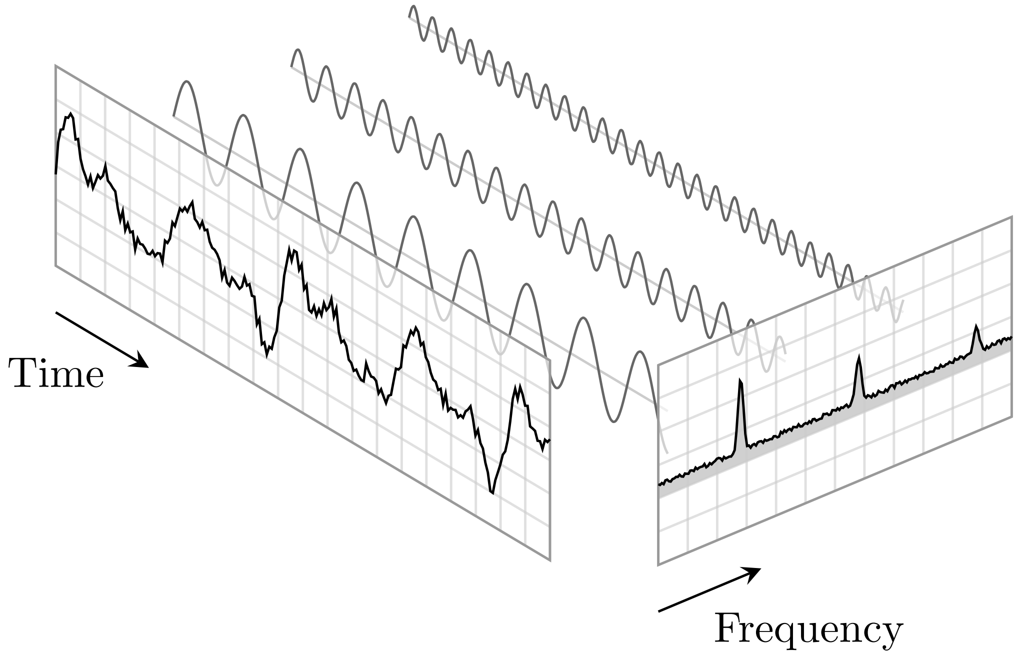
CHAPTER 5

Harmonic Analysis 101

In this lecture, we introduce the quantum Fourier transform, which is $O((\log N)^2)$, i.e., exponentially faster than the classical fast Fourier transform in $O(N \log N)$. As is perhaps familiar, the simple intuition for the classical Fourier transform is basically as a change of basis, or perhaps a duality transformation. Typically we think of it as taking us from the original function domain to its corresponding frequency domain, where certain properties, such as which frequencies are present in a signal, are easier to analyze. This is directly translated to the quantum Fourier transform, which is a change of basis from the computational basis to the Fourier basis. For example, as we will see below, in the simplest case of one qubit the quantum Fourier transform is simply the Hadamard gate. Which we have already seen is a change of basis from the computational basis to the Hadamard basis $|\pm\rangle$. Before we get into the quantum Fourier transform, we will begin with the classical case, and in particular try to explain some harmonic analysis in general.



Harmonic analysis as we need it requires discrete samples and finite fields. The corresponding, perhaps seemingly obscure parts of harmonic analysis have also led to classical breakthroughs, such as multiplication of n-bit integers in time $O(n \log n)$ [Harvey and Van Der Hoeven, 2021] and multiplication of polynomials over finite fields [Harvey and Van Der Hoeven, 2022] in the same time. The idea is that we can think of the Fourier transform in a very general way as a function on a group or over a finite field. The only difference between things like the full continuous Fourier transform, the discrete-time Fourier transform and the discrete Fourier transform are then simply the choice of group. For a very nice introduction to Harmonic analysis on finite groups, see Peyré [2020].

1. Discrete Fourier Transform

The discrete Fourier transform (DFT) maps an N-vector $\mathbf{x}$ of complex numbers to an N-vector $\mathbf{X}$ of complex numbers:

$$X_k = \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}}, \tag{5.1}$$

up to a normalization $\frac{1}{\sqrt{N}}$. (This is sometimes called the analysis formula.) Let us assume $N = 2^n$ throughout, where n is a constant.

One way to think of the N-vector $\mathbf{x}$ is to see those as samples of a periodic function with period T, i.e., $f(t) = f(t+T)$. In particular, one would sample f uniformly at points $j\Delta t$, where $\Delta T = T/N$ and $j = 0, 1, ..., N-1$.

