Problem solving by search II

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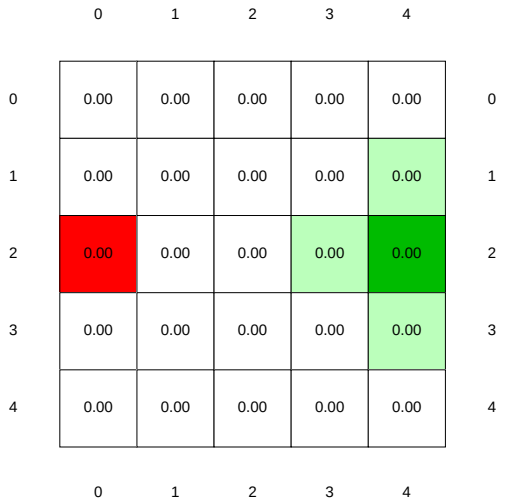
Outline

- Graph search
- Heuristics (how to search faster)
- Greedy
- A*. A-star search.

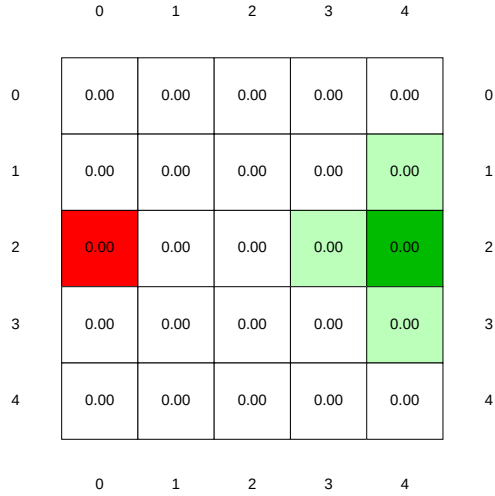
A Maze, what could possibly go wrong?



https://youtu.be/WKSoedfRZQ4

Tree search the maze

function TREE_SEARCH(env) **return** a solution or failure
    initialize the frontier
    while frontier **do**
        node ← frontier.pop()
        if goal in node **then return** node
        end if
        child_nodes ← env.expand(node.state)
        Add child_nodes to frontier
    end while
end function



A graph search

function GRAPH_SEARCH(env) **return** a solution or failure
    init frontier by the start state
    initialize the explored set to be empty
    while frontier **do**
        node ← frontier.pop()
        add node.state to explored
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        child_nodes ← env.expand(node.state)
        for all child_nodes **do**
            if *child_node.state not in explored* **then**
                add nodes to frontier
            end if
        end for
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Do not forget: node is not the same as state!

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▷ What about frontier?

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The BFS graph search

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function BFS_GRAPH_SEARCH(env) return a solution or failure
    node ← env.observe()
    frontier ← FIFOqueue(node)
    explored ← set()
    while frontier not empty do
        node ← frontier.pop()
        explored.add(node.state)                          ▷ Add state, not node!
        child_nodes ← env.expand(node.state)
        for all child_nodes do
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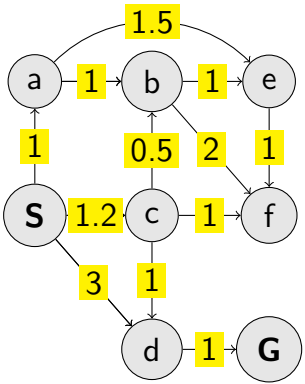
Checking `frontier` in uniform costs graph search?



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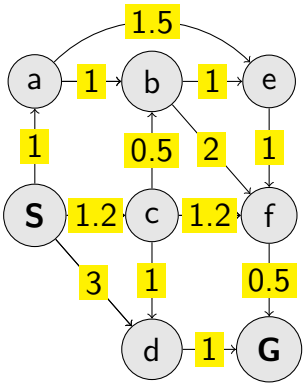

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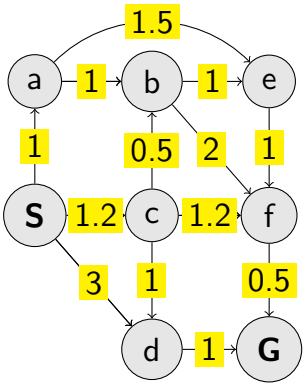

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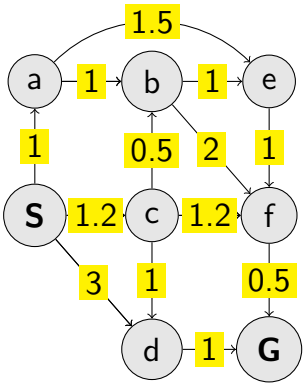

