# Iterative Closest Point SLAM

Karel Zimmermann

# Absolute orientation problem in **SE(2)**

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Can be always zero by appropriate choice of $\mathbf{t}$

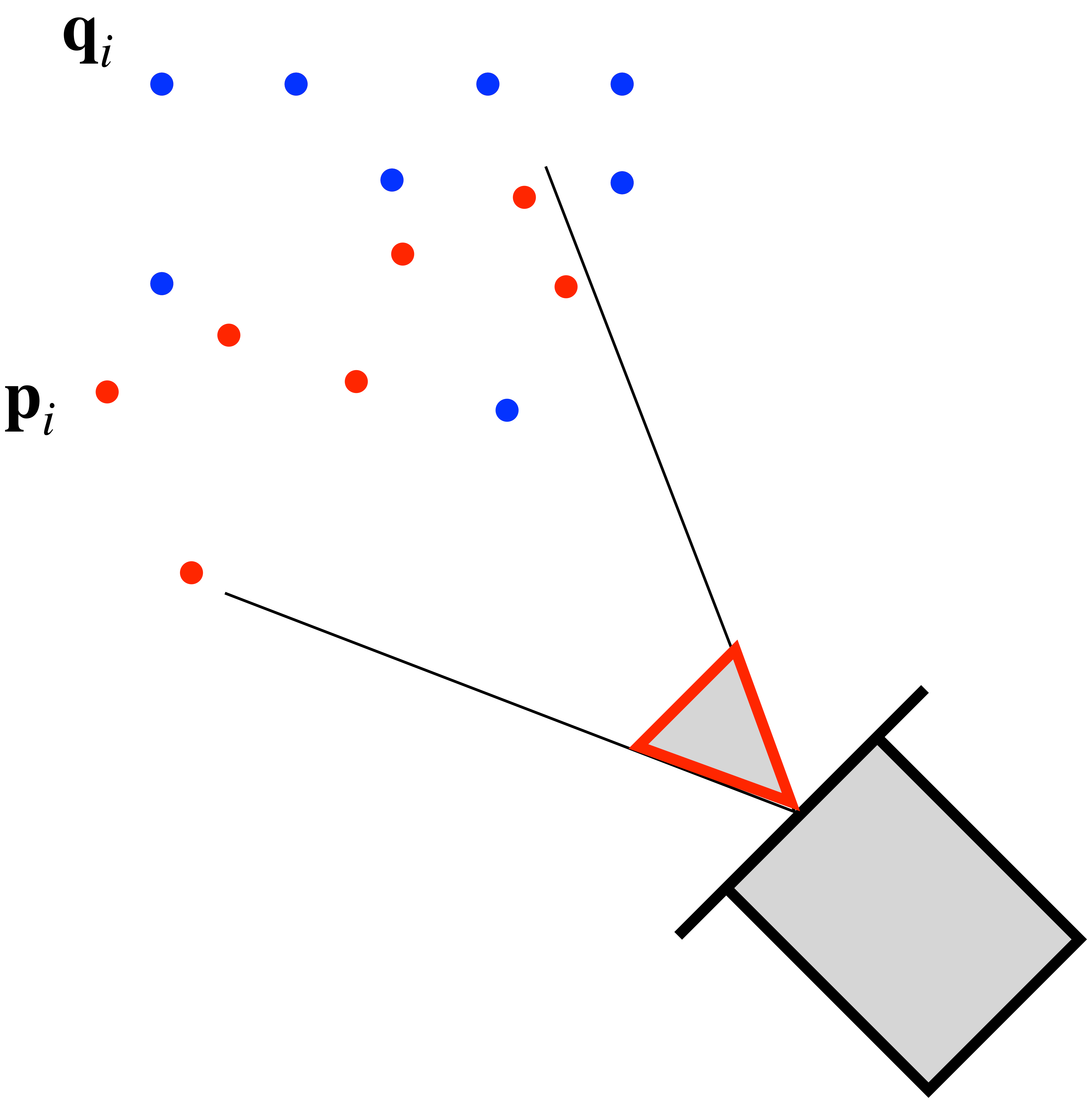
Substitution: $$\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$$

Solution: $$\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'^{\top}_i$$ … covariance matrix

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 = \arctan\left(\frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}}\right)$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star} \tilde{\mathbf{p}}$$

# Pose from **known** correspondences

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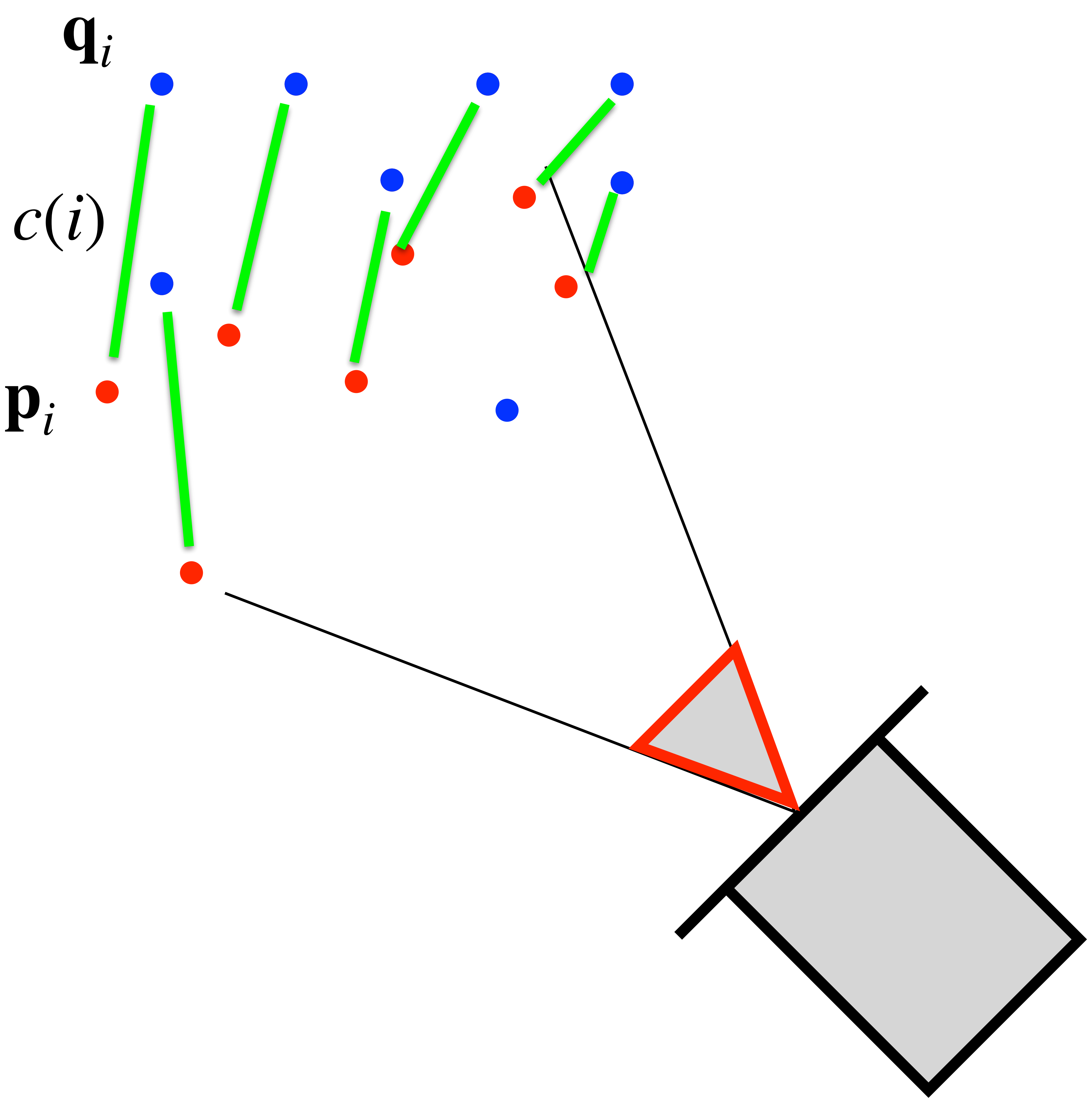


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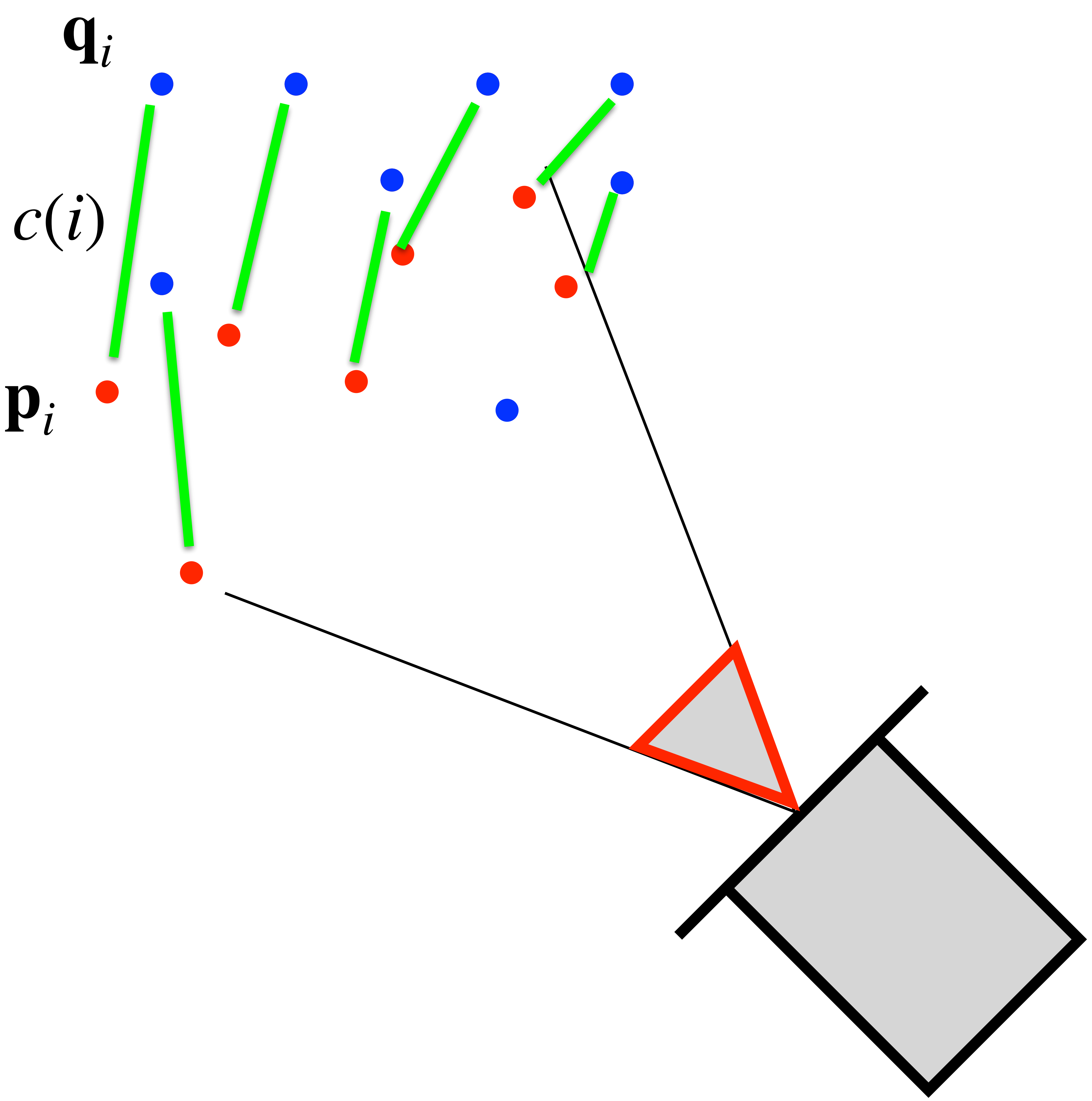