Alternatively, one could see discrete Fourier transform as a function on a finite cyclic group $G_q = \{1, g, g^2, \ldots, g^{q-1}\} \cong \mathbb{Z}/q\mathbb{Z}$. A simple representation of the elements of this group is as q^{th} roots of unity, $g^n = \exp\left(\frac{2\pi in}{q}\right)$, with $n = 0, \ldots, q-1$, where the group multiplication is simply ordinary complex multiplication. By visualizing this group action on the unit circle we can see that it is the rotational symmetry group of the q-polygon.

For any group G, and especially for cyclic groups $\mathbb{Z}_q$, it may be tempting to identify the group with its elements $g \in G$ and consider $f(g)$ only, or to identify the cyclic groups $\mathbb{Z}_q$ with the set $\{0, \ldots, q-1\}$ and modulo q addition. One could do much better, however, if one considers the group's symmetries.

Alternatively, one could see the analysis formula (5.1) as a matrix equation $\mathbf{X} = \mathbf{F}\mathbf{x}$. Thus, a discrete Fourier transform can be expressed as a so-called Vandermonde matrix (Sylvester, 1867),

$$\mathbf{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 1} & \cdots & \omega_N^{0\cdot(N-1)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 1} & \cdots & \omega_N^{1\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(N-1)\cdot 0} & \omega_N^{(N-1)\cdot 1} & \cdots & \omega_N^{(N-1)\cdot(N-1)} \end{bmatrix} \tag{5.2}$$

where $\omega_N^{m\cdot n} = e^{-i2\pi mn/N}$ and the mn is the usual product of the integers.

Notice that:

- because ω depends only on the product of frequency m, and position n, the DFT $\mathbf{F}$ is symmetric. Notice that it is also unitary: $\mathbf{F}^{-1} = \mathbf{F}^*$ and $|\det(\mathbf{F})| = 1$.
- $\mathbf{X}$ is the inner product of $\mathbf{x}$ with the m-th row of $\mathbf{F}$. Conversely, $\mathbf{f}$ is a linear combination of the columns of $\mathbf{F}$, where the mth column is weighted by X_m.
- the vectors $u_m = \left[e^{\frac{i2\pi mn}{N}} \;\middle|\; n = 0, 1, \ldots, N-1\right]^{\mathsf{T}}$ form an orthogonal basis over the set of N-dimensional complex vectors.
- $\mathbf{F}^2$ reverses the input, while $\mathbf{F}^4 = \mathbf{I}$. The eigenvalues satisfy: $\lambda^4 = 1$ and thus are the fourth roots of unity: $+1, -1, +i, or - i$.

1.1. The Hadamard Transform.

We can define the 1×1 Hadamard transform $H_0 = 1$ as the identity, and then define H_m for $m > 0$ by:

$$H_m = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{bmatrix}.$$

Other than the normalization, the Hadamard matrices are made up of 1 and -1. Notice that Hadamard

$$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is a discrete Fourier transform; indeed, we have $\omega_1^{0\cdot 0} = \omega_1^{0\cdot 1} = \omega_1^{1\cdot 0} = e^0 = +1$ and $\omega_1^{1\cdot 1} = e^{-i\pi} = -1$. As a further example, the next Hadamard matrix is

$$H_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Note that this is not a DFT as defined by (5.2).

Notice that classically, we can compute the fast Hadamard transform algorithm in $O(n \log n)$ while performing only sign-flips. Quantumly, the Hadamard transform can be computed in time $O(1)$, in many commonly used gate sets.

1.2. The z-Transform. If you know the z-transform, notice that the X_k can also be seen as evaluation of the z-transform $X(z) = \sum_{j=0}^{N-1} x_j z^{-j}$ at points ω_N^{-j}, i.e., $X_j = X(z)_{z=\omega_N^{-j}}$.

1.3. Examples of Discrete Fourier Transform. Let us see:

$$\mathbf{F}_0 = H_0 = 1$$

$$\mathbf{F}_1 = H_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{F}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^{1\cdot 1} & \omega_3^{1\cdot 2} \\ 1 & \omega_3^{2\cdot 1} & \omega_3^{2\cdot 2} \end{bmatrix}$$

While the omega notation may obscure the nature of the DFT, see that the column correspond to passes along the unit circle, clockwise, expressed in the corresponding complex number (e.g., 1, $-i$, -1, i for $\mathbf{F}_3$) at varying frequency (e.g., 0, 1, 2, 3 for $\mathbf{F}_3$):

$$\mathbf{F}_3 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

Exercise 5.1. Visualise $\mathbf{F}_3$, $\mathbf{F}_4$ on the unit circle.