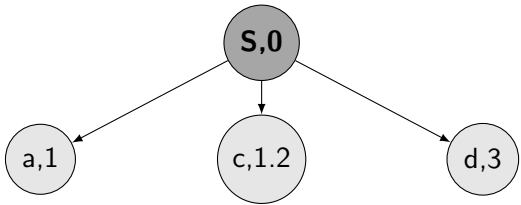

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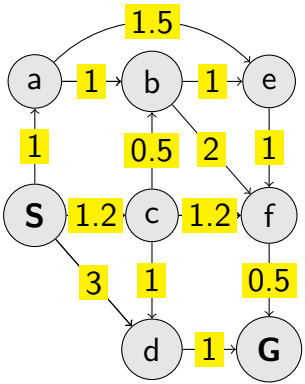

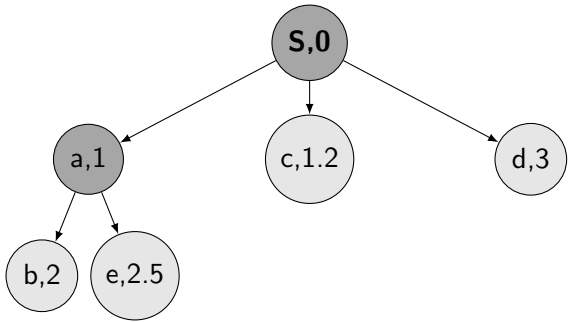

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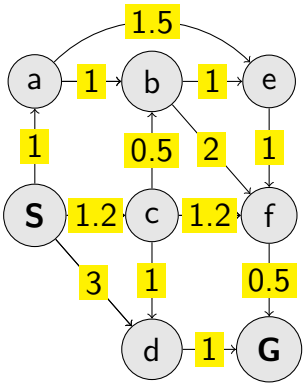

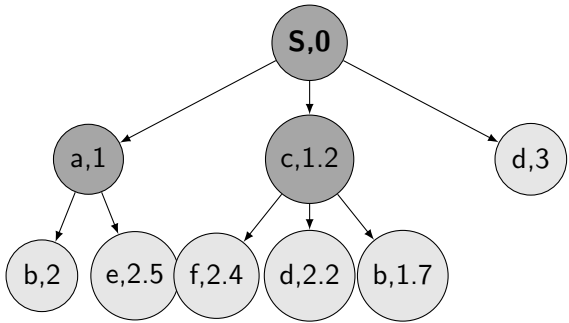

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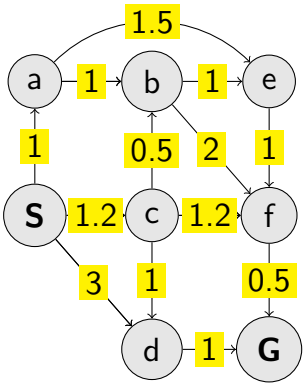

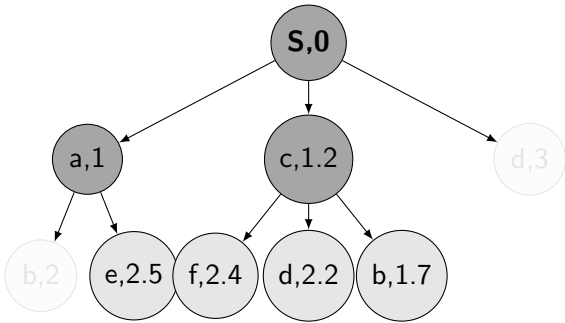

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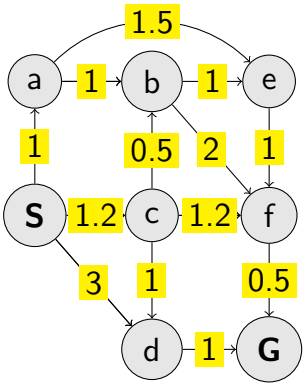

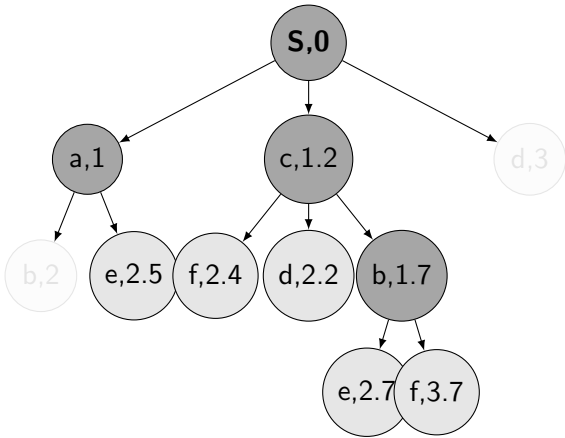

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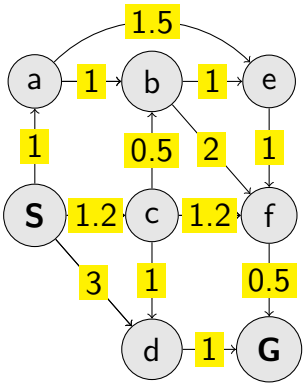

