boldsymbol{w}$ so that $\boldsymbol{w} = (w_0, w_1, \ldots, w_D)$, then

$$\widehat{y} = \delta(\boldsymbol{x}) = w_0 \cdot 1 + w_1 x_1 + \ldots + w_D x_D = \langle \boldsymbol{w}, \boldsymbol{x} \rangle = \boldsymbol{w}^\top \boldsymbol{x}.$$

Matrix notation: If we organize the data $\mathcal{T}$ into matrices $\boldsymbol{X}$ and $\boldsymbol{y}$, such that

$$\boldsymbol{X} = \begin{pmatrix} 1 & \ldots & 1 \\ \boldsymbol{x}^{(1)} & \ldots & \boldsymbol{x}^{(N)} \end{pmatrix} \quad \text{and} \quad \boldsymbol{y} = \left(y^{(1)}, \ldots, y^{(N)}\right),$$

and similarly with $\widehat{\boldsymbol{y}}$, then we can write a batch computation of predictions for all data in $\boldsymbol{X}$ as

$$\widehat{\boldsymbol{y}} = \left(\delta(\boldsymbol{x}^{(1)}), \ldots, \delta(\boldsymbol{x}^{(N)})\right) = \left(\boldsymbol{w}^\top \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{w}^\top \boldsymbol{x}^{(N)}\right) = \boldsymbol{w}^\top \boldsymbol{X}.$$

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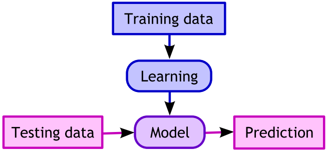
Two operation modes

Any ML model has 2 operation modes:

1. learning (training, fitting) of δ and
2. application of δ (testing, making predictions).



The dec. strategy δ can be viewed as a function of 2 variables: $\delta(\boldsymbol{x}, \boldsymbol{w})$.

Model application: (Inference) Given $\boldsymbol{w}$, we can manipulate $\boldsymbol{x}$ to make predictions:

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$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{w}} R_{\mathrm{emp}}(\delta_{\boldsymbol{w}}) = \operatorname*{argmin}_{\boldsymbol{w}} J(\boldsymbol{w}, \mathcal{T})$$

$J(\boldsymbol{w}, \mathcal{T})$ and $\ell(\boldsymbol{w}, \mathcal{T})$ are closely related. Optimization criterium $J()$ is a broader term. $\ell()$ essentially measures discrepancy between true data and the predictions. How to train the model?

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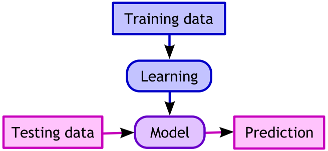

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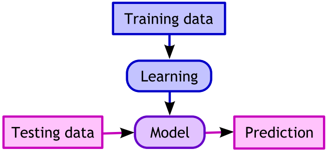

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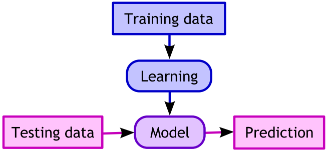

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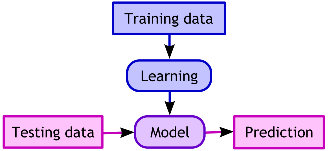

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Simple (univariate) linear regression

Simple regression

- $\boldsymbol{x}^{(i)} = x^{(i)}$, i.e., the examples are described by a single feature (they are 1-dimensional).
- Find parameters w_0, w_1 of a linear model $\hat{y} = w_0 + w_1 x$ given a training (multi)set $\mathcal{T} = \{(x^{(i)}, y^{(i)})\}_{i=1}^{N}$.

How to fit a line depending on the number of training examples N:

- $N = 1$ (1 equation, 2 parameters) $\Rightarrow \infty$ linear functions with zero error
- $N = 2$ (2 equations, 2 parameters) $\Rightarrow$ 1 linear function with zero error
- $N \geq 3$ (> 2 equations, 2 parameters) $\Rightarrow$ no linear function with zero error (in general) $\Rightarrow$ a line which minimizes the "size" of error $y - \widehat{y}$ can be fitted:

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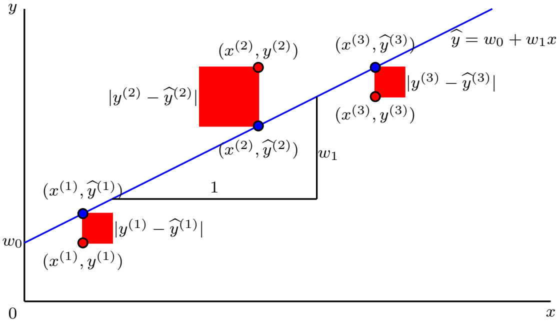
The least squares method

Choose such parameters $\boldsymbol{w}$ which minimize the *mean squared error* (MSE)

$$J_{MSE}(\boldsymbol{w}) = \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \widehat{y}^{(i)}\right)^2$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left(y^{(i)} - \delta_{\boldsymbol{w}}(\boldsymbol{x}^{(i)})\right)^2 .$$



Is there a (closed-form) solution? Explicit solution:

$$w_1 = \frac{\sum_{i=1}^{N}(x^{(i)} - \bar{x})(y^{(i)} - \bar{y})}{\sum_{i=1}^{N}(x^{(i)} - \bar{x})^2} = \frac{s_{xy}}{s_x^2} = \frac{\text{covariance of } X \text{ and } Y}{\text{variance of } X} \qquad w_0 = \bar{y} - w_1\bar{x}$$

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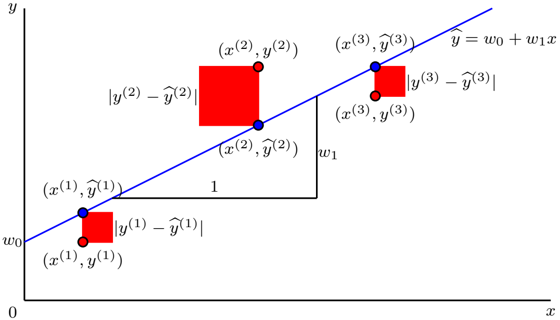

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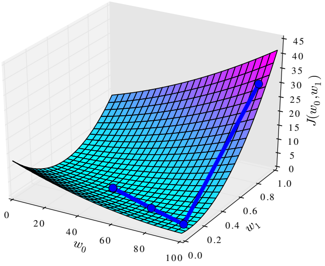
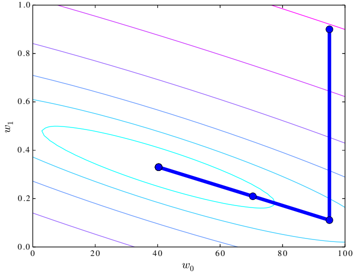
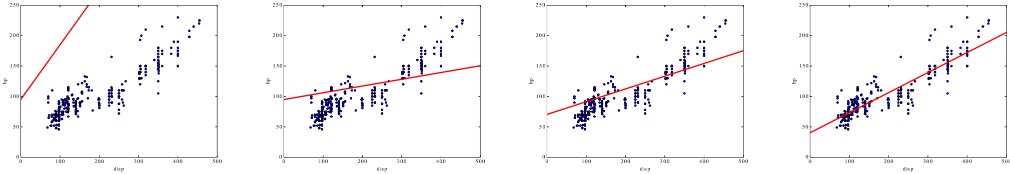
Universal fitting method: minimization of cost function J

The landscape of J in the space of parameters w_0 and w_1:





Gradually better linear models found by an optimization method (BFGS):



Gradient descent algorithm

Given a function $J(w_0, w_1)$ that should be minimized,

- start with a guess of w_0 and w_1 and
- change it, so that $J(w_0, w_1)$ decreases, i.e.
- update our current guess of w_0 and w_1 by taking a step in the direction opposite to the gradient:

$$\boldsymbol{w} \leftarrow \boldsymbol{w} - \alpha \nabla J(w_0, w_1), \text{ i.e.}$$
$$w_i \leftarrow w_i - \alpha \frac{\partial}{\partial w_i} J(w_0, w_1),$$

where all w_is are updated simultaneously and α is a learning rate (step size).