Let us further consider some simple examples:

$$\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.3}$$

$$\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \tag{5.4}$$

$$\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{5.5}$$

2. Fast Fourier Transform

2.1. The Many Fast Fourier Transforms. A straightforward implementation of DFT as a matrix-vector product requires $O(N^2)$ operations. In the so-called fast Fourier transform (Cooley and Tukey, 1965), one requires only $O(N \log_2 N) = O(2^n n)$ operations. There are a number of variants, all based on the divide-and-conquer approach. As we assume $N = 2^n$, we will present a variant known as the radix-2 decimation in time (DIT) algorithm.

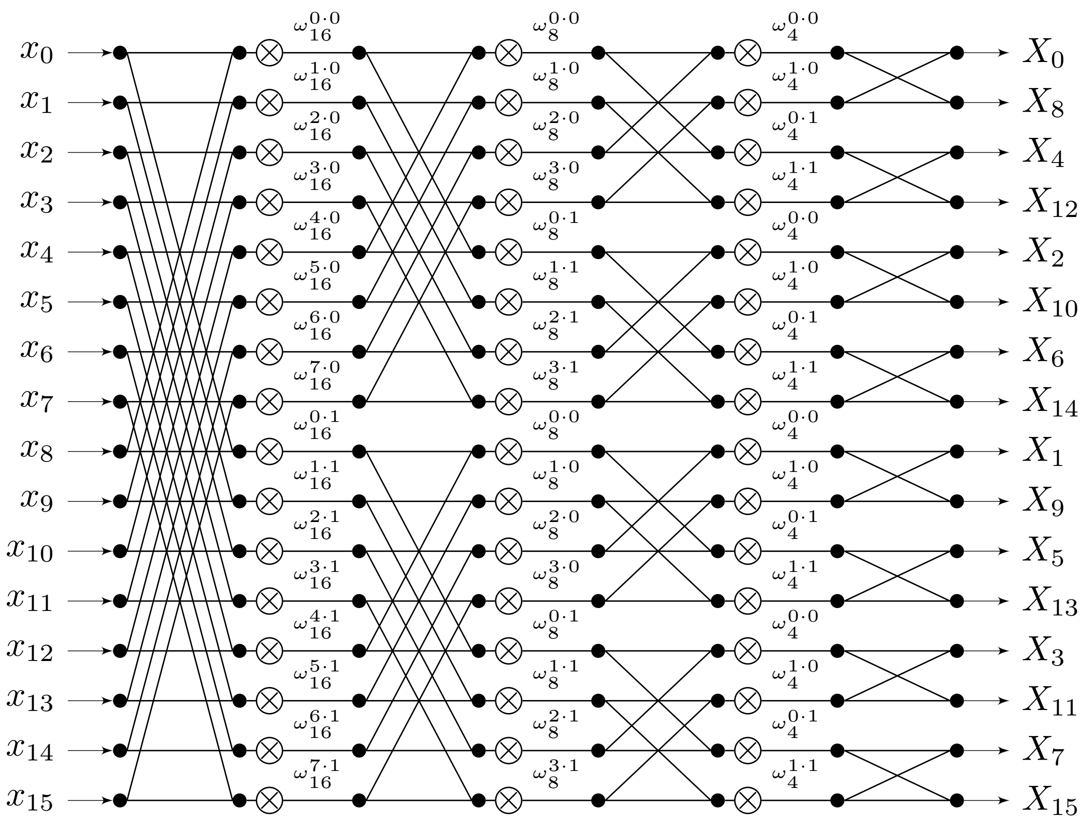
This speedup is achieved by this variant of the divide-and-conquer approach, where we consider subsets of the initial sequence, take the DFT of these subsequences, and reconstruct the DFT of the original sequence from the results on the subsequences. One option is based on the following insight:

$$X_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}} \tag{5.6}$$

$$= \frac{1}{\sqrt{N}} \left(\sum_{\text{even } j} x_j \cdot e^{\frac{2\pi jki}{N}} + \sum_{\text{odd } j} x_j \cdot e^{\frac{2\pi (j-1)ki}{N}} \right) \tag{5.7}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{N/2}} \sum_{\text{even } j} x_j \cdot e^{\frac{2\pi (j/2)ki}{N/2}} \right) + \left(\frac{1}{\sqrt{N/2}} \sum_{\text{odd } j} x_j \cdot e^{\frac{2\pi (j-1)/2ki}{N/2}} \right) \tag{5.8}$$

The divide-and-conquer approach can be illustrated on $N = 16$ with the following cartoon.