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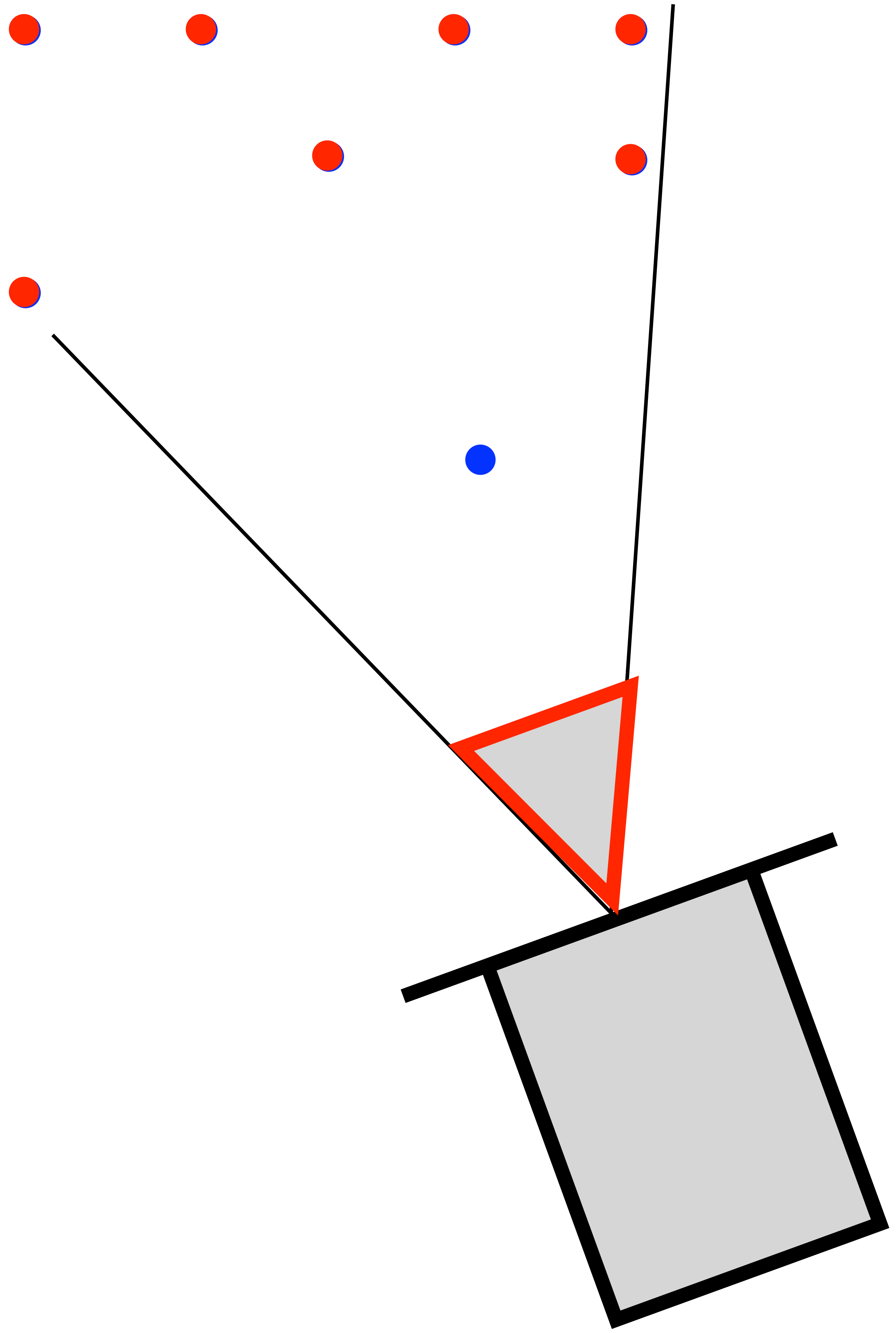

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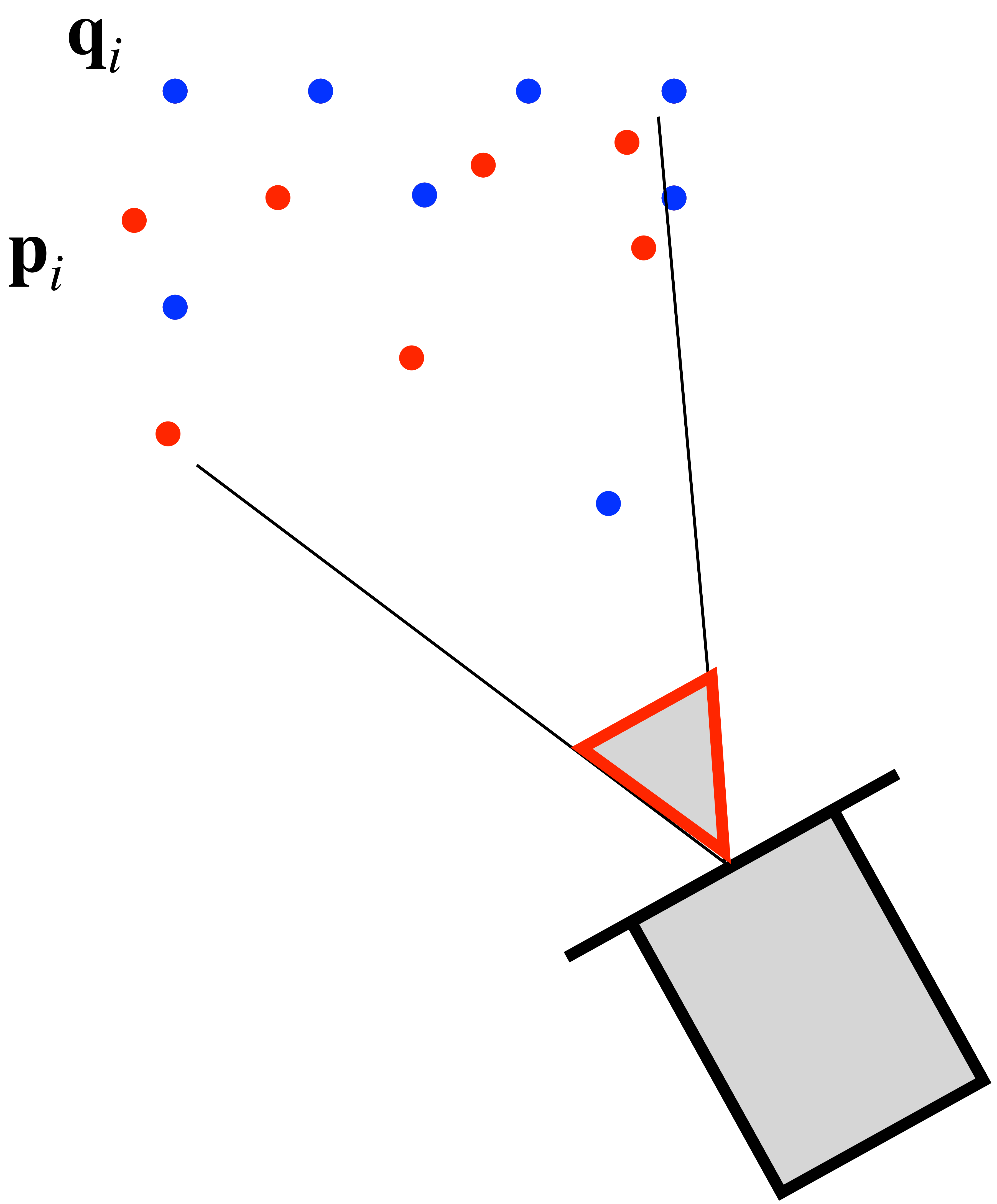


