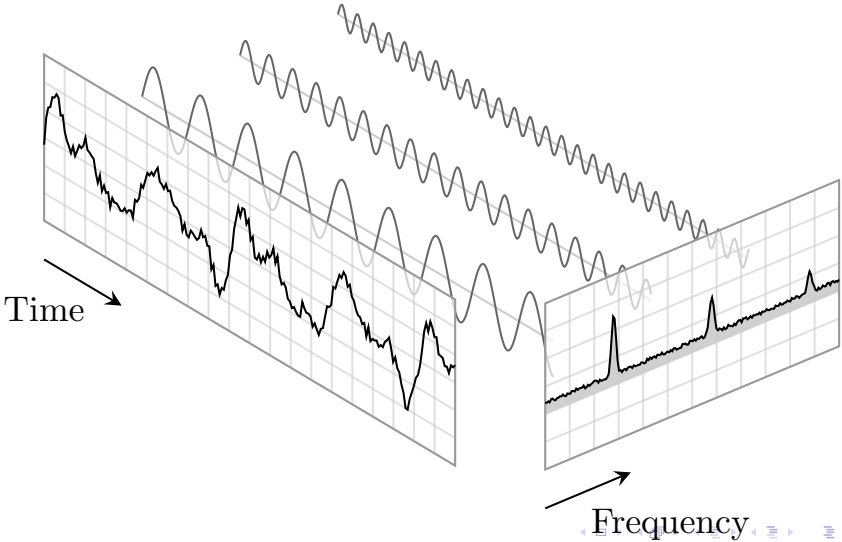
Lecture Topic: Harmonic Analysis 101

Harmonic Analysis as you Know it



Harmonic Analysis as we Need it

Harmonic analysis as we need it requires

- discrete samples
- finite fields
- complex probability amplitudes on the input and output.

The corresponding, perhaps seemingly obscure parts of harmonic analysis have also led to classical breakthroughs, such as multiplication of n-bit integers in time $O(n \log n)$ and multiplication of polynomials over finite fields in the same time. The idea is that we can think of the Fourier transform in a very general way as a function on a group or over a finite field. The only difference between things like the full continuous Fourier transform, the discrete-time Fourier transform and the discrete Fourier transform are then simply the choice of group.

Discrete Fourier Transform

The discrete Fourier transform (DFT) maps an N-vector $\mathbf{x}$ of complex numbers to an N-vector $\mathbf{X}$ of complex numbers:

$$X_k = \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}}, \tag{1.1}$$

up to a normalization $\frac{1}{\sqrt{N}}$. (This is sometimes called the analysis formula.) Let us assume $N = 2^n$ throughout, where n is a constant.

via Discretisation

One way to think of the N-vector $\mathbf{x}$ is to see those as samples of a periodic function with period T, i.e., $f(t) = f(t + T)$. In particular, one would sample f uniformly at points $j\Delta t$, where $\Delta T = T/N$ and $j = 0, 1, ..., N - 1$.

Cyclic groups

Alternatively, one could see discrete Fourier transform as a function on a finite cyclic group $G_q = \{1, g, g^2, \ldots, g^{q-1}\} \cong \mathbb{Z}/q\mathbb{Z}$. A simple representation of the elements of this group is as q^{th} roots of unity, $g^n = \exp\left(\frac{2\pi i n}{q}\right)$, with $n = 0, \ldots, q-1$, where the group multiplication is simply ordinary complex multiplication. By visualizing this group action on the unit circle we can see that it is the rotational symmetry group of the q-polygon.

For any group G, and especially for cyclic groups $\mathbb{Z}_q$, it may be tempting to identify the group with its elements $g \in G$ and consider $f(g)$ only, or to identify the cyclic groups $\mathbb{Z}_q$ with the set $\{0, \ldots, q-1\}$ and modulo q addition. One could do much better, however, if one considers the group's symmetries.

Discrete Fourier Transform

Alternatively, one could see the analysis formula as a matrix equation $\mathbf{X} = \mathbf{F}\mathbf{x}$. Thus, a discrete Fourier transform can be expressed as a so-called Vandermonde matrix (Sylvester, 1867),

$$\mathbf{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 1} & \cdots & \omega_N^{0\cdot (N-1)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 1} & \cdots & \omega_N^{1\cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(N-1)\cdot 0} & \omega_N^{(N-1)\cdot 1} & \cdots & \omega_N^{(N-1)\cdot (N-1)} \end{bmatrix} \tag{1.2}$$

where $\omega_N^{m\cdot n} = e^{-i2\pi mn/N}$ and the mn is the usual product of the integers.