If you know the z-transform, you should see that $X(z) = \sum_{j=0}^{N-1} x_j z^{-j} = \sum_{l=0}^{r-1} \sum_{j \in \mathbf{I}_l} x_j z^{-j}$ for some partition $\mathbf{I}$ of $\{0, 1, \ldots, N-1\}$ into r subsets, and that one can also normalise the terms. This way, one can define a variety of recursions similar to the one above, as long as the subset are chosen to be similar to the initial sequence in terms of their periodicity. This is very nicely explained in Duhamel and Vetterli [1990].

2.2. Fast Fourier Transform as a Factorization. Alternatively, Camps et al. [2021] sees the Fast Fourier Transform as a certain matrix factorization. This is both important to understand FFT, but also to understand the QFT later. In particular, the $2^n \times 2^n$ DFT matrix $\mathbf{F}_n$ can be factored as:

$$\mathbf{F} = \mathbf{P}_n \mathbf{A}_n^{(0)} \mathbf{A}_n^{(1)} \cdots \mathbf{A}_n^{(n-1)}, \tag{5.9}$$

where,

- $\mathbf{P}_n$ is some permutation matrix
- $\mathbf{A}_n^{(k)} = \mathbf{I}_{n-k-1} \otimes \mathbf{B}_{k+1}$,
- $\mathbf{B}_{k+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_k & \mathbf{I}_k \\ \mathbf{\Omega}_k & -\mathbf{\Omega}_k \end{bmatrix}$ with $\mathbf{I}_k$ the $k \times k$ identity matrix
- $\mathbf{\Omega}_k \coloneqq \mathbf{\Omega}_{2^k}$ is a $2^k \times 2^k$ diagonal matrix:

$$\mathbf{\Omega}_{2^k} \coloneqq \begin{bmatrix} \omega_{2^{k+1}}^0 & & & \\ & \omega_{2^{k+1}}^1 & & \\ & & \ddots & \\ & & & \omega_{2^{k+1}}^{2^k-1} \end{bmatrix}, \tag{5.10}$$

  where $\omega_{2^{k+1}}$ is $e^{\frac{-2\pi i}{2^{k+1}}}$ as before.

Notice that each matrix $\mathbf{A}_n^{(i)}$ has two non-zero elements on every row. Consequently, the matrix-vector product $\mathbf{A}_n^{(i)}\mathbf{x}$ can be computed in $O(2^n)$ operations, resulting in $O(2^n n)$ operations, when one includes the permutation.

3. Quantum Fourier Transform

In general, $\mathbf{F}$ is an $N \times N$ unitary matrix, and thus we can implement it on a quantum computer as an n-qubit unitary for $N = 2^n$. As such, it maps an N-dimensional vector of amplitudes to an N-dimensional vector of amplitudes. This is called the quantum Fourier transform (QFT). Shor [1994], Kitaev [1995] presented the first polynomial, $O(n^2)$ quantum algorithms for QFT over certain finite fields and arbitrary finite Abelian groups, respectively. This is exponentially faster than the classical fast Fourier transform, which takes $O(N \log N)$ steps.

Recall that the DFT is:

$$X_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}}.$$

In contrast, the QFT on an orthonormal basis $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ is a linear operator:

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi jki}{N}} |k\rangle.$$

An alternative representation of the QFT utilizes the *product form*:

$$|j_1, j_2, \ldots, j_n\rangle \to \frac{\left(|0\rangle + e^{2\pi i\, 0.j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i\, 0.j_{n-1}j_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 \cdots j_n}|1\rangle\right)}{2^{n/2}},$$

where $|j_1, j_2, \ldots, j_n\rangle$ is a binary representation of a basis state j and $0.j_1 j_2 \cdots j_n$ is a notation for binary fraction $j_1/2 + j_2/4 \cdots j_n/2^{n+1}$. This is actually easy enough to derive:

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_j \cdot e^{\frac{2\pi jki}{N}} |k\rangle \tag{5.11}$$

$$\frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi jki}{2^n}} |k\rangle \tag{5.12}$$