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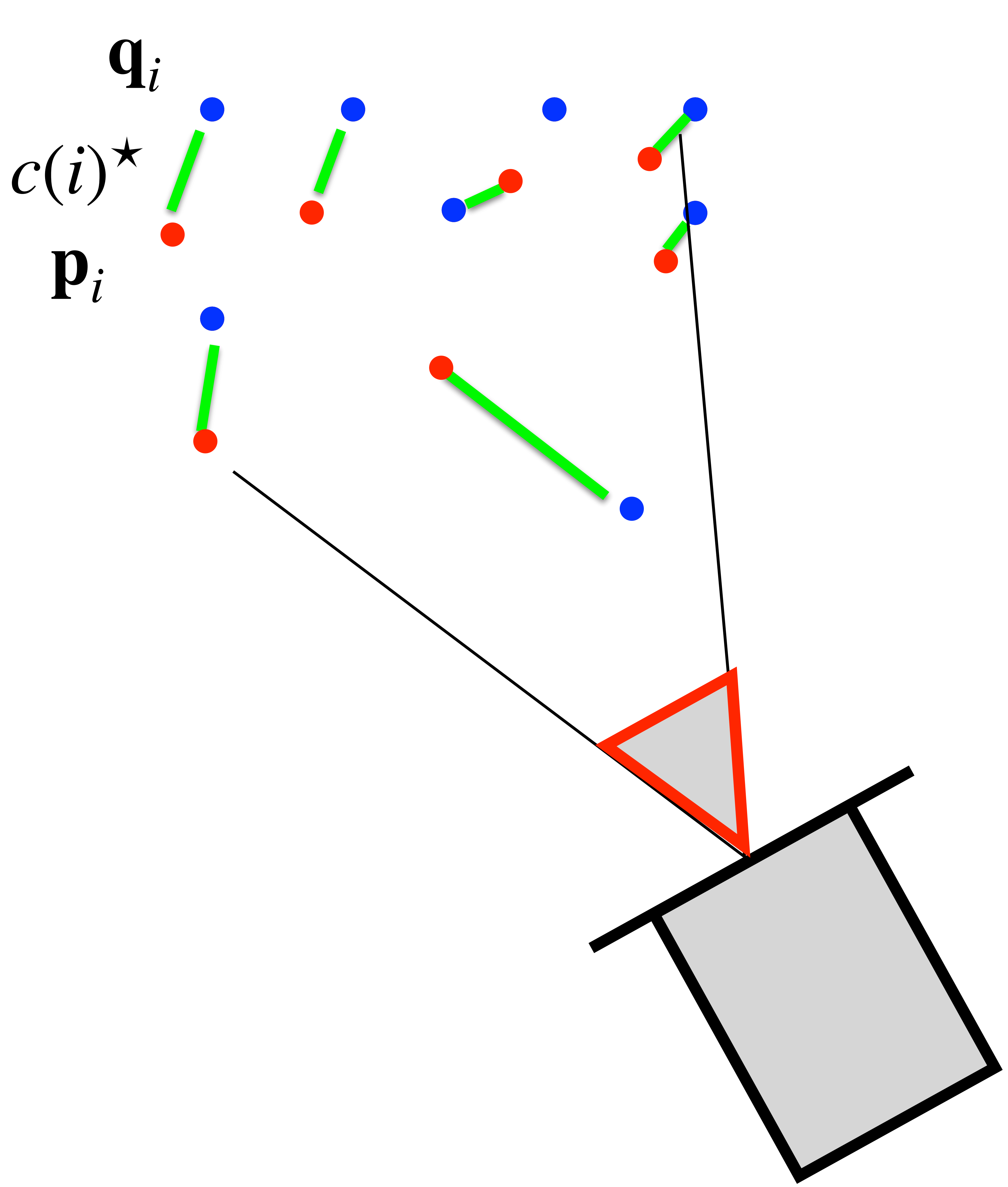


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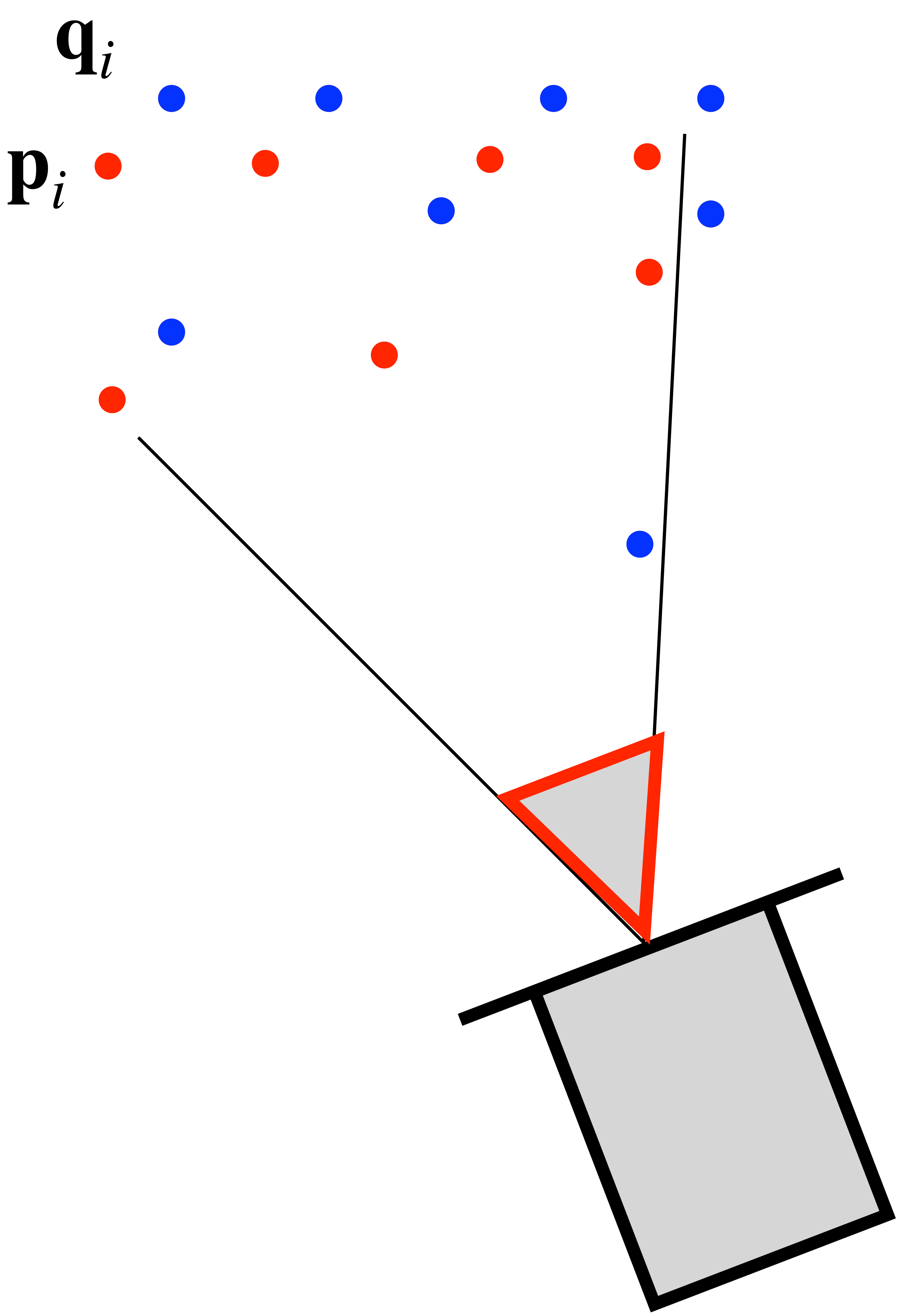


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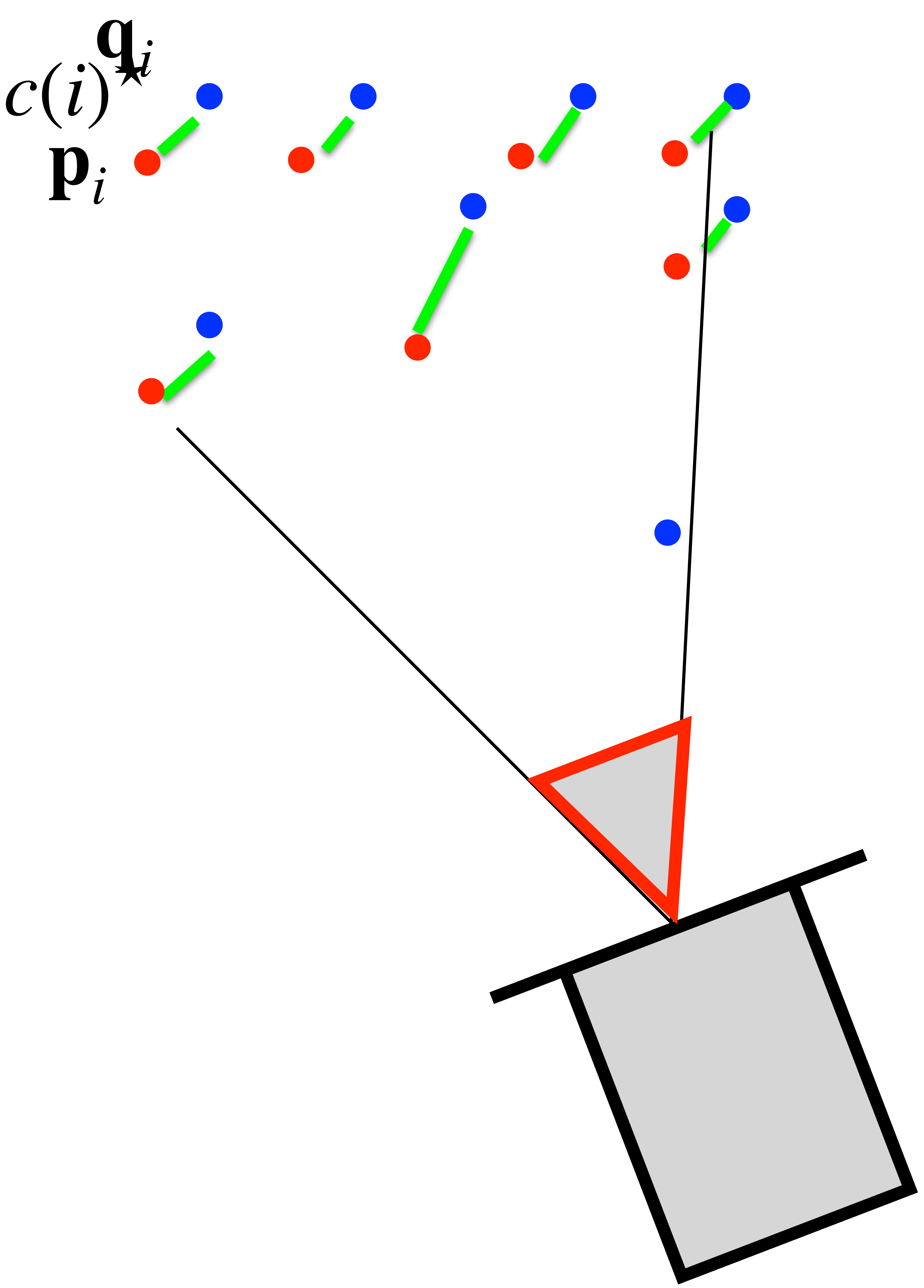