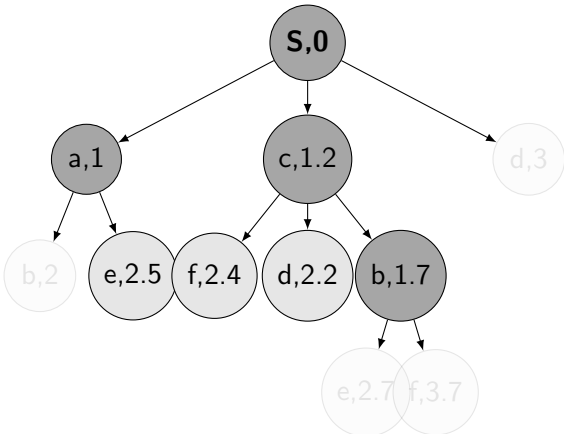

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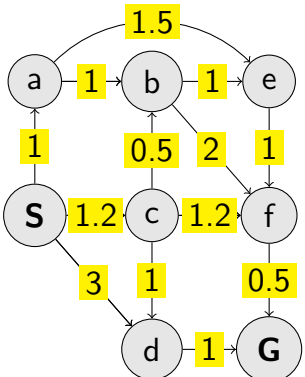

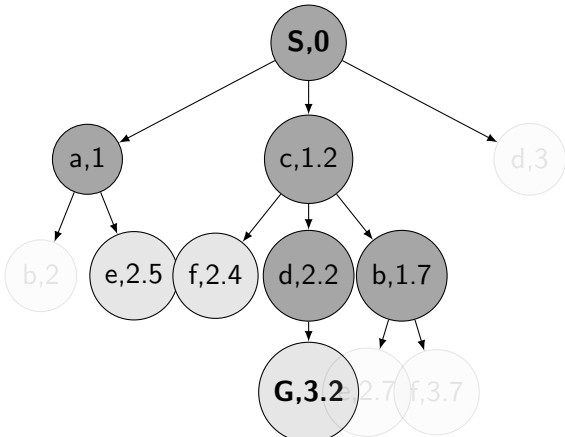

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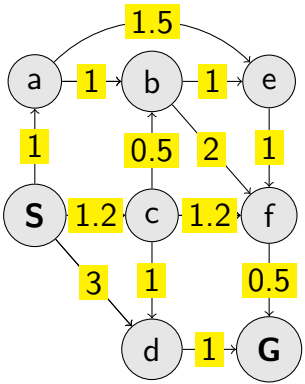

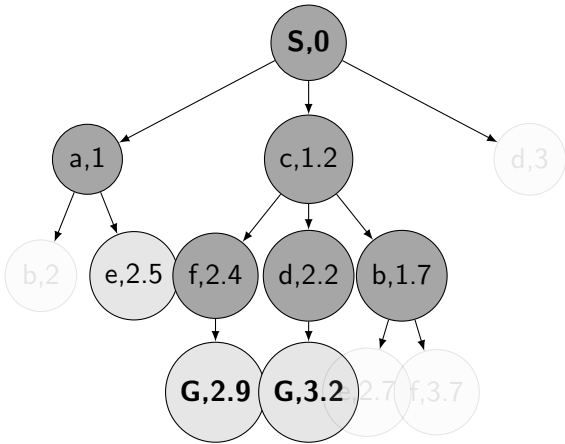

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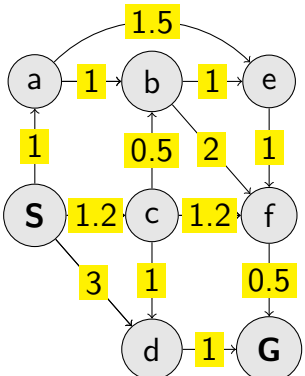

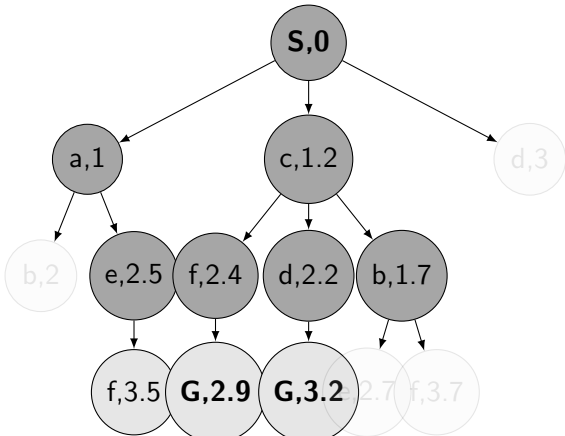

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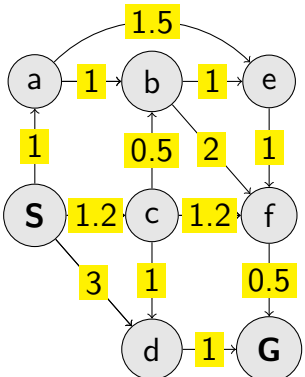