Gradient descent for MSE minimization

For the cost function

$$J(w_0, w_1) = \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - \delta_{\boldsymbol{w}}(x^{(i)}) \right)^2 = \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - (w_0 + w_1 x^{(i)}) \right)^2 ,$$

the gradient can be computed as

$$\frac{\partial}{\partial w_0} J(w_0, w_1) = -\frac{2}{N} \sum_{i=1}^{N} \left(y^{(i)} - \delta_{\boldsymbol{w}}(x^{(i)}) \right)$$

$$\frac{\partial}{\partial w_1} J(w_0, w_1) = -\frac{2}{N} \sum_{i=1}^{N} \left(y^{(i)} - \delta_{\boldsymbol{w}}(x^{(i)}) \right) x^{(i)}$$

Multivariate linear regression

- $\boldsymbol{x}^{(i)} = (x_1^{(i)}, \ldots, x_D^{(i)})^\top$, i.e. the examples are described by more than 1 feature (they are D-dimensional).
- Find parameters $\boldsymbol{w} = (w_0, \ldots, w_D)^\top$ of a linear model $\widehat{y} = \boldsymbol{w}^\top \boldsymbol{x}$ given the training (multi)set $\mathcal{T} = \{(\boldsymbol{x}^{(i)}, y^{(i)})\}_{i=1}^N$.

Training: foreach (i): $y^{(i)} = \boldsymbol{w}^\top \boldsymbol{x}^{(i)}$.
In the matrix form:

$$\boldsymbol{y} = \boldsymbol{w}^\top \boldsymbol{X}$$

What is the dimension of $\boldsymbol{X}$?

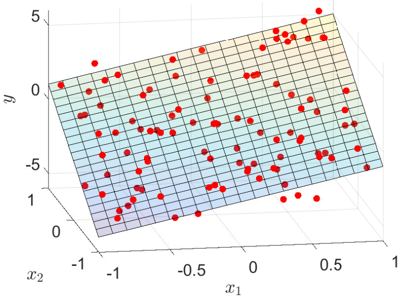
A $(D+1) \times (D+1)$

B $(D+1) \times N$

C $N \times (D+1)$

D $N \times N$

The model is a *hyperplane* in the $(D+1)$ dimensional space.



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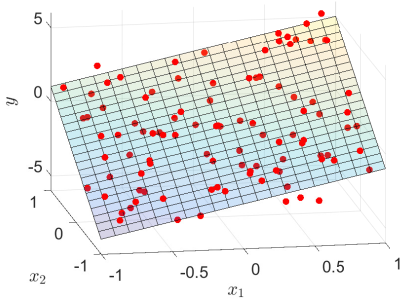

Multivariate linear regression: learning

1. Numeric optimization of $J(\boldsymbol{w}, T)$:
   - Works as for simple regression, it only searches a space with more dimensions.
   - Sometimes one needs to tune some parameters of the optimization algorithm to work properly (learning rate in gradient descent, etc.).
   - May be slow (many iterations needed), but works even for very large D.

2. Normal equation:

$$\boldsymbol{w}^* = (\boldsymbol{X}\boldsymbol{X}^\top)^{-1}\boldsymbol{X}\boldsymbol{y}^\top$$

   - Method to solve for the optimal $\boldsymbol{w}^*$ analytically!
   - No need to choose optimization algorithm parameters. No iterations.
   - Needs to compute $(\boldsymbol{X}\boldsymbol{X}^\top)^{-1}$, which is $O((D+1)^3)$. Becomes intractable for large D.

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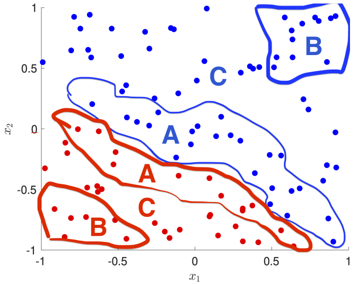
Classification

- Binary classification
- Discriminant function
- Classification as a regression problem (linear, logistic regression)
- What is the right loss function?
- Etalon classifier (meeting nearest neighbour and linear classifier)
- Acuracy vs precision

Quiz: Importance of training examples



Intuitively, which of the training data points should have the biggest influence on the decision whether a new, unlabeled data point shall be red or blue?

A Those which are closest to data points with the opposite color.

B Those which are farthest from the data points of the opposite color.

C Those which are near the middle of the points with the same color.

D None. All of the data points have the same importance.

Binary classification task

Let's have a training dataset $T = \{(\boldsymbol{x}^{(1)}, y^{(1)}), \ldots, (\boldsymbol{x}^{(N)}, y^{(N)})$:

- each example described by a vector $\boldsymbol{x} = (x_1, \ldots, x_D)$,
- labeled with the correct class $y \in \{+1, -1\}$.

The goal:

- Find the classifier (decision strategy/rule) δ that minimizes the empirical risk $R_{\mathrm{emp}}(\delta)$.

Discriminant function

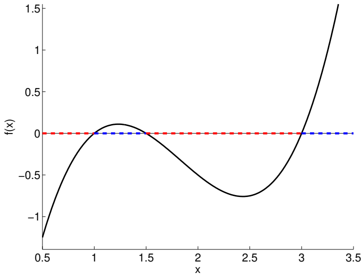
Discriminant function $f(\boldsymbol{x})$:

- It assigns a real number to each observation $\boldsymbol{x}$, may be linear or non-linear.
- For 2 classes, 1 discriminant function is enough.
- It is used to create a decision rule (which then assigns a class to an observation):

$$\widehat{y} = \delta(\boldsymbol{x}) = \begin{cases} +1 & \text{iff} \quad f(\boldsymbol{x}) > 0, \textit{ and} \\ -1 & \text{iff} \quad f(\boldsymbol{x}) < 0. \end{cases}$$

i.e. $\widehat{y} = \delta(\boldsymbol{x}) = \operatorname{sign}(f(\boldsymbol{x}))$.

- Decision boundary: $\{\boldsymbol{x} | f(\boldsymbol{x}) = 0\}$
- Linear classification: the decision boundaries must be linear.
- *Learning* then amounts to finding (suitable parameters of) function f.