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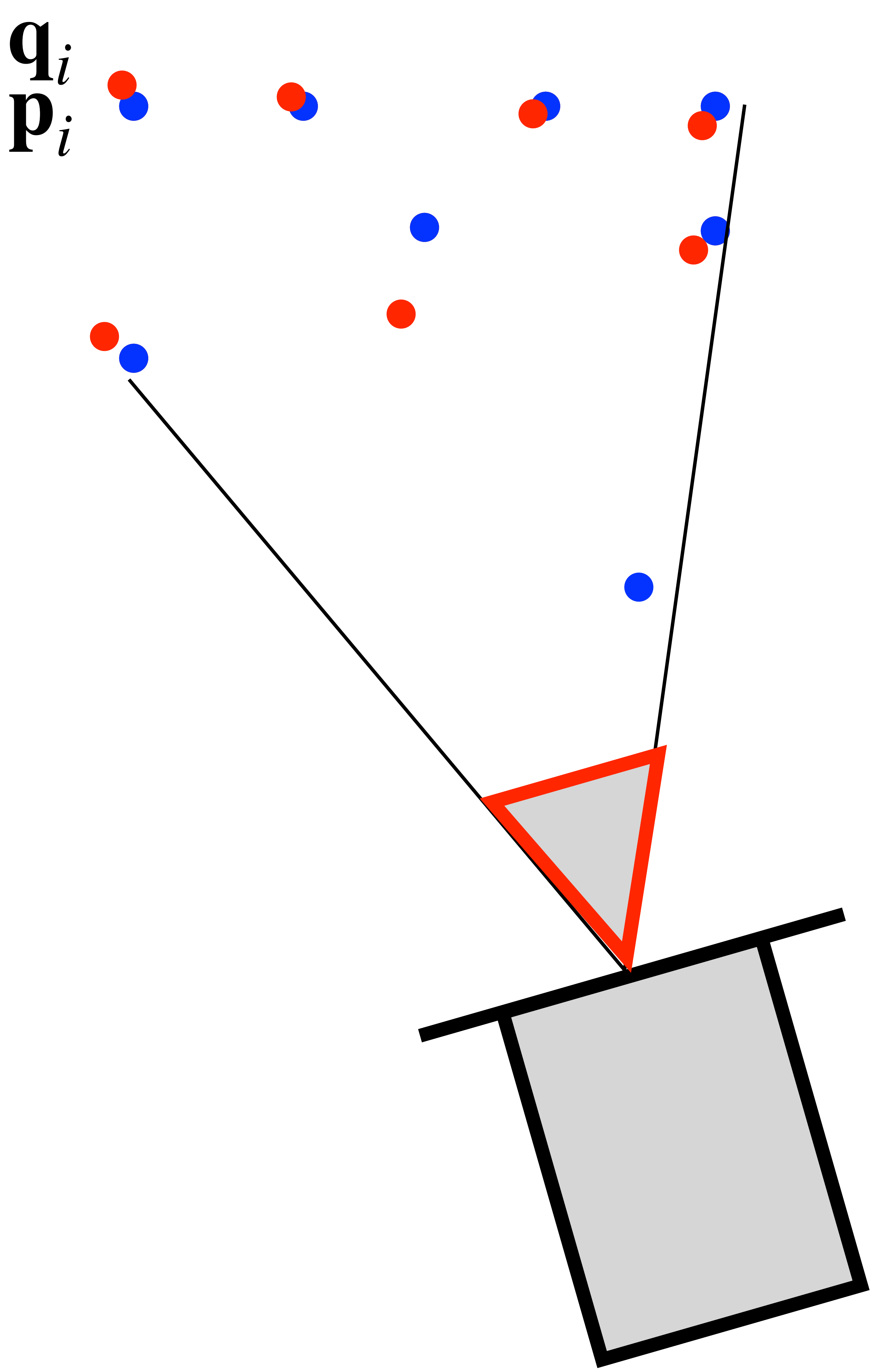


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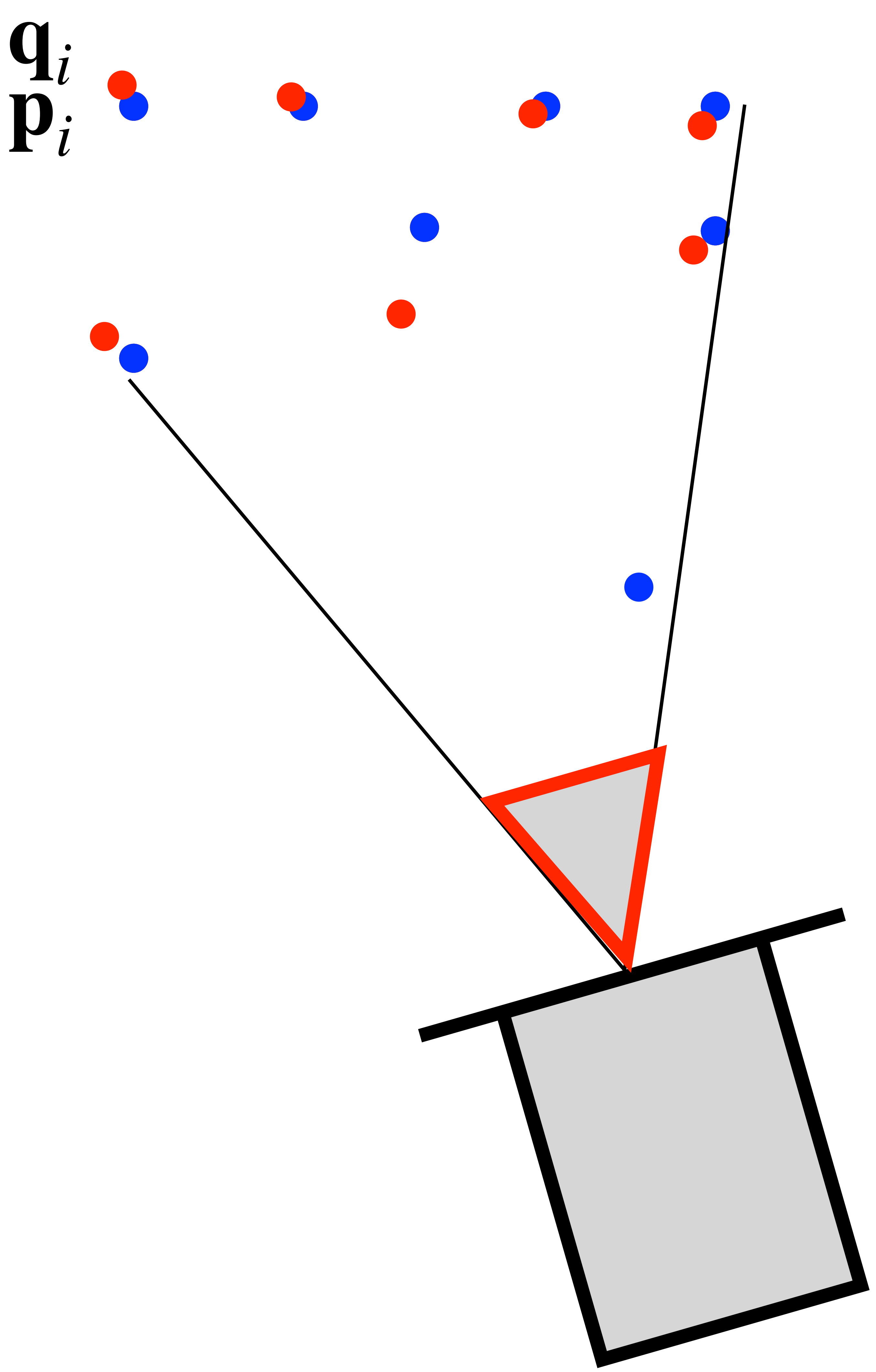