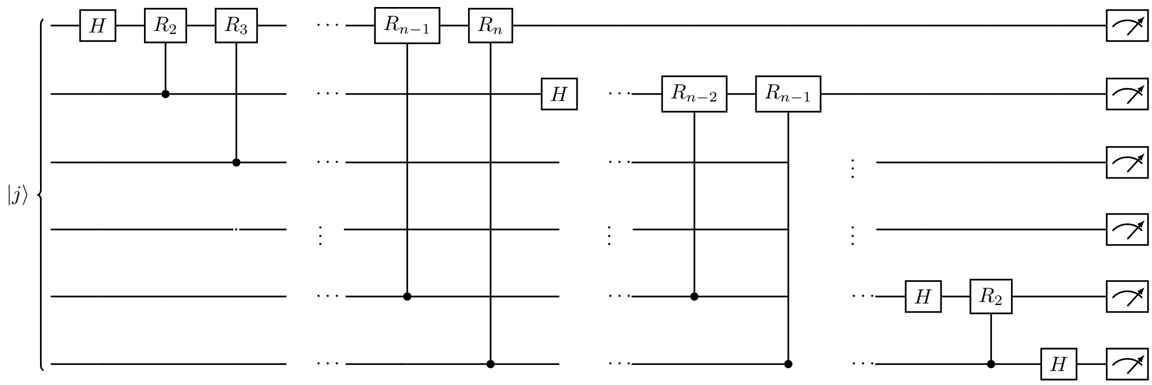
$$\frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi j(\sum_{l=1}^{n} k_l 2^{-l})i} |k_1 k_2 \ldots k_n\rangle \tag{5.13}$$

$$\frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi j k_l 2^{-l} i} |k_l\rangle \tag{5.14}$$

$$\frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi j k_l 2^{-l} i} |k_l\rangle \right] \tag{5.15}$$

$$\frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi j 2^{-l} i} |1\rangle \right] \tag{5.16}$$

A simplistic illustration of the quantum circuit for QFT, omitting swaps at the end and normalization:



Its derivation is very nicely given in Camps et al. [2021], using the framework of matrix decompositions above. Instead of a proper derivation, let us consider the workings of the circuit step by step. The key to its understanding is the phase kickback, which we have seen earlier.

Applying the Hadamard gate to the first qubit of the input state $|j_1 \ldots j_n\rangle$ gives

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1} |1\rangle \right) |j_2 \ldots j_n\rangle \tag{5.17}$$

since $e^{2\pi i \ 0.j_1}$ equals $+1$ when $j_1 = 0$ and equals -1 when $j_1 = 1$. We define a unitary gate R_k as

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix} \tag{5.18}$$

The controlled-R_2 gate applied on the first qubit, conditional on j_2, now gives

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2} |1\rangle \right) |j_2 \ldots j_n\rangle \tag{5.19}$$

Applying further the controlled-R_3, R_4 ... R_n gates, conditional on j_3, j_4 etc., we get

$$\frac{1}{2^{1/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \ldots j_n} |1\rangle \right) |j_2 \ldots j_n\rangle \tag{5.20}$$

Next we perform a similar procedure onto the second qubit. The Hadamard gate produces the state

$$\frac{1}{2^{2/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \ldots j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i \ 0.j_2} |1\rangle \right) |j_3 \ldots j_n\rangle \tag{5.21}$$

and the controlled-R_2 through R_{n-1} gates yield the state

$$\frac{1}{2^{2/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \ldots j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i \ 0.j_2 \ldots j_n} |1\rangle \right) |j_3 \ldots j_n\rangle \tag{5.22}$$

We continue this procedure for each qubit, obtaining a final state

$$\frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \ldots j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i \ 0.j_2 \ldots j_n} |1\rangle \right) \ldots \left(|0\rangle + e^{2\pi i \ 0.j_n} |1\rangle \right) \tag{5.23}$$

Eventually, we use the $SWAP$ operations to reverse the order of the qubits to obtain the state in the desired product form

$$\frac{1}{2^{n/2}} \left(|0\rangle + e^{2\pi i \ 0.j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i \ 0.j_{n-1} j_n} |1\rangle \right) \ldots \left(|0\rangle + e^{2\pi i \ 0.j_1 j_2 \ldots j_n} |1\rangle \right) \tag{5.24}$$

Thus, we have clearly used at most $O(n^2)$ gates.

3.1. Even Faster QFT. Cleve and Watrous [2000], Hales and Hallgren [2000], and others improved this to $O(n \log n)$ depth, if one allows for some error. This is based on the realization that R_s for $s \ll \log n$ are very close to the identity and can be omitted.

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