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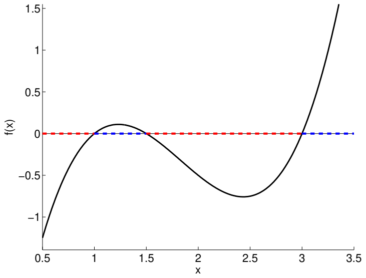
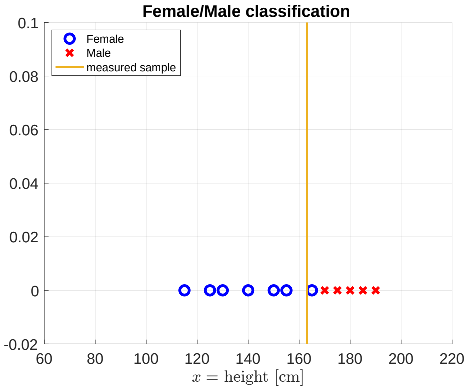


Example: Female/Male classification based on height

Training (multi)set $\mathcal{T} = \{(x^{(i)}, s^{(i)})\}_{i=1}^{N}$, $x^{(i)} \in \mathcal{X}$, $s^{(i)} \in \mathcal{S} = \{F, M\}$

i	1	2	3	4	5	6	7	8	9	10	11	12
Height $x^{(i)}$	115	125	130	140	150	155	165	170	175	180	185	190
Gender $s^{(i)}$	F	F	F	F	F	F	F	M	M	M	M	M



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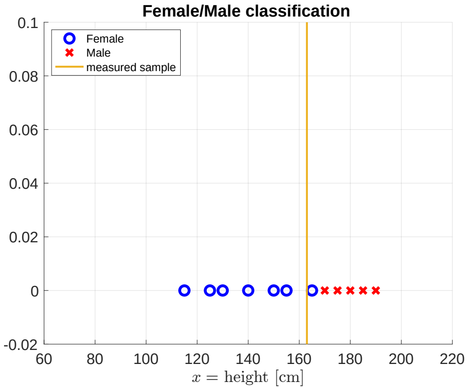
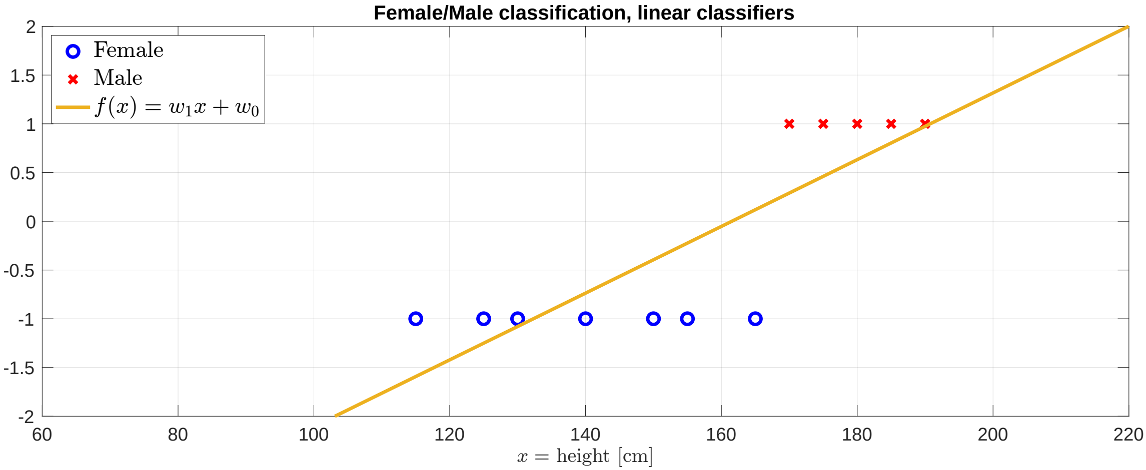


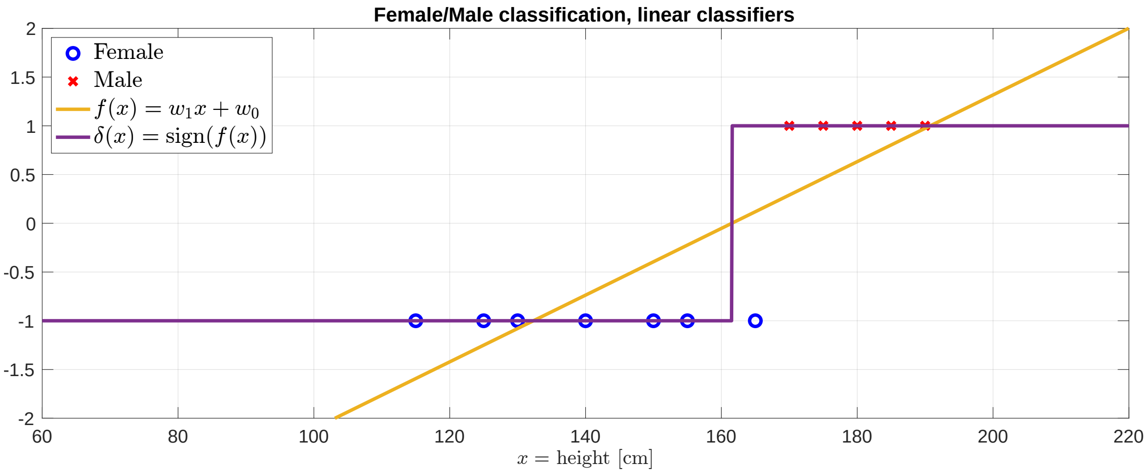
A new point to clasify: $x^Q = 163$

Which class does x^Q belong to? $d^Q = ?$

Linear function LSQ fit



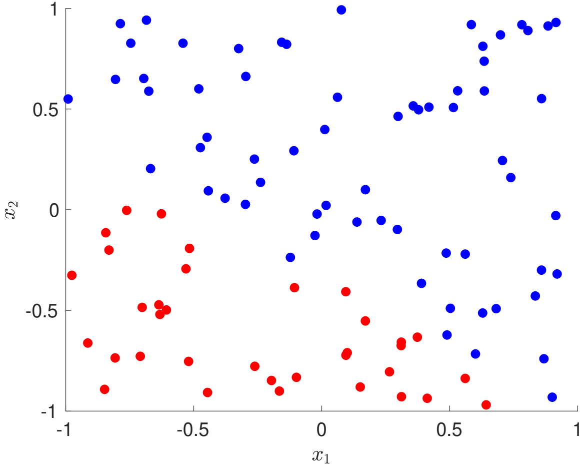
Linear function LSQ fit, discriminant function



Can we do better than fitting a linear function?

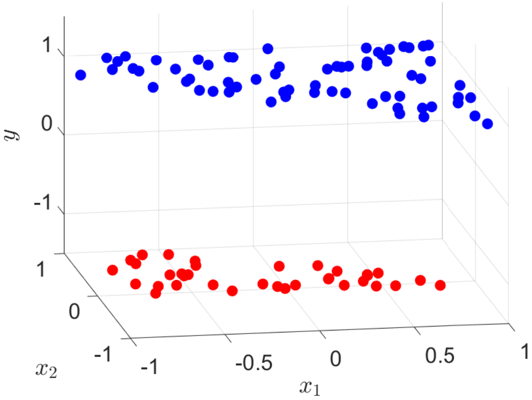
Recap the naive linear approach first.