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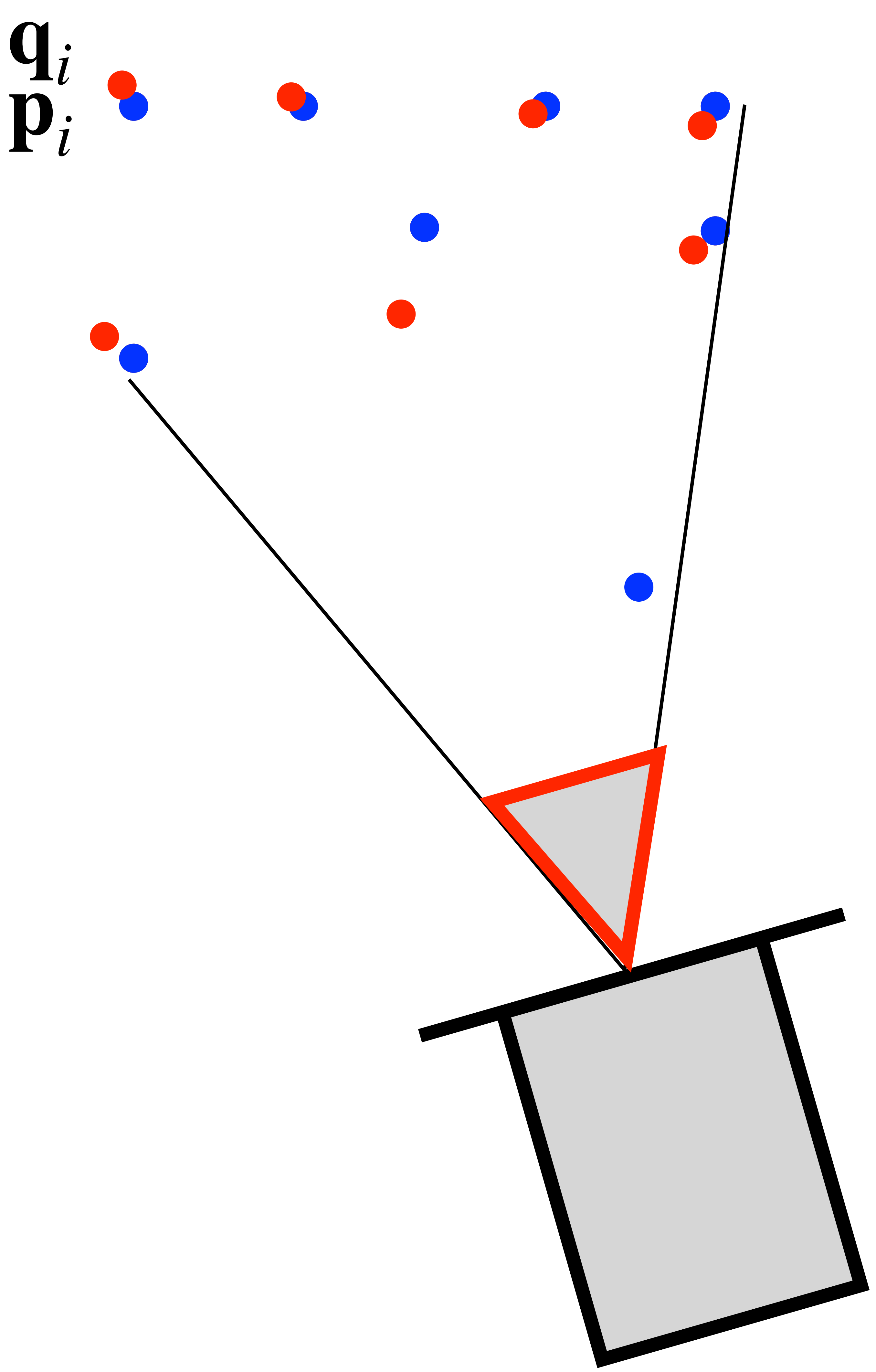


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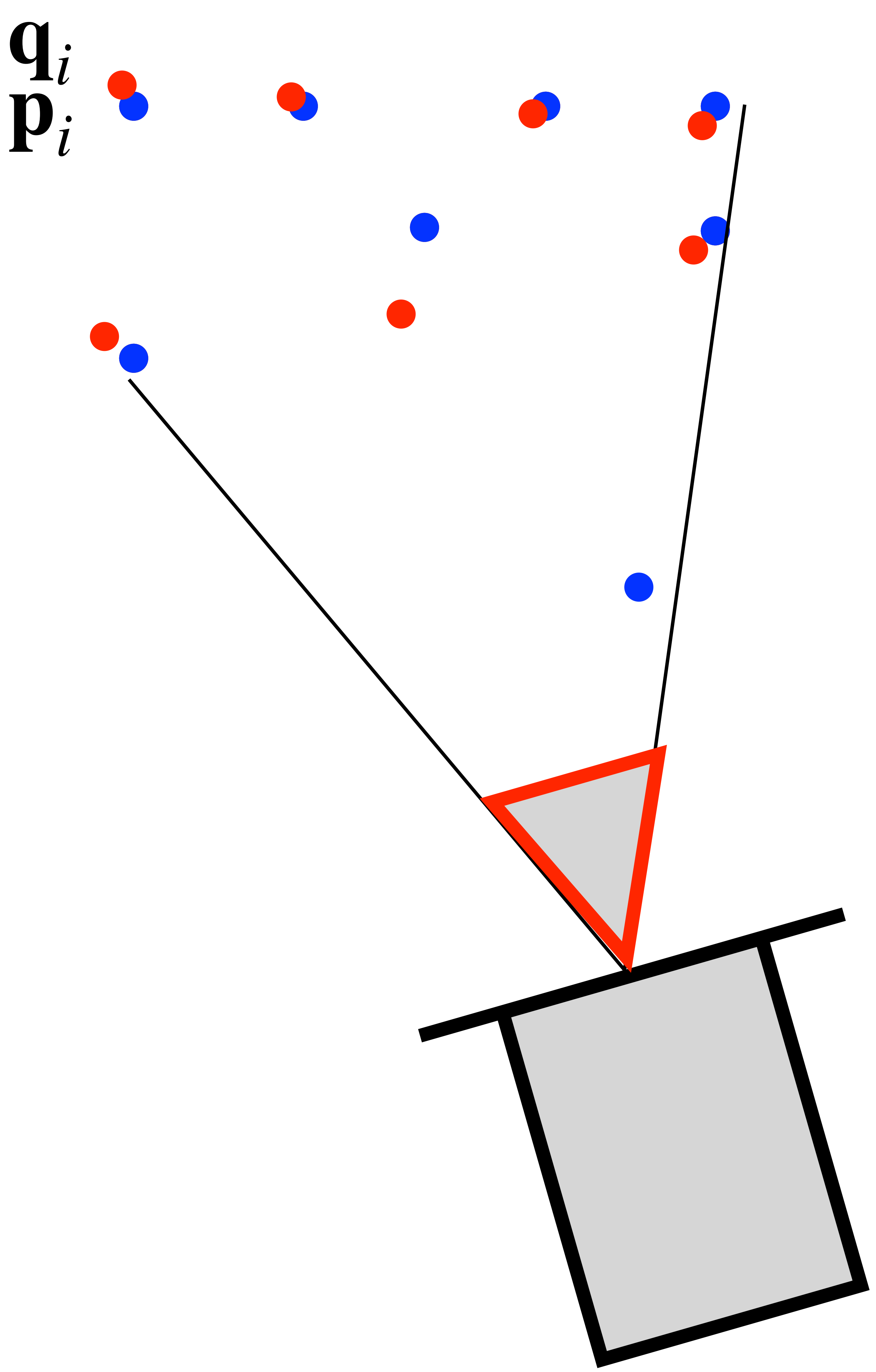


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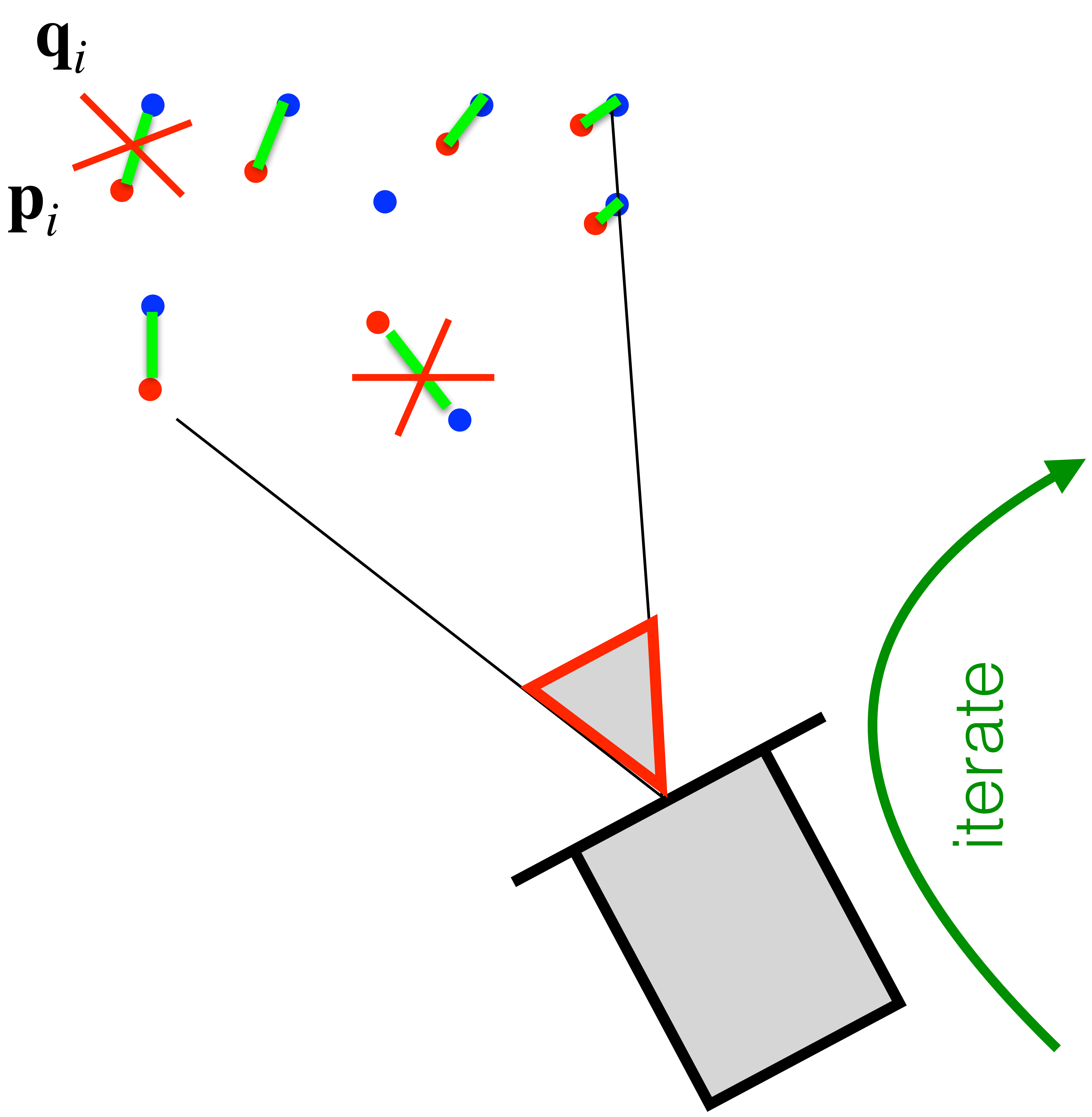


