

# Homework 1

## The 2D harmonic oscillator

The classical 2D harmonic oscillator has a potential energy given by

$$V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2). \quad (1)$$

The classical equations of motion has the solution

$$\begin{aligned} x(t) &= A_x \cos(\omega t + \phi_x), \\ y(t) &= A_y \cos(\omega t + \phi_y), \end{aligned} \quad (2)$$

for some constants  $A_x$ ,  $A_y$ ,  $\phi_x$  and  $\phi_y$ .

When we go to the quantum system we find the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2), \quad (3)$$

which we obviously can write simply as the sum of two 1D harmonic oscillator Hamiltonians,  $\hat{H}_x = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$  and  $\hat{H}_y = \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2\hat{y}^2$ . For this reason we can write our energy eigenstate as

$$|\psi\rangle = |\psi_x\rangle \otimes |\psi_y\rangle, \quad (4)$$

and the corresponding energy eigenvalues will just be

$$E = \hbar\omega \left( n_x + \frac{1}{2} \right) + \hbar\omega \left( n_y + \frac{1}{2} \right) = \hbar\omega(n_x + n_y + 1), \quad (5)$$

for  $n_x, n_y$  non-negative integers. The wave function in the position basis is similarly given simply by the product of the two copies of the 1D harmonic oscillator wave functions,

$$\psi_{n_x, n_y}(x, y) = \frac{1}{\sqrt{2^{n_x+n_y}(n_x)!(n_y)!}} \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega(x^2+y^2)}{2\hbar}} H_{n_x} \left( x\sqrt{m\omega/\hbar} \right) H_{n_y} \left( y\sqrt{m\omega/\hbar} \right), \quad (6)$$

where  $H_n$  are the physicist's Hermite polynomials. These results all of course depend on the powerful technique of separation of variables in solving the differential equations, which you are perhaps well familiar with.

### Questions:

1. Plot the classical potential energy (for some values of the constants). What is the total energy of the system?
2. Does  $\hat{H}_x$  and  $\hat{H}_y$  commute? Hint: The commutator for the position and momentum variables is that  $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar$ , while  $[\hat{x}, \hat{y}] = [\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = [\hat{p}_x, \hat{p}_y] = 0$ .
3. Can you construct some operator, using combinations of  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{p}_x$  and  $\hat{p}_y$ , that commutes with the full Hamiltonian  $\hat{H} = \hat{H}_x + \hat{H}_y$ ? If possible, what does this tell you?
4. Plot the probability amplitudes and probability densities for the energy levels  $E = \hbar\omega, 2\hbar\omega, 3\hbar\omega$  and  $4\hbar\omega$ . Compare it with the classical case (look at the turning points) and with the 1D case. What is same, what is different?

## The transmon qubit

Read about the transmon qubit, for example here: <https://qiskit.org/textbook/ch-quantum-hardware/transmon-physics.html>.

Notice the resemblance with the harmonic oscillator. Unfortunately, the Schrödinger equation for the transmon qubit, or the anharmonic oscillator, is not exactly solvable (as was the case for the ordinary quantum harmonic oscillator). Let us make a very crude approximation for the lowest energy wave function.

### Questions:

1. Why is the transmon system more suitable than the ordinary harmonic oscillator for implementing a qubit?