Learning linear classifier: naive approach, illustration



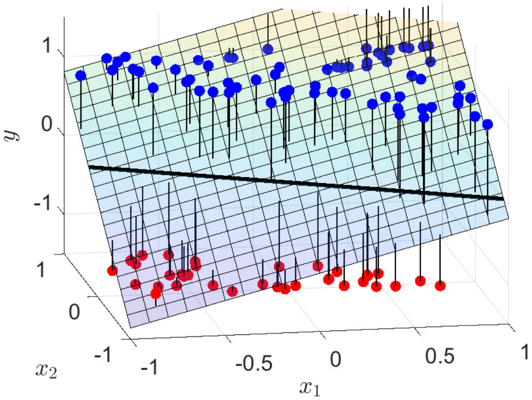
Given a dataset of input vectors $\boldsymbol{x}^{(i)}$ and their classes $y^{(i)}$ ...

Learning linear classifier: naive approach, illustration



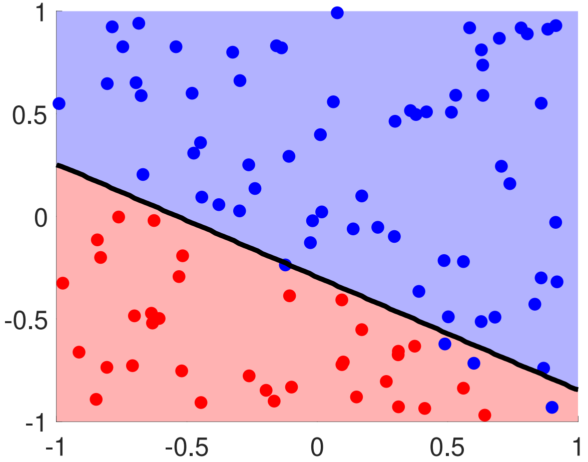
. . . we shall encode the class label as $y = -1$ and $y = 1$ . . .

Learning linear classifier: naive approach, illustration



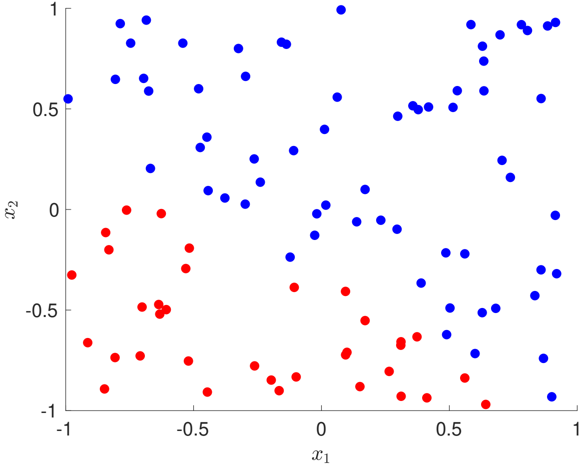
...and fit a linear discriminant function by minimizing MSE as in regression. The contour line $y = 0$ ...

Learning linear classifier: naive approach, illustration



…then forms a linear decision boundary in the original 2D space.
But is such a classifier good in general?

Fitting a better function: Logistic regression



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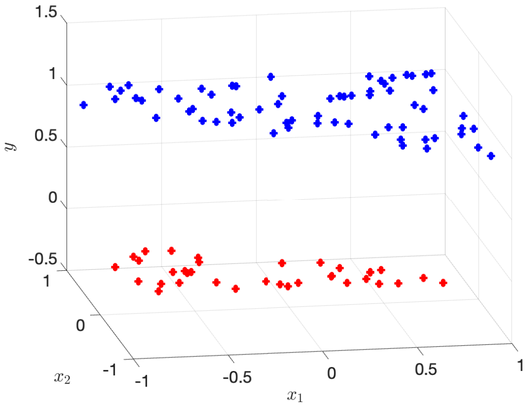
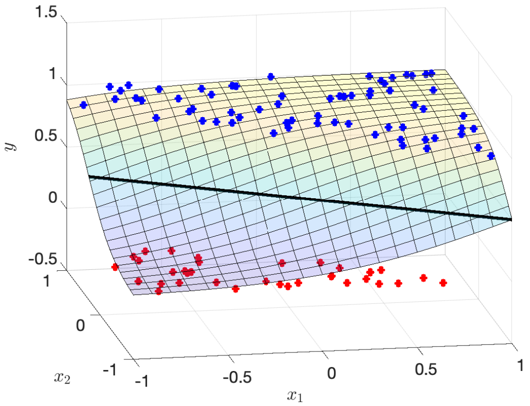


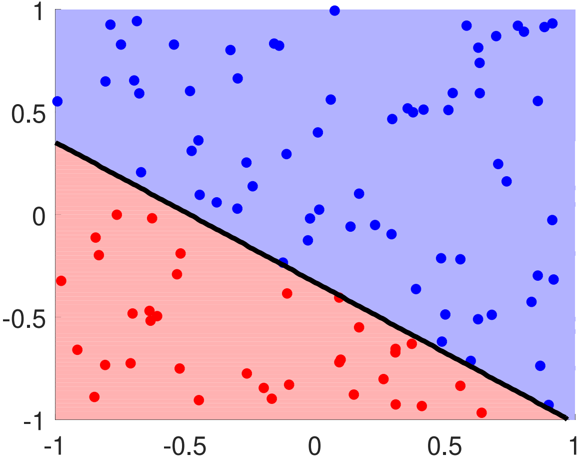
…we shall encode the class label as $y = 0$ and $y = 1$ …

Fitting a better function: Logistic regression



. . . and fit a sigmoidal discriminant function with the threshold 0.5 . . .

Fitting a better function: Logistic regression



. . . which forms a linear decision boundary in the original 2D space.

Logistic regression model

Logistic regression uses a discriminant function which is a nonlinear transformation of the values of a linear function

$$f_{\boldsymbol{w}}(\boldsymbol{x}) = g(\boldsymbol{w}^\top \boldsymbol{x}) = \frac{1}{1 + e^{-\boldsymbol{w}^\top \boldsymbol{x}}},$$

where $g(z) = \dfrac{1}{1 + e^{-z}}$ is the sigmoid function (a.k.a logistic function).

Interpretation of the model:

- $f_{\boldsymbol{w}}(\boldsymbol{x})$ is interpretted as an estimate of the probability that $\boldsymbol{x}$ belongs to class 1.
- The decision boundary is defined using a different level-set: $\{\boldsymbol{x} : f_{\boldsymbol{w}}(\boldsymbol{x}) = 0.5\}$.
- Logistic *regression* is a *classification model!*
- The discriminant function $f_{\boldsymbol{w}}(\boldsymbol{x})$ itself is not linear anymore; but the *decision boundary is still linear!*
- Thanks to the sigmoidal transformation, logistic regression is much less influenced by examples far from the decision boundary!