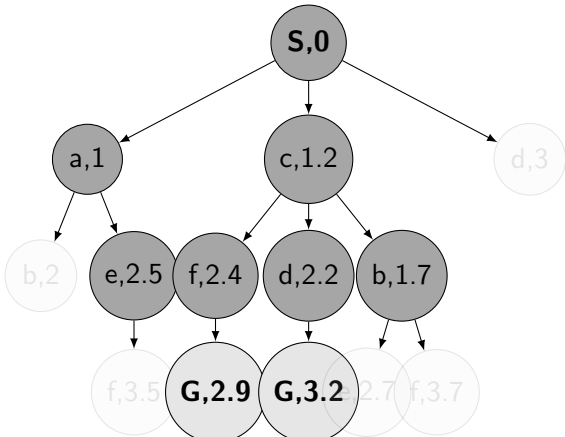

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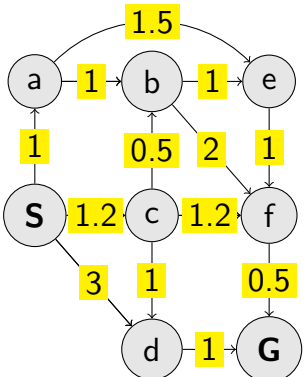

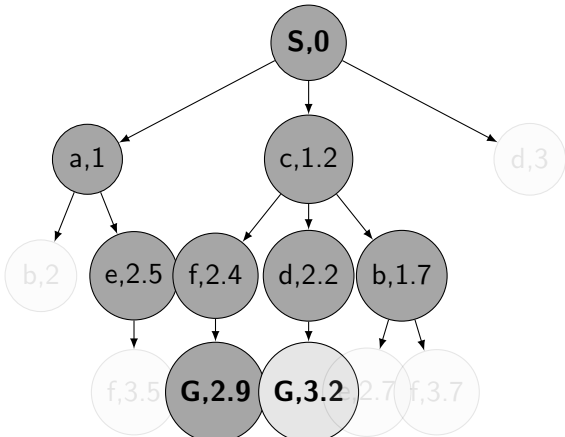

The UCS graph search

function UCS_GRAPH_SEARCH(env) **return** a solution or failure
    node ← env.observe()
    frontier ← priority_queue(node) ▷ path_cost for ordering
    explored ← set()
    while frontier not empty **do**
        node ← frontier.pop()
        if node contains Goal **then return** node ▷ check here!
        end if
        explored.add(node.state)
        child_nodes ← env.expand(node.state)
        for all child_nodes **do**
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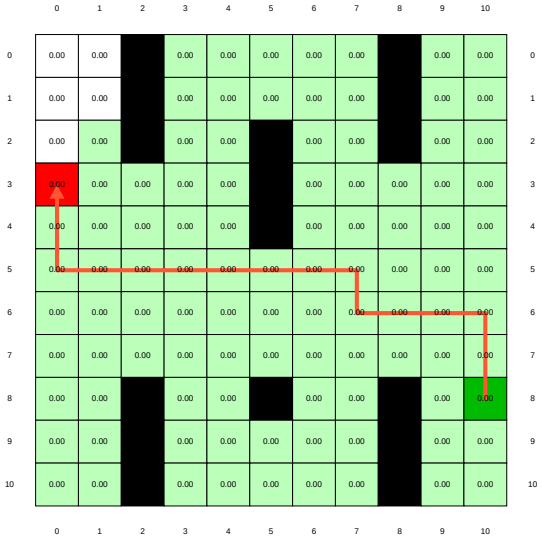
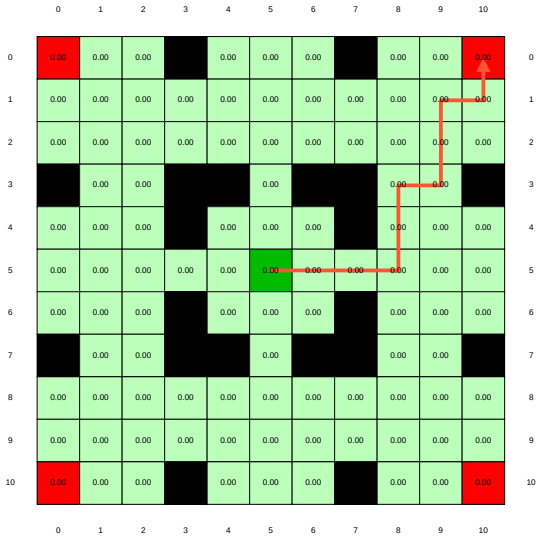
Few examples of search strategies so far



Run the demos.