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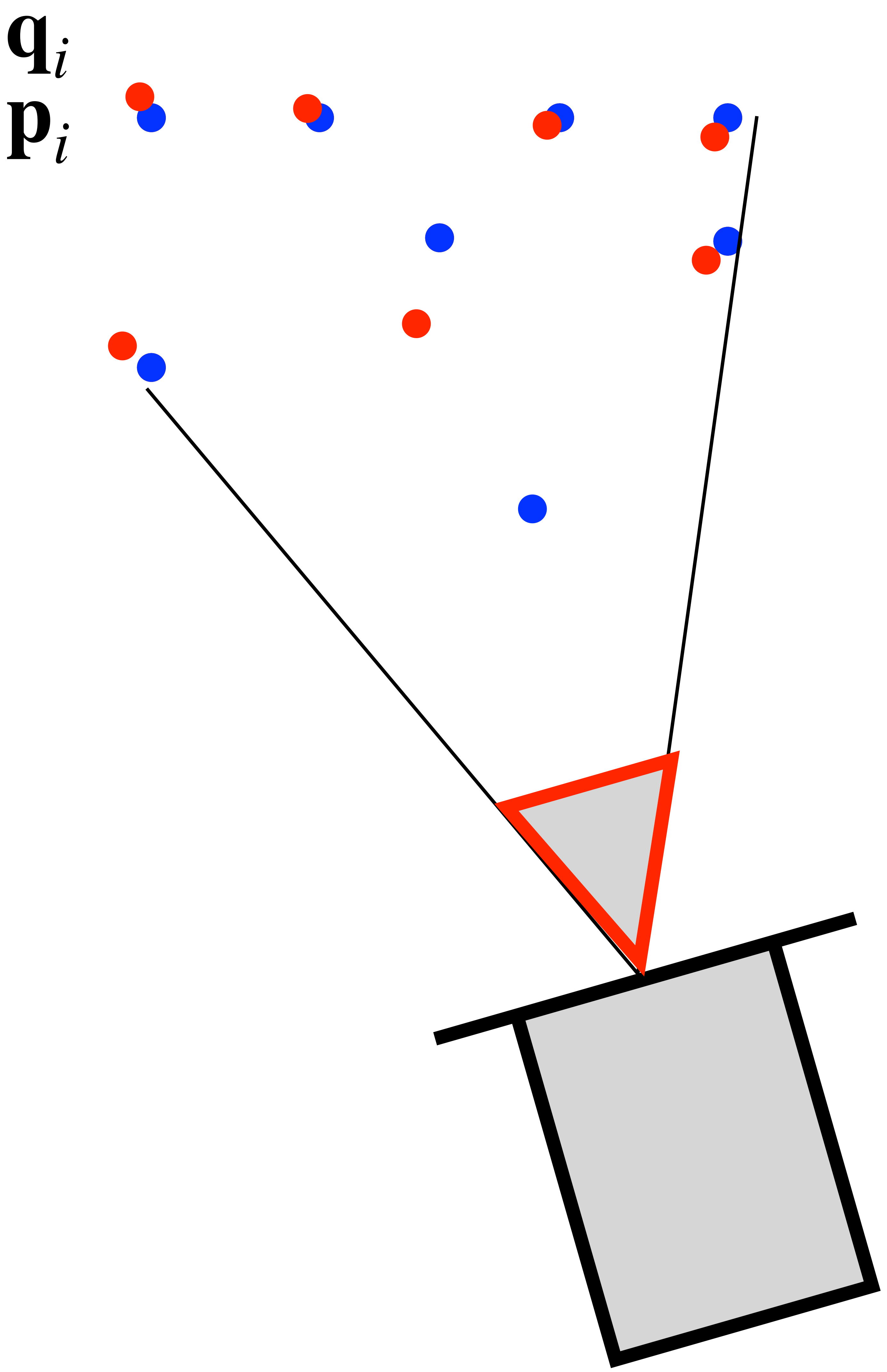


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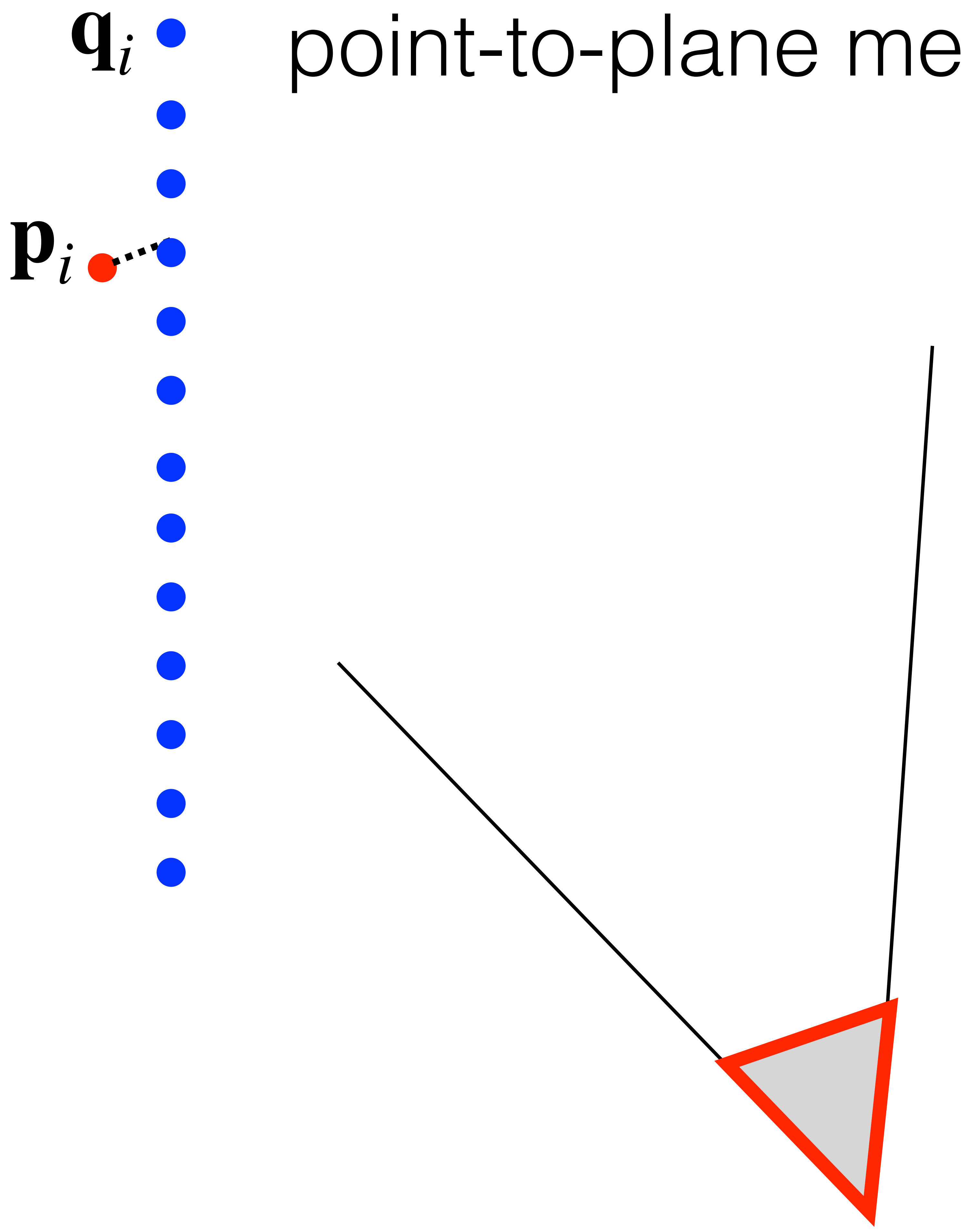


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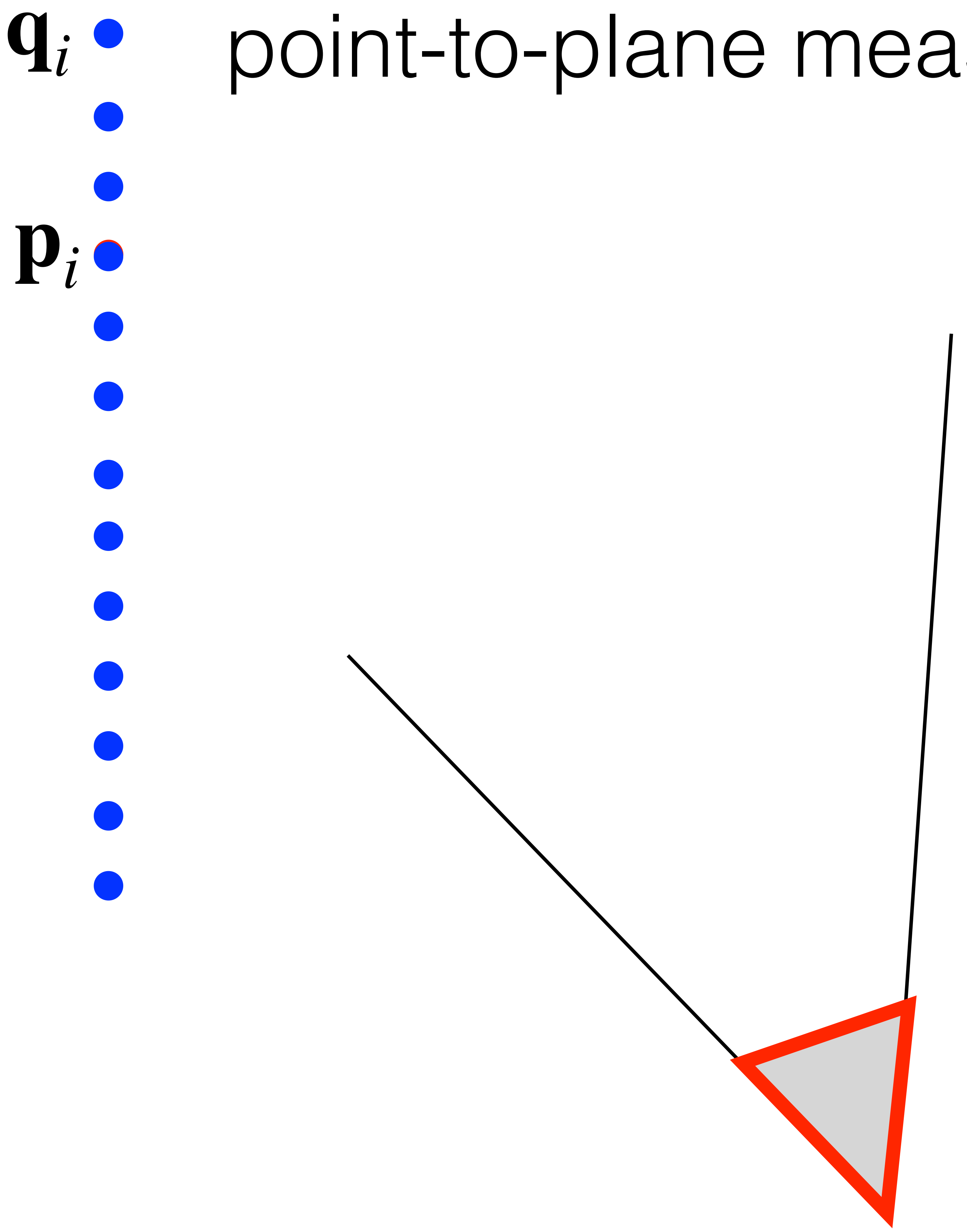