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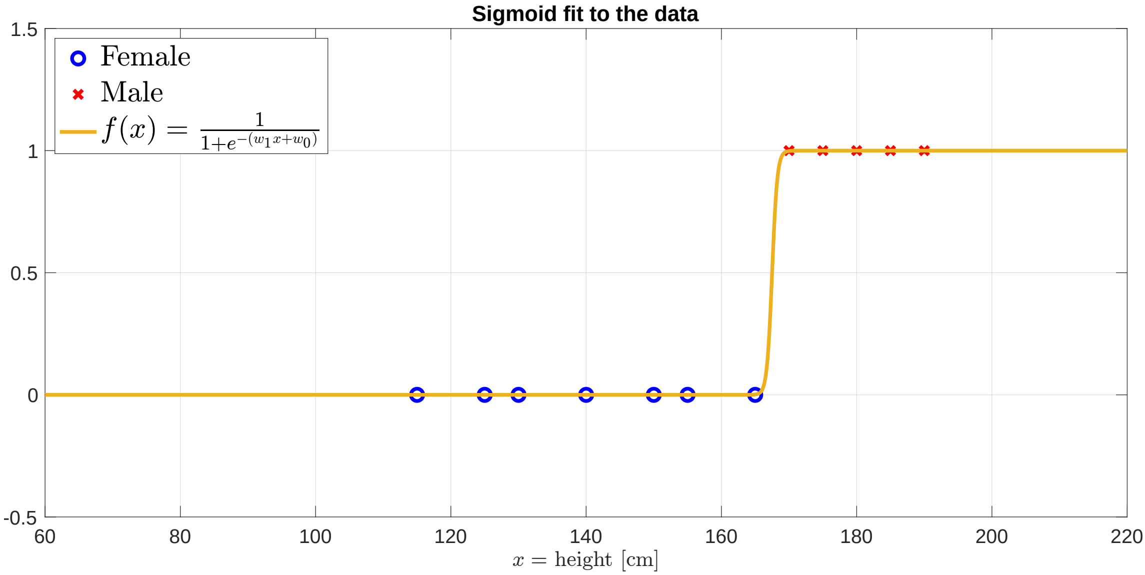
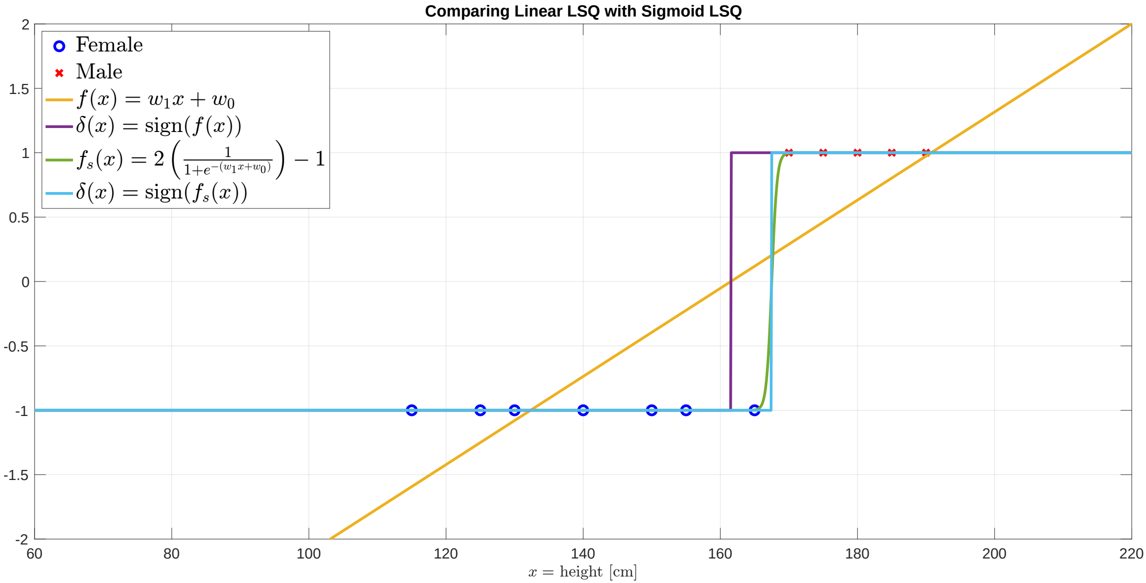
Sigmoid LSQ fit



Comparing Linear and Sigmoid LSQ fit



What is the proper loss function ℓ?

To train the logistic regression model, one can minimize the J_{MSE} criterion:

- results in a non-convex, multimodal landscape which is hard to optimize.

Log. reg. uses a loss function called "cross-entropy":

$$J(\boldsymbol{w}, T) = \frac{1}{N}\sum_{i=1}^{N} \ell(y^{(i)}, f_{\boldsymbol{w}}(\boldsymbol{x}^{(i)})), \text{ where}$$

$$\ell(y, \widehat{y}) = \begin{cases} -\log(\widehat{y}) & \text{if } y = 1 \\ -\log(1-\widehat{y}) & \text{if } y = 0 \end{cases},$$

which can be rewritten in a single expression as

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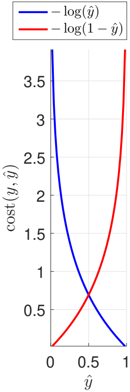

- simpler to optimize for numerical solvers.

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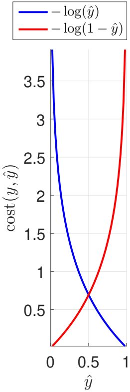

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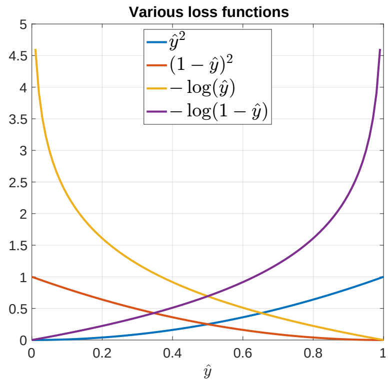
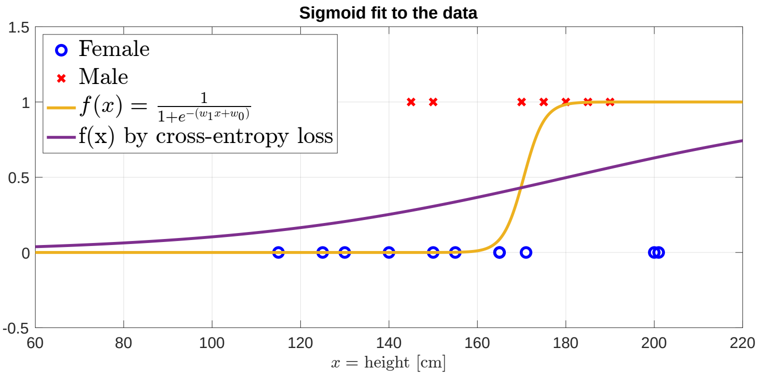
MSE vs cross entropy loss



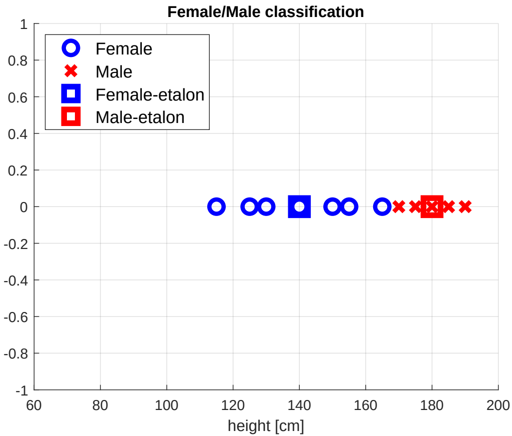


Sigmoidal $f(x)$ can be also interpreted as $p(s = \text{Male} \mid x)$ – Learning Dicriminative model directly.