What is wrong with UCS and other strategies?



Run the demo, or see https://youtu.be/TT5MY8xCgAg

Node selection, take argmin $f(n)$

Selecting next node to expand/visit (pop operation):

$$\texttt{node} \leftarrow \underset{n \in \texttt{frontier}}{\operatorname{argmin}} \; f(n)$$

What is $f(n)$ for DFS, BFS, and UCS?

- DFS:
- BFS:
- UCS:

- $f(n) = n.path\_cost$
- $f(n) = n.depth$
- $f(n) = -n.depth$

The good: (one) frontier as a priority queue
(I.e., priority queue will work universally. Still, stack (LIFO) and queue (FIFO) are (conceptually) the perfect data structures for DFS and BFS, respectively.)
The bad: All the $f(n)$ correspond to the cost from n to the start - only backward cost; cost-to-come (to n).

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Selecting next node to expand/visit (pop operation):

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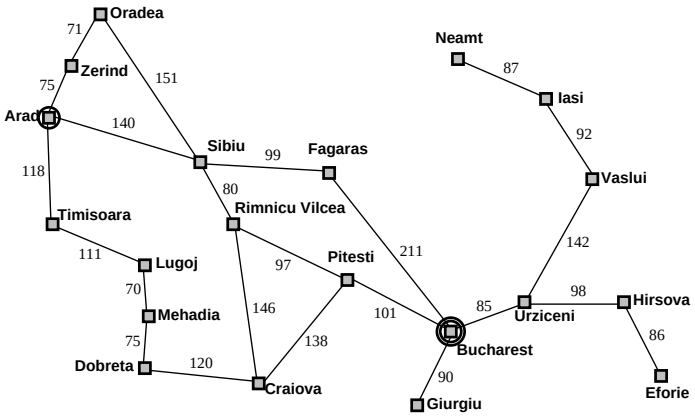
How far are we from the goal cost-to-go ? – Heuristics

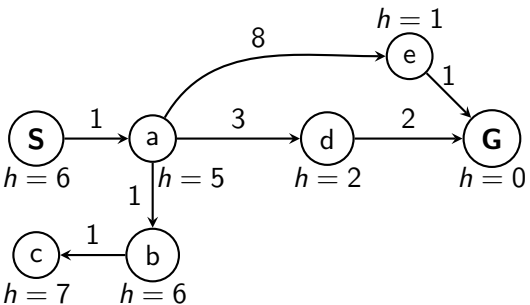
- A function that estimates how close a *state* is to the goal.
- Designed for a particular problem.
- $h(n.state)$ – it is function of the state (attribute of node)
- It is often shortened as $h(n)$ – heuristic value of node n.

Example of heuristics



Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

Greedy, take the $n^* = \operatorname{argmin}_{n \in \texttt{frontier}} h(n)$



What is wrong (and nice) with the Greedy?

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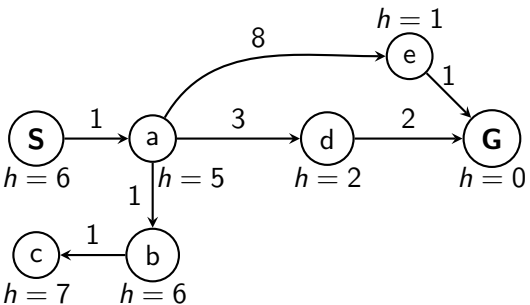

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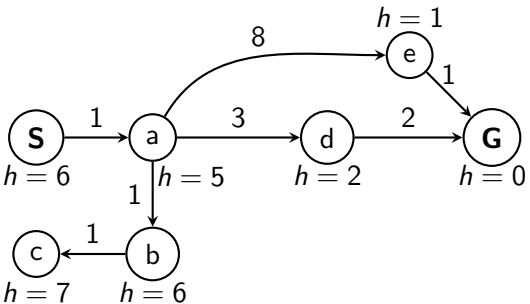
A* combines UCS and Greedy



UCS orders by backward (path) cost $g(n)$
Greedy uses heuristics (goal proximity) $h(n)$

A* orders nodes by: $f(n) = g(n) + h(n)$

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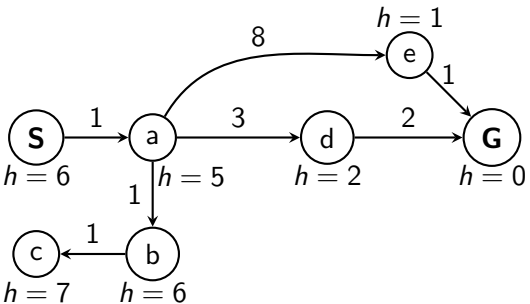