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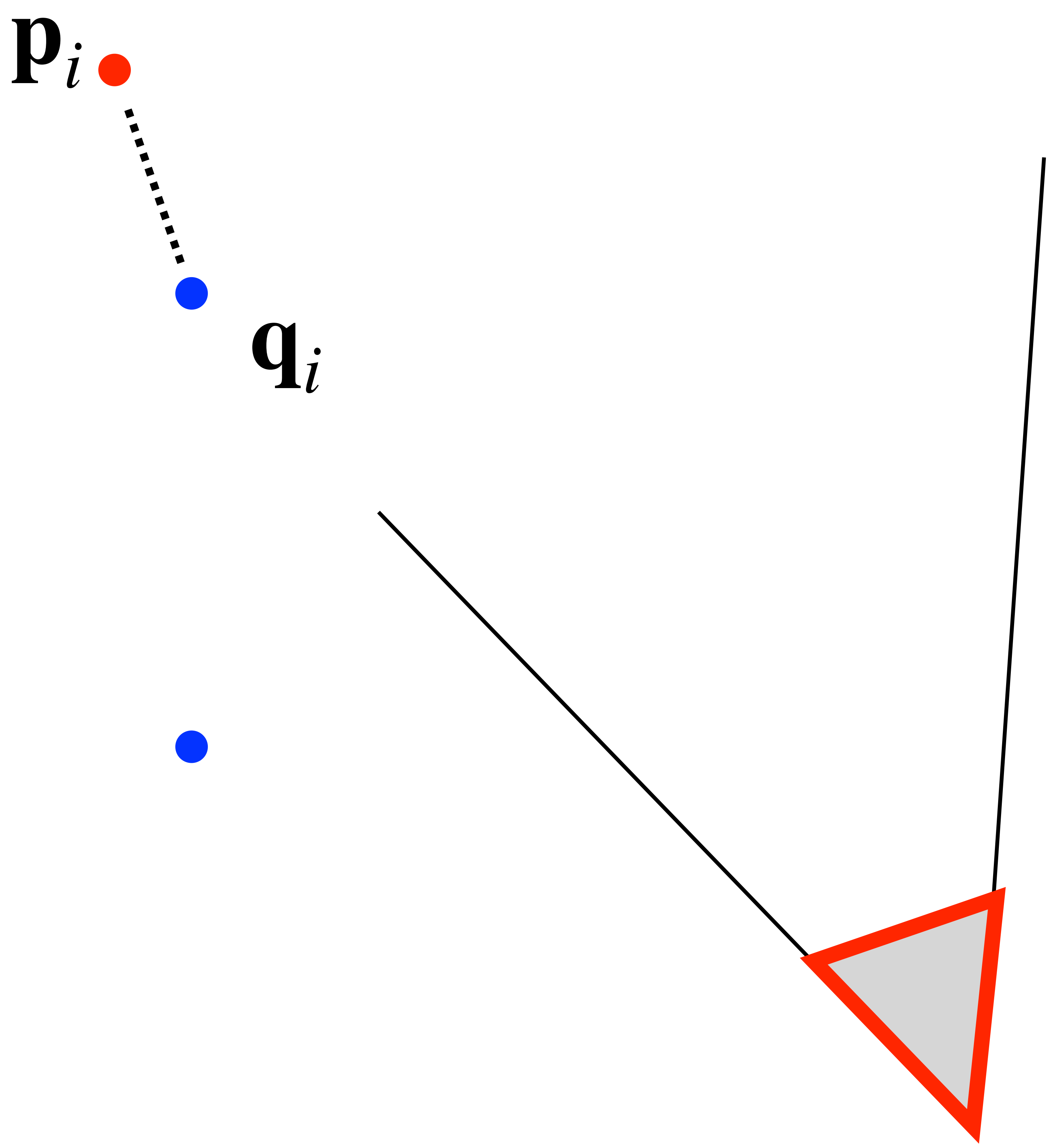


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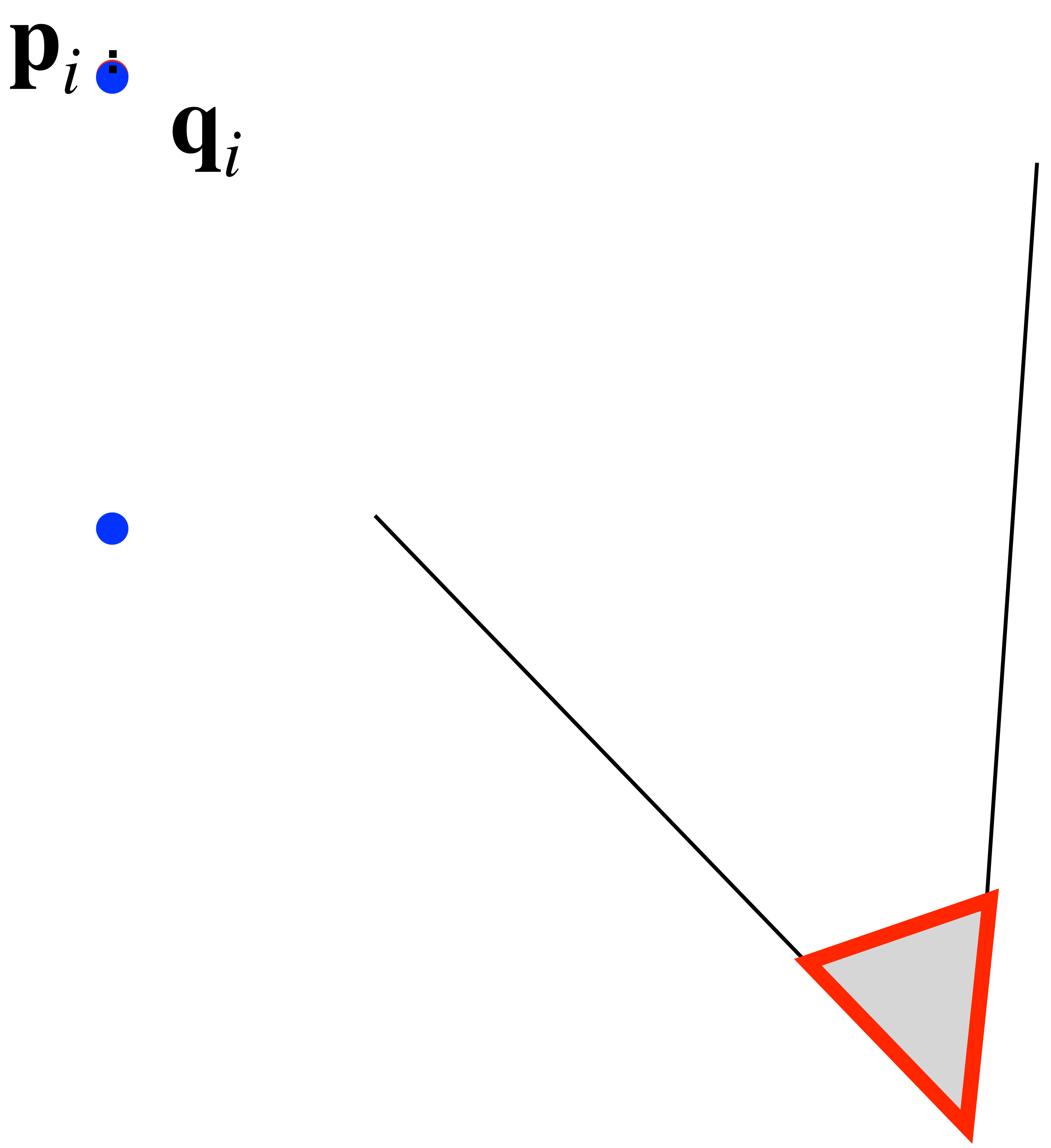