Cross-entropy loss strongly penalizes hard errors, complete mismatches.

Alternative idea: F/M classification – Etalons

Represent each class by a single example called *etalon*! (Or by a very small number of etalons.)



$$e_F = \mathrm{ave}(\{x^{(i)} : s^{(i)} = F\}) = 140$$
$$e_M = \mathrm{ave}(\{x^{(i)} : s^{(i)} = M\}) = 180$$

Based on etalons: $d_Q = ?$

- A $d^Q = F$
- B $d_Q = M$
- C Both classes equally likely
- D Cannot provide any decision

Classify as $d^Q = \mathrm{argmin}_{s \in S} \mathrm{dist}(x^Q, e_s)$

What type of function is $\mathrm{dist}(x^Q, e_s)$?

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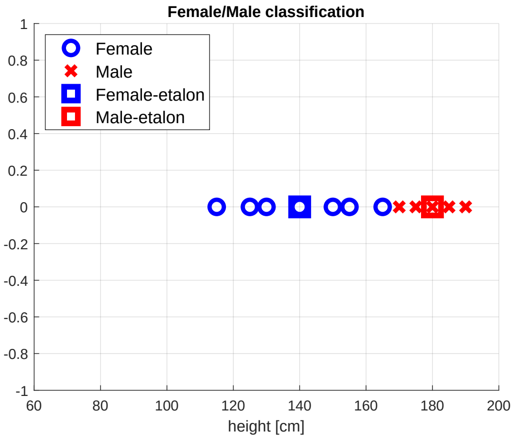

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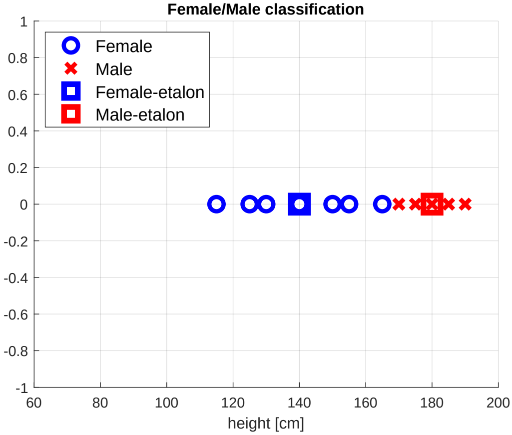

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Etalon classifier is a Linear classifier

Assuming $\text{dist}(x, e) = (x - e)^2$, then

$$\operatorname*{argmin}_{s \in S} \text{dist}(x, e_s) = \operatorname*{argmin}_{s \in S} (x - e_s)^2 = \operatorname*{argmin}_{s \in S} (\underbrace{x^2}_{\text{const.}} - 2e_s x + e_s^2) =$$

$$= \operatorname*{argmin}_{s \in S} (-2e_s x + e_s^2) = \operatorname*{argmax}_{s \in S} (\underbrace{e_s x - \frac{1}{2} e_s^2}_{\text{linear function of } x})$$

Multiclass classification: each class s has a linear discriminant function $f_s(x) = a_s x + b_s$ and

$$\delta(x) = \operatorname*{argmax}_{s \in S} f_s(x)$$

Binary classification: a single linear discriminant function $g(x)$ is sufficient and

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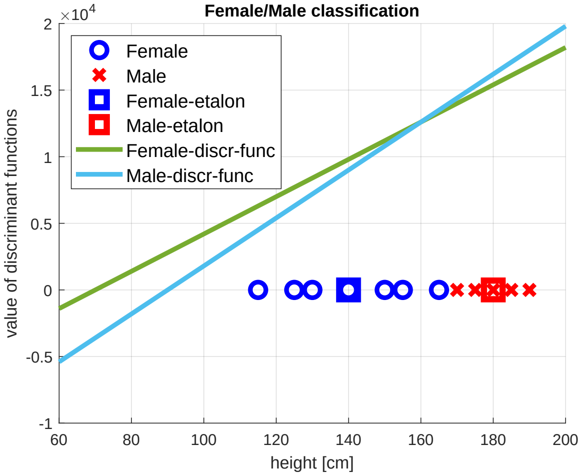
Example: F/M – Linear discriminant functions based on etalons



Discriminant functions for 2 classes:

$$f_F(x) = a_F x + b_F =$$
$$= e_F x - \frac{1}{2} e_F^2 = 140x - 9800$$
$$f_M(x) = a_M x + b_M =$$
$$= e_M x - \frac{1}{2} e_M^2 = 180x - 16200$$

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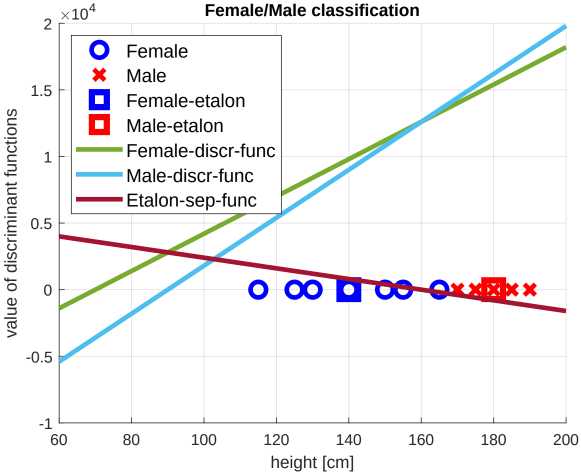

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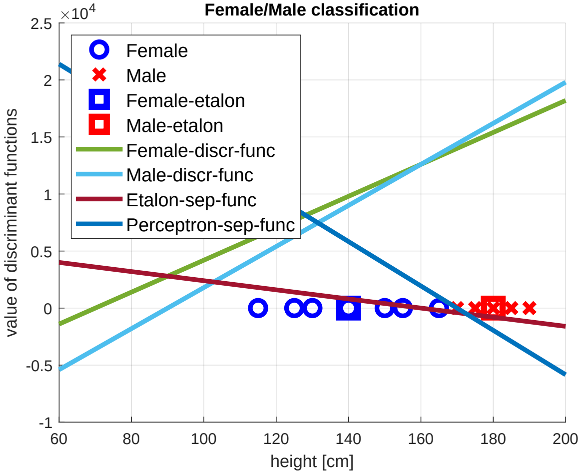
$$f_M(x) = a_M x + b_M = $$
$$= e_M x - \frac{1}{2} e_M^2 = 180x - 16200$$