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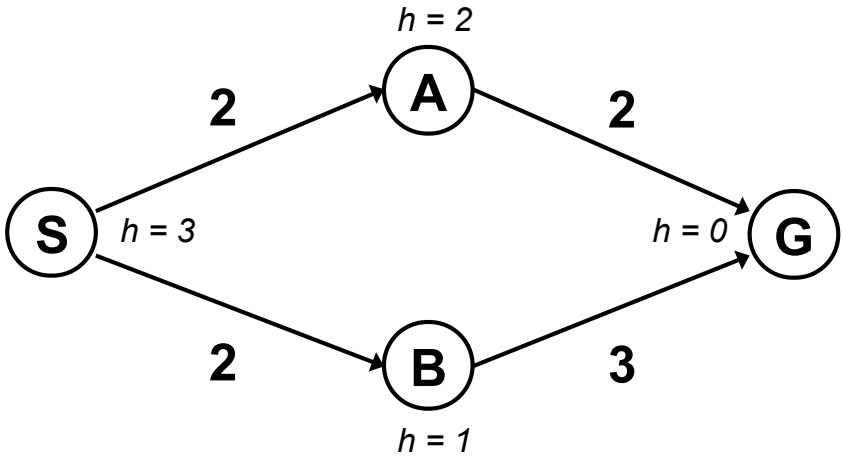
When to stop A*?

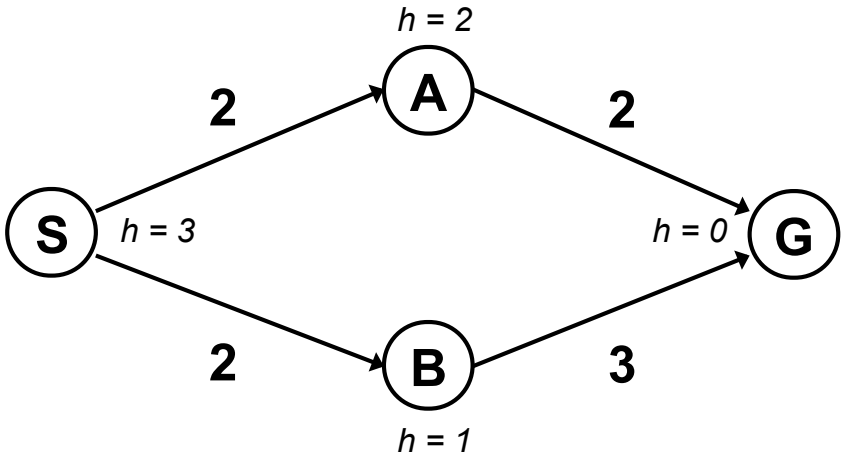


1

When poping G from `frontier`,

[1] Graph example: Dan Klein and Pieter Abbeel

When to stop A*?



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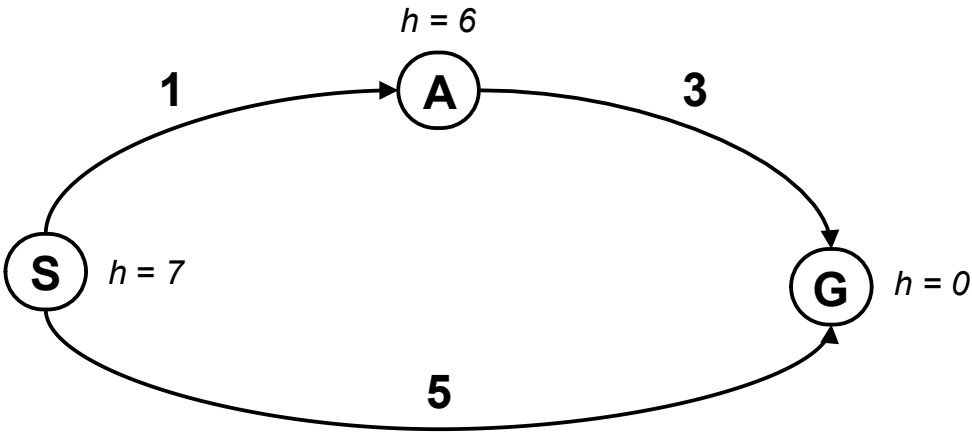
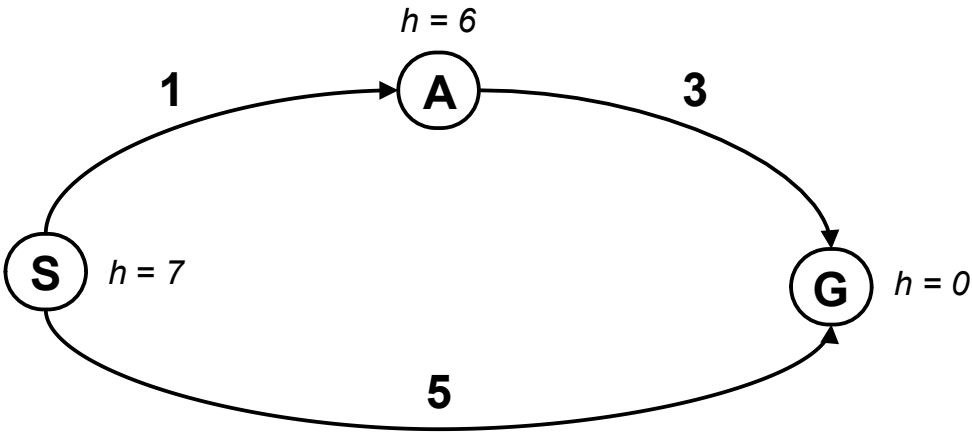


2

What is the problem?