Discrete Fourier Transform

Notice that:

- because ω depends only on the product of frequency m, and position n, the DFT $\mathbf{F}$ is symmetric. Notice that it is also unitary: $\mathbf{F}^{-1} = \mathbf{F}^*$ and $|\det(\mathbf{F})| = 1$.
- $\mathbf{X}$ is the inner product of $\mathbf{x}$ with the m-th row of $\mathbf{F}$. Conversely, $\mathbf{f}$ is a linear combination of the columns of $\mathbf{F}$, where the mth column is weighted by X_m.
- the vectors $u_m = \left[e^{\frac{i2\pi mn}{N}} \;\middle|\; n = 0, 1, \ldots, N-1 \right]^{\mathsf{T}}$ form an orthogonal basis over the set of N-dimensional complex vectors.
- $\mathbf{F}^2$ reverses the input, while $\mathbf{F}^4 = \mathbf{I}$. The eigenvalues satisfy: $\lambda^4 = 1$ and thus are the fourth roots of unity: $+1, -1, +i, or - i$.

The Hadamard Transform

We can define the 1×1 Hadamard transform $H_0 = 1$ as the identity, and then define H_m for $m > 0$ by:

$$H_m = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{bmatrix}.$$

Other than the normalization, the Hadamard matrices are made up of 1 and -1.

Notice that Hadamard

$$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is a discrete Fourier transform; indeed, we have $\omega_1^{0 \cdot 0} = \omega_1^{0 \cdot 1} = \omega_1^{1 \cdot 0} = e^0 = +1$ and $\omega_1^{1 \cdot 1} = e^{-i\pi} = -1$. As a further example, the next Hadamard matrix is

$$H_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

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The z-Transform

If you know the z-transform, notice that the X_k can also be seen as evaluation of the z-transform $X(z) = \sum_{j=0}^{N-1} x_j z^{-j}$ at points ω_N^{-j}, i.e., $X_j = X(z)_{z=\omega_N^{-j}}$.

The Discrete Fourier Transform for $N = 1, 2, 3$

Let us see:

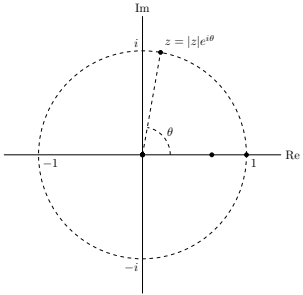
$$\mathbf{F}_0 = H_0 = 1$$

$$\mathbf{F}_1 = H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{F}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{1\cdot 1} & \omega_3^{1\cdot 2} \\ 1 & \omega_3^{2\cdot 1} & \omega_3^{2\cdot 2} \end{bmatrix}$$

The Discrete Fourier Transform for $N = 4$

While the omega notation may obscure the nature of the DFT, see that the column correspond to passes along the unit circle, clockwise, expressed in the corresponding complex number (e.g., 1, $-i$, -1, i for $\mathbf{F}_3$) at varying frequency (e.g., 0, 1, 2, 3 for $\mathbf{F}_3$):



$$\mathbf{F}_3 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

Exercise 1

Exercise

Visualise $\mathbf{F}_3$, $\mathbf{F}_4$ on the unit circle.

The Discrete Fourier Transform for $N = 4$

Let us further consider some simple examples:

$$\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.3}$$

The Discrete Fourier Transform for $N = 4$

$$\frac{1}{\sqrt{4}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{bmatrix}\begin{bmatrix}1\\ -i\\ -1\\ i\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 2\end{bmatrix} \tag{1.4}$$

The Discrete Fourier Transform for $N = 4$

$$\frac{1}{\sqrt{4}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -i & -1 & i\\ 1 & -1 & 1 & -1\\ 1 & i & -1 & -i\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 0\\ -1\\ 0\end{bmatrix} \tag{1.5}$$

Fast Fourier Transform

A straightforward implementation of DFT as a matrix-vector product requires $O(N^2)$ operations. In the so-called fast Fourier transform (Cooley and Tukey, 1965), one requires only $O(N \log_2 N) = O(2^n n)$ operations. There are a number of variants, all based on the divide-and-conquer approach. As we assume $N = 2^n$, we will present a variant known as the radix-2 decimation in time (DIT) algorithm.

Fast Fourier Transform

This speedup is achieved by this variant of the divide-and-conquer approach, where we consider subsets of the initial sequence, take the DFT of these subsequences, and reconstruct the DFT of the original sequence from the results on the subsequences. One option is based on the following insight:

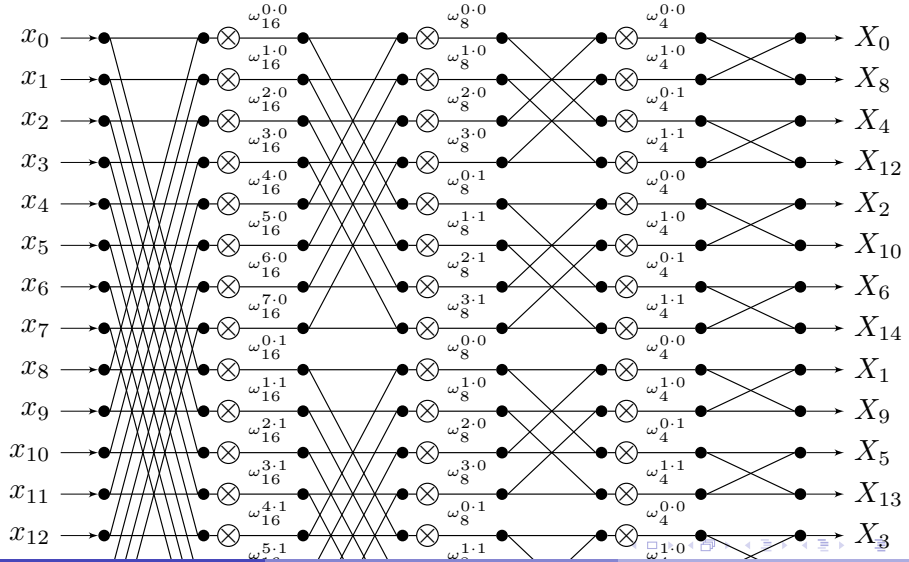
$$X_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}} \tag{2.1}$$

$$= \frac{1}{\sqrt{N}} \left(\sum_{\text{even } j} x_j \cdot e^{\frac{2\pi jki}{N}} + \sum_{\text{odd } j} x_j \cdot e^{\frac{2\pi (j-1)ki}{N}} \right) \tag{2.2}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{N/2}} \sum_{\text{even } j} x_j \cdot e^{\frac{2\pi (j/2)ki}{N/2}} \right) + \left(\frac{1}{\sqrt{N/2}} \sum_{\text{odd } j} x_j \cdot e^{\frac{2\pi (j-1)/2ki}{N/2}} \right) \tag{2.3}$$

The divide-and-conquer approach can be illustrated on $N = 16$ with the following cartoon.

The Discrete Fourier Transform for $N = 16$



The z-Transform

If you know the z-transform, you should see that $X(z) = \sum_{j=0}^{N-1} x_j z^{-j} = \sum_{l=0}^{r-1} \sum_{j \in \mathbf{I}_l} x_j z^{-j}$ for some partition $\mathbf{I}$ of $\{0, 1, \ldots, N-1\}$ into r subsets, and that one can also normalise the terms. This way, one can define a variety of recursions similar to the one above, as long as the subset are chosen to be similar to the initial sequence in terms of their periodicity.

Fast Fourier Transform as a Factorization

Alternatively, one could see the Fast Fourier Transform as a certain matrix factorization. This is both important to understand FFT, but also to understand the QFT later. In particular, the $2^n \times 2^n$ DFT matrix $\mathbf{F}_n$ can be factored as:

$$\mathbf{F} = \mathbf{P}_n \mathbf{A}_n^{(0)} \mathbf{A}_n^{(1)} \cdots \mathbf{A}_n^{(n-1)}, \tag{2.4}$$

where,

- $\mathbf{P}_n$ is some permutation matrix
- $\mathbf{A}_n^{(k)} = \mathbf{I}_{n-k-1} \otimes \mathbf{B}_{k+1}$,
- $\mathbf{B}_{k+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_k & \mathbf{I}_k \\ \mathbf{\Omega}_k & -\mathbf{\Omega}_k \end{bmatrix}$ with $\mathbf{I}_k$ the $k \times k$ identity matrix
- $\mathbf{\Omega}_k \coloneqq \mathbf{\Omega}_{2^k}$ is a $2^k \times 2^k$ diagonal matrix:

$$\mathbf{\Omega}_{2^k} \coloneqq \begin{bmatrix} \omega_{2^{k+1}}^0 & & & \\ & \omega_{2^{k+1}}^1 & & \\ & & \ddots & \\ & & & \omega_{2^{k+1}}^{2^k-1} \end{bmatrix}, \tag{2.5}$$

Quantum Fourier Transform

In general, $\mathbf{F}$ is an $N \times N$ unitary matrix, and thus we can implement it on a quantum computer as an n-qubit unitary for $N = 2^n$. As such, it maps an N-dimensional vector of amplitudes to an N-dimensional vector of amplitudes. This is called the quantum Fourier transform (QFT).

Shor, Kitaev presented the first polynomial, $O(n^2)$ quantum algorithms for QFT over certain finite fields and arbitrary finite Abelian groups, respectively. This is exponentially faster than the classical fast Fourier transform, which takes $O(N \log N)$ steps.