A single discriminant function separating 2 classes:

$$g(x) = f_F(x) - f_M(x) = $$
$$= -40x + 6400$$

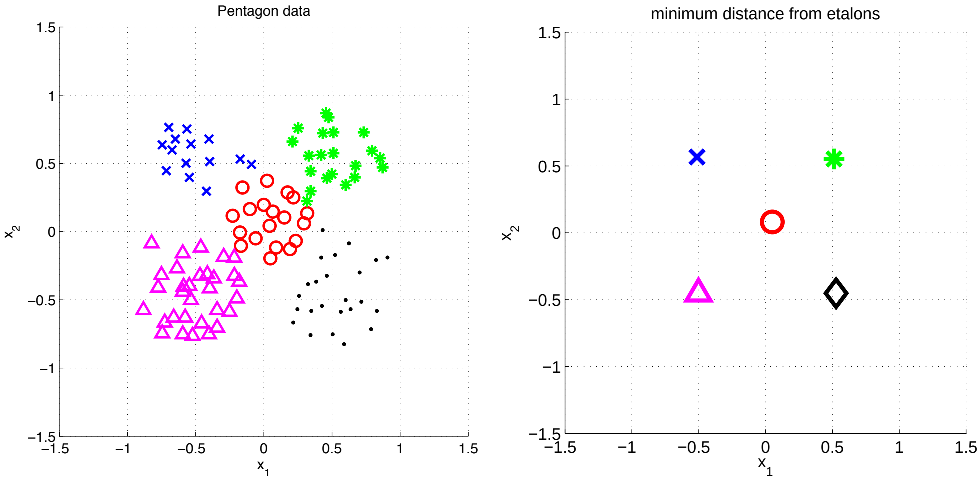
Example: F/M – Can we do better etalons?



Etalon-based linear classifier makes some errors.

A perceptron algorithm may be used to find a zero-error classifier (if one exists).

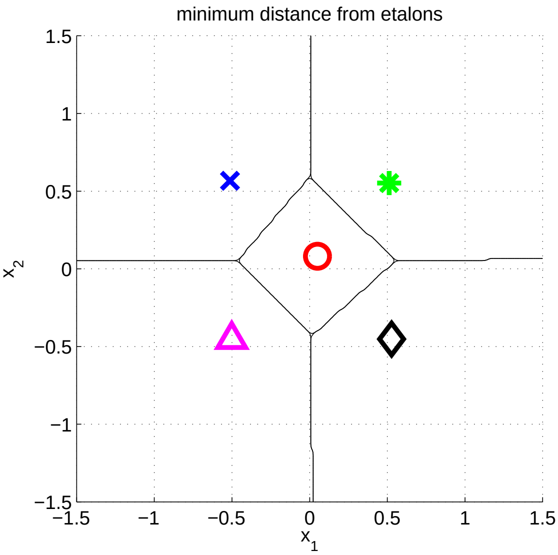
Etalon based classification



Represent $\vec{x}$ by etalon , $\vec{e}_s$ per each class $s \in S$.

Separate etalons

$$s^* = \arg\min_{s \in S} \|\vec{x} - \vec{e}_s\|^2$$

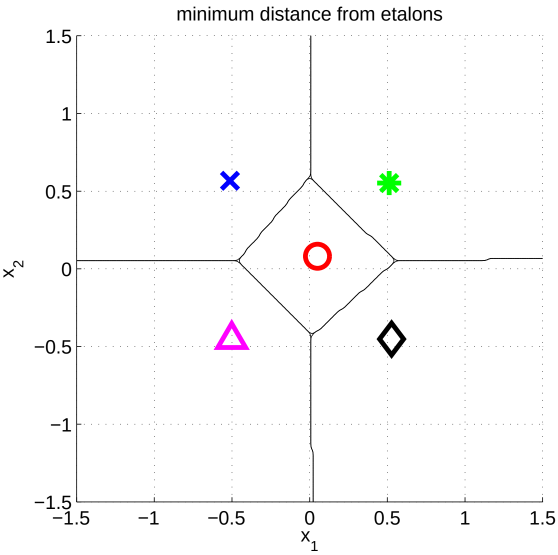


What etalons?

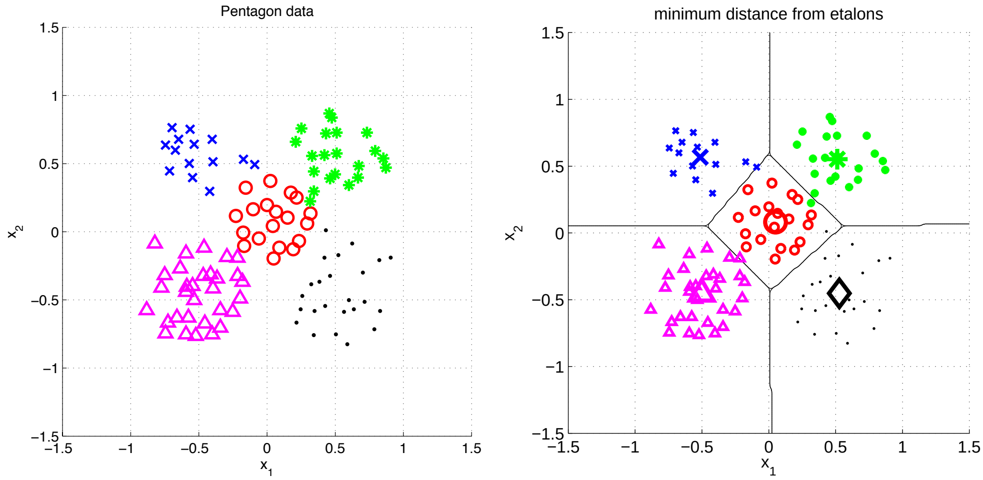
If $\mathcal{N}(\vec{x}|\vec{\mu},\Sigma)$; all classes same covariance matrices, then

$$\vec{e}_s \stackrel{\text{def}}{=} \vec{\mu}_s = \frac{1}{|\mathcal{X}^s|}\sum_{i\in\mathcal{X}^s} \vec{x}_i^s$$

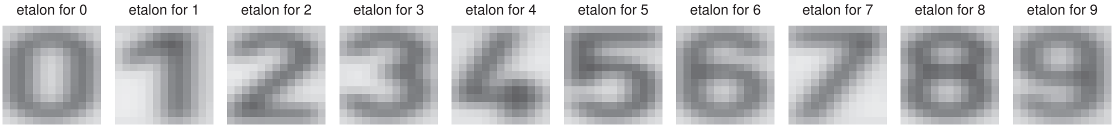
and separating hyperplanes halve distances between pairs.



Etalon based classification, $\vec{e}_s = \vec{\mu}_s$



Digit recognition - etalons $\vec{e}_s = \vec{\mu}_s$



Figures from [7].