[2] Graph example: Dan Klein and Pieter Abbeel

Is A* optimal?



[2]

What is the problem?

[2] Graph example: Dan Klein and Pieter Abbeel

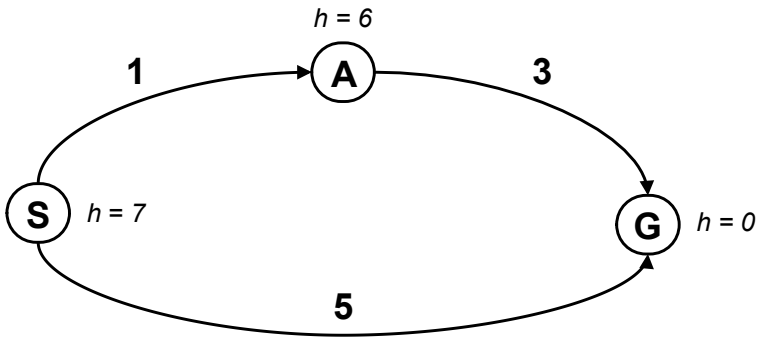
What is the right $h(A)$?

A: $0 \leq h(A) \leq 4$

B: $h(A) \leq 3$

C: $0 \leq h(A) \leq 3$

D: $0 \leq h(A)$



Admissible heuristics

A heuristic function h is admissible if:

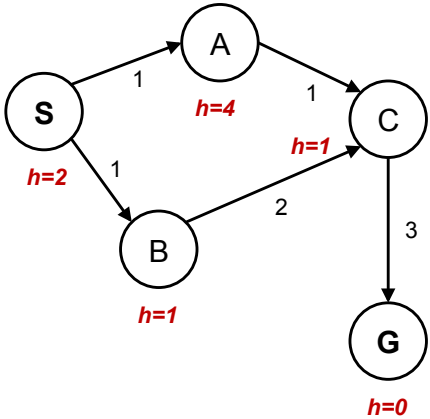
$$\begin{aligned} h(n) &\leq \text{cost}(n.state, \text{Goal}_{\text{nearest}}) \\ h(\text{Goal}) &= 0 \end{aligned}$$

Optimality of A* tree search

A* is optimal if $h(n)$ is admissible.

A* graph search

```
function GRAPH_SEARCH(env)
    frontier.insert(startnode)
    explored ← set()
    while frontier do
        node ← frontier.pop()
        if goal in node then return node
        end if
        child_nodes ← env.expand(node.state)
        explored.add(node.state)
        for all child_nodes do
            if child_node.state not in explored then
                frontier.insert(child_node)
            end if
        end for
    end while
end function
```



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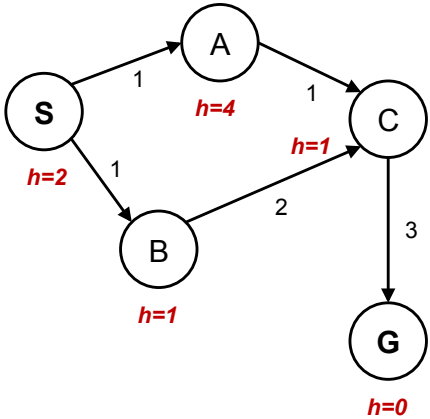


What went wrong?

Graph example: Dan Klein and Pieter Abbeel.

What would be the proper $h(A)$?

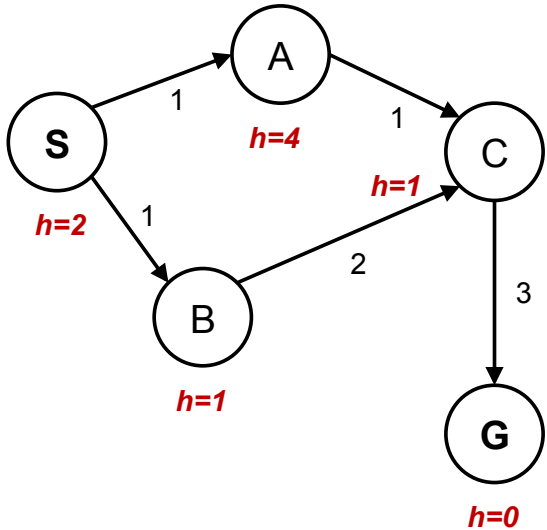
Consider other $h(s)$ fixed.