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Quantum Fourier Transform

Recall that the DFT is:

$$X_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}}.$$

In contrast, the QFT on an orthonormal basis $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ is a linear operator:

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi jki}{N}} |k\rangle.$$

Quantum Fourier Transform

An alternative representation of the QFT utilizes the *product form*:

$$|j_1, j_2, \ldots, j_n\rangle \rightarrow \frac{\left(|0\rangle + e^{2\pi i\, 0.j_n}\,|1\rangle\right)\left(|0\rangle + e^{2\pi i\, 0.j_{n-1}j_n}\,|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 \cdots j_n}\,|1\rangle\right)}{2^{n/2}},$$

where $|j_1, j_2, \ldots, j_n\rangle$ is a binary representation of a basis state j and $0.j_1 j_2 \cdots j_n$ is a notation for binary fraction $j_1/2 + j_2/4 \cdots j_n/2^{n+1}$. This is actually easy enough to derive:

Derivation of the QFT

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}} |k\rangle \tag{3.1}$$

$$\frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi jki}{2^n}} |k\rangle \tag{3.2}$$

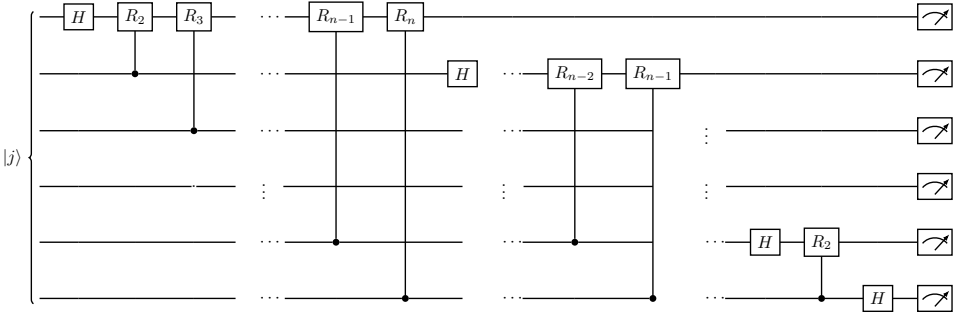
$$\frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi j(\sum_{l=1}^{n} k_l 2^{-l})i} |k_1 k_2 \ldots k_n\rangle \tag{3.3}$$

$$\frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi j k_l 2^{-l} i} |k_l\rangle \tag{3.4}$$

$$\frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi j k_l 2^{-l} i} |k_l\rangle \right] \tag{3.5}$$

Quantum Circuit for QFT

A simplistic illustration of the quantum circuit for QFT, omitting swaps at the end and normalization:



Let us consider the workings of the circuit step by step. The key to its understanding is the phase kickback, which we have seen earlier.

Working of the QFT

Applying the Hadamard gate to the first qubit of the input state $|j_1 \dots j_n\rangle$ gives

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1} |1\rangle \right) |j_2 \dots j_n\rangle \tag{3.7}$$

since $e^{2\pi i \ 0.j_1}$ equals $+1$ when $j_1 = 0$ and equals -1 when $j_1 = 1$.
We define a unitary gate R_k as

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix} \tag{3.8}$$

The controlled-R_2 gate applied on the first qubit, conditional on j_2, now gives

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2} |1\rangle \right) |j_2 \dots j_n\rangle \tag{3.9}$$

Applying further the controlled-R_3, R_4 ... R_n gates, conditional on j_3, j_4 etc., we get

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \dots j_n} |1\rangle \right) |j_2 \dots j_n\rangle \tag{3.10}$$

Working of the QFT

Next we perform a similar procedure onto the second qubit. The Hadamard gate produces the state

$$\frac{1}{2^{2/2}}\left(|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i\ 0.j_2}|1\rangle\right)|j_3 \ldots j_n\rangle \tag{3.11}$$

and the controlled-R_2 through R_{n-1} gates yield the state

$$\frac{1}{2^{2/2}}\left(|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i\ 0.j_2 \ldots j_n}|1\rangle\right)|j_3 \ldots j_n\rangle \tag{3.12}$$

We continue this procedure for each qubit, obtaining a final state

$$\frac{1}{2^{n/2}}\left(|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i\ 0.j_2 \ldots j_n}|1\rangle\right)\ldots\left(|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle\right) \tag{3.13}$$

Eventually, we use the *SWAP* operations to reverse the order of the qubits to obtain the state in the desired product form

$$\frac{1}{2^{n/2}}\left(|0\rangle + e^{2\pi i\ 0.j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i\ 0.j_{n-1} j_n}|1\rangle\right)\ldots\left(|0\rangle + e^{2\pi i\ 0.j_1 j_2 \ldots j_n}|1\rangle\right) \tag{3.14}$$

Approximate Quantum Circuits for QFT

Above, we have clearly used at most $O(n^2)$ gates.

Cleve and Watrous, Hales and Hallgren, and others improved this to $O(n \log n)$ depth, if one allows for some error. This is based on the realization that R_s for $s \ll \log n$ are very close to the identity and can be omitted.

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