Bayesian Discriminant functions $f(\vec{x}, s)$, $g_s(\vec{x})$

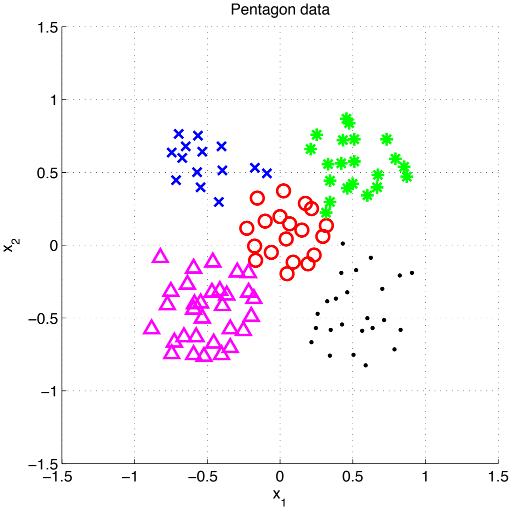
$$s^* = \operatorname*{argmax}_{s \in \mathcal{S}} f(\vec{x}, s)$$

Bayes:

$$s^* = \operatorname*{argmax}_{s \in \mathcal{S}} P(s|\vec{x}) = \frac{P(\vec{x} \mid s)P(s)}{P(\vec{x})}$$

Discriminant function:

$$f(\vec{x}, s) = g_s(\vec{x}) = P(\vec{x} \mid s)P(s)$$



Etalon classifier – Linear classifier, generalization to higher dimensions

$$
\begin{aligned}
s^* &= \arg\min_{s \in S} \|\vec{x} - \vec{e}_s\|^2 = \arg\min_{s \in S} (\vec{x}^\top \vec{x} - 2\,\vec{e}_s^\top \vec{x} + \vec{e}_s^\top \vec{e}_s) = \\
&= \arg\min_{s \in S} \left(\vec{x}^\top \vec{x} - 2\left(\vec{e}_s^\top \vec{x} - \frac{1}{2}(\vec{e}_s^\top \vec{e}_s)\right)\right) = \\
&= \arg\min_{s \in S} \left(\vec{x}^\top \vec{x} - 2\left(\vec{e}_s^\top \vec{x} + b_s\right)\right) = \\
&= \boxed{\arg\max_{s \in S} (\vec{e}_s^\top \vec{x} + b_s)} = \arg\max_{s \in S} g_s(\vec{x}). \qquad b_s = -\frac{1}{2}\vec{e}_s^\top \vec{e}_s
\end{aligned}
$$

Linear function (plus offset)

$$g_s(\mathbf{x}) = \mathbf{w}_s^\top \mathbf{x} + w_{s0}$$

Learning and decision

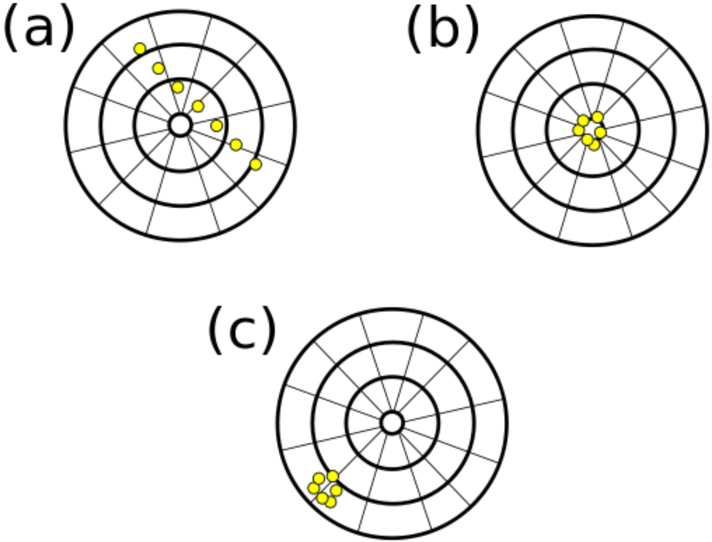
Learning stage - learning models/function/parameters from data.

Decision stage - decide about a query $\vec{x}$.

What to learn?

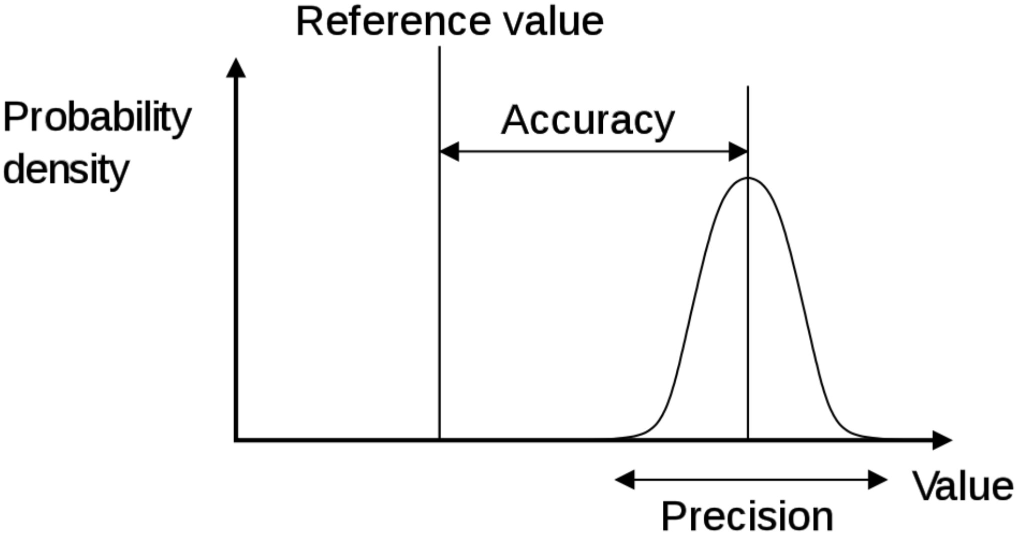
- Generative model : Learn $P(\vec{x}, s)$. Decide by computing $P(s|\vec{x})$.
- Discriminative model : Learn $P(s|\vec{x})$.
- Discriminant function : Learn $g(\vec{x})$ which maps $\vec{x}$ directly into class labels.

Accuracy vs precision



https://commons.wikimedia.org/wiki/File:Precision_versus_accuracy.svg

Accuracy vs precision



https://en.wikipedia.org/wiki/Accuracy_and_precision

References I

Further reading: Chapter 18 of [6], or chapter 4 of [1], or chapter 5 of [2]. Many figures created with the help of [3]. You may also play with demo functions from [7].
Human deciding and predicting under noise, [4] (in Czech [5])

[1] Christopher M. Bishop.
Pattern Recognition and Machine Learning.
Springer Science+Bussiness Media, New York, NY, 2006.
https://www.microsoft.com/en-us/research/uploads/prod/2006/01/Bishop-Pattern-Recognition-and-Machine-Learning-2006.pdf.

[2] Richard O. Duda, Peter E. Hart, and David G. Stork.
Pattern Classification.
John Wiley & Sons, 2nd edition, 2001.

[3] Vojtěch Franc and Václav Hlaváč.
Statistical pattern recognition toolbox.
http://cmp.felk.cvut.cz/cmp/software/stprtool/index.html.

References II

[4] D. Kahneman, O. Sibony, and C.R. Sunstein.
Noise: A Flaw in Human Judgment.
Little Brown Spark, 2021.

[5] D. Kahneman, O. Sibony, and C.R. Sunstein.
Šum, O chybách v lidském úsudku.
Jan Melvil Publishing, 2021.

[6] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
http://aima.cs.berkeley.edu/.

[7] Tomáš Svoboda, Jan Kybic, and Hlaváč Václav.
Image Processing, Analysis and Machine Vision — A MATLAB Companion.
Thomson, Toronto, Canada, 1st edition, September 2007.
http://visionbook.felk.cvut.cz/.