A: $h(A) = 1$

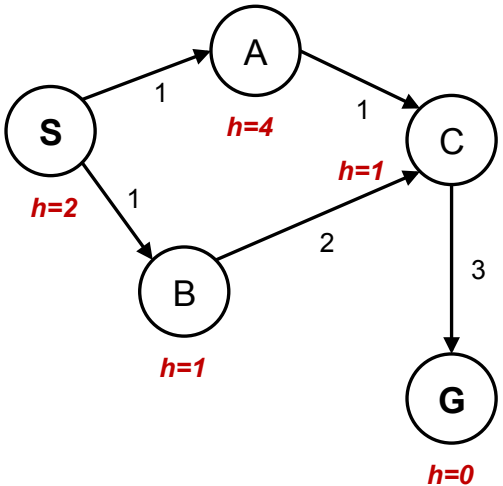
B: $h(A) = 2$

C: $1 \leq h(A) \leq 2$

D: $0 \leq h(A) \leq 1$



Consistent heuristics



Admissible h:
$h(A) \leq$ true cost $A \to G$

Consistent h:
$h(A) - h(C) \leq$ true cost $A \to C$
in general:
$h(n) - h(s) \leq$ true cost $n \to s$ for any pair: node n and its successor s

$f(n) = g(n) + h(n)$ along a path never decreases!

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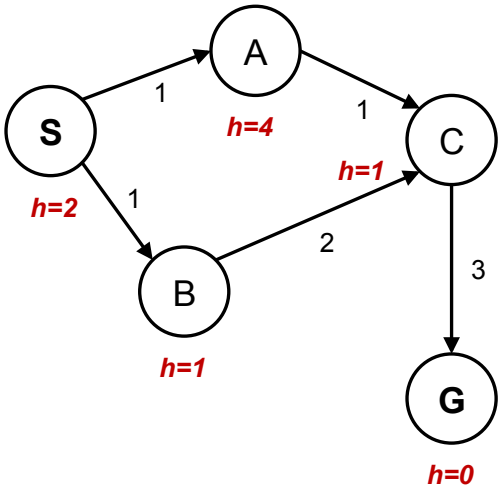

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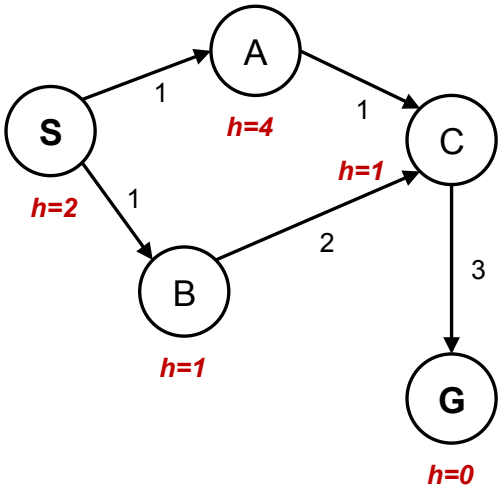

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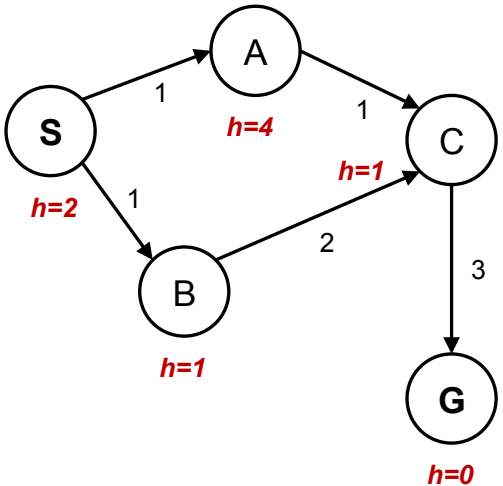

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Optimality of A*

- admissible h for tree search
- consistent h for graph search
- Are all consistent heuristics also admissible?

$h(A) - h(C) \leq \text{cost}(A \to C)$

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Summary

- Graph vs Tree search – avoid repeating expansions
- Effectivness – adding heuristic estimates of cost-to-go
- Not all heuristics are equally good (admissibility, consistence, informativeness)

References, further reading

Some figures from [2]. Chapter 2 in [1] provides a compact/dense intro into search algorithms. (State space) Search algoritmhs are ubiquitous, explanations in many (text)books about Algorithms.
Nice online course from UC Berkeley (CS 188 Into to AI):
http://ai.berkeley.edu/lecture_videos.html Lecture: Informed Search.

[1] Steven M. LaValle.
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[2] Stuart Russell and Peter Norvig.
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