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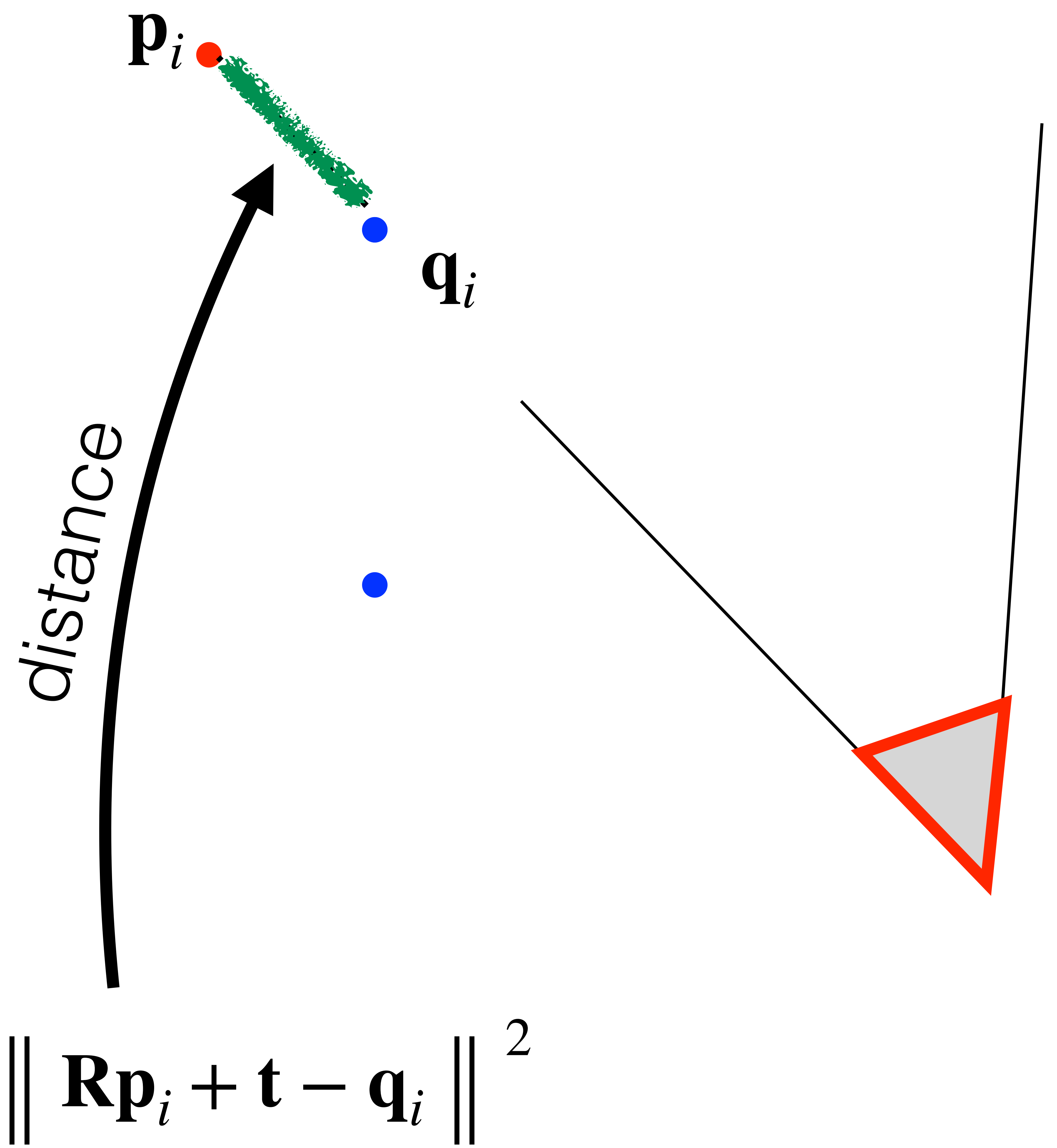


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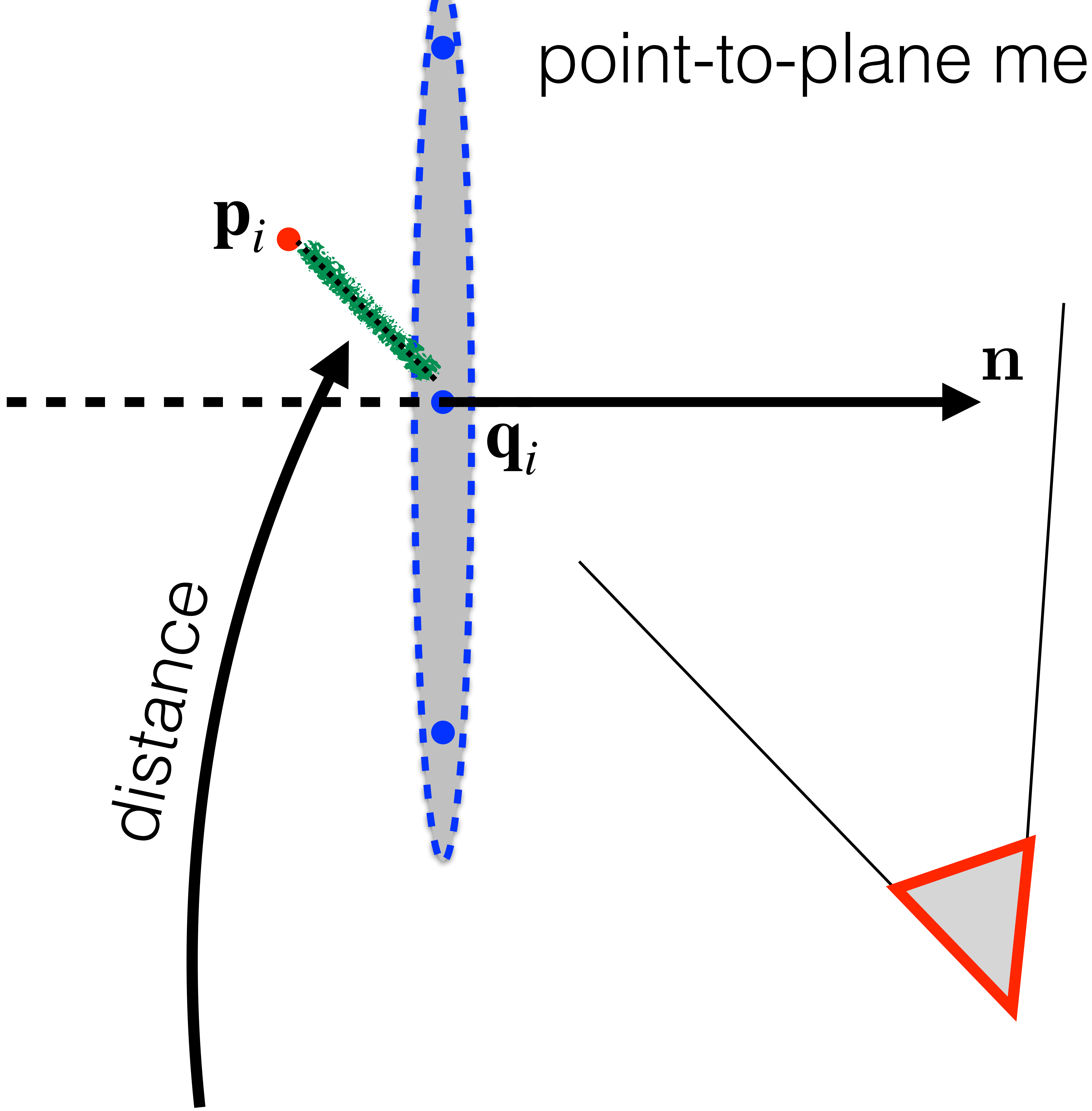


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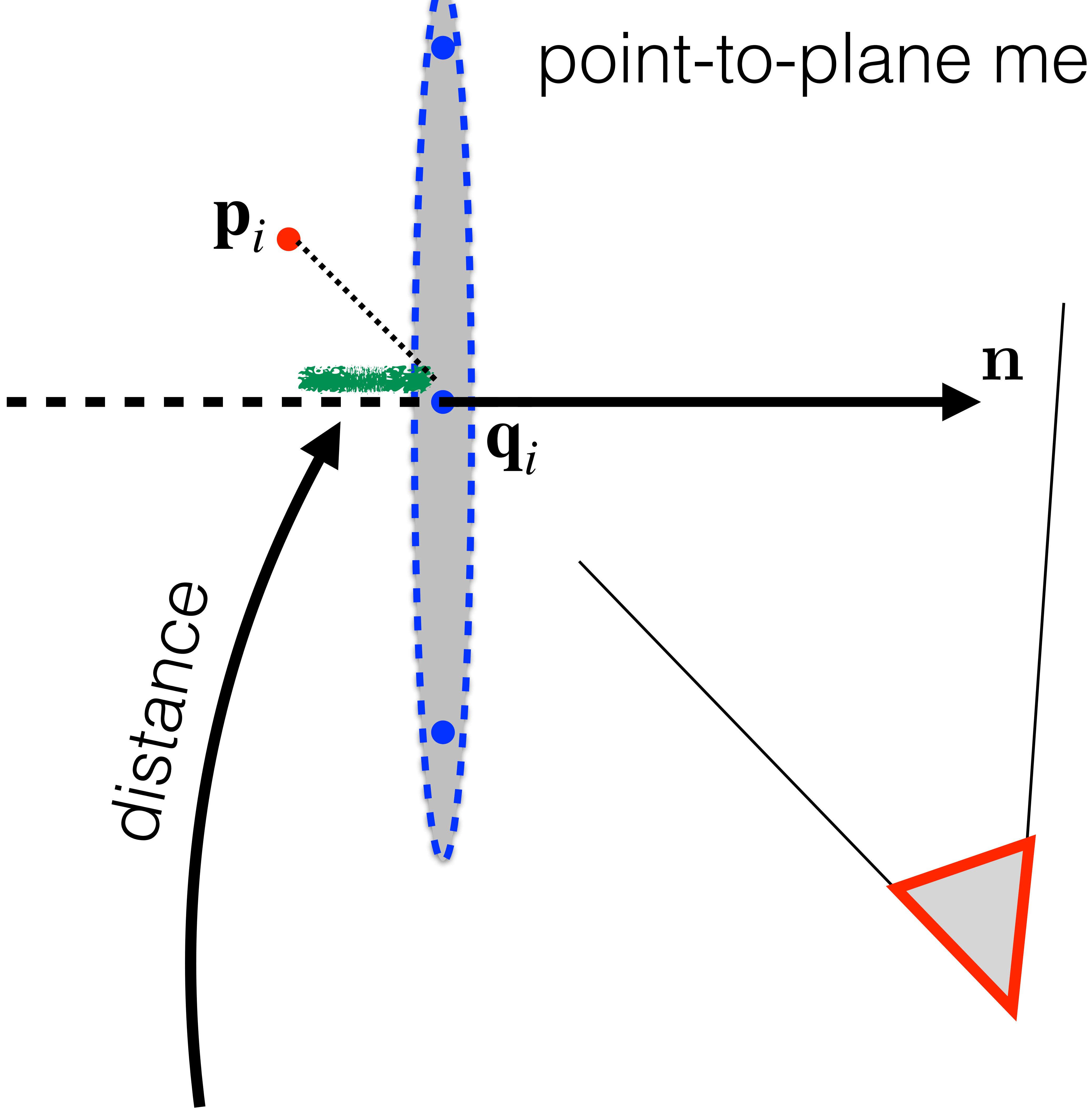


$$\left\| \mathbf{n}^\top(\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i) \right\|^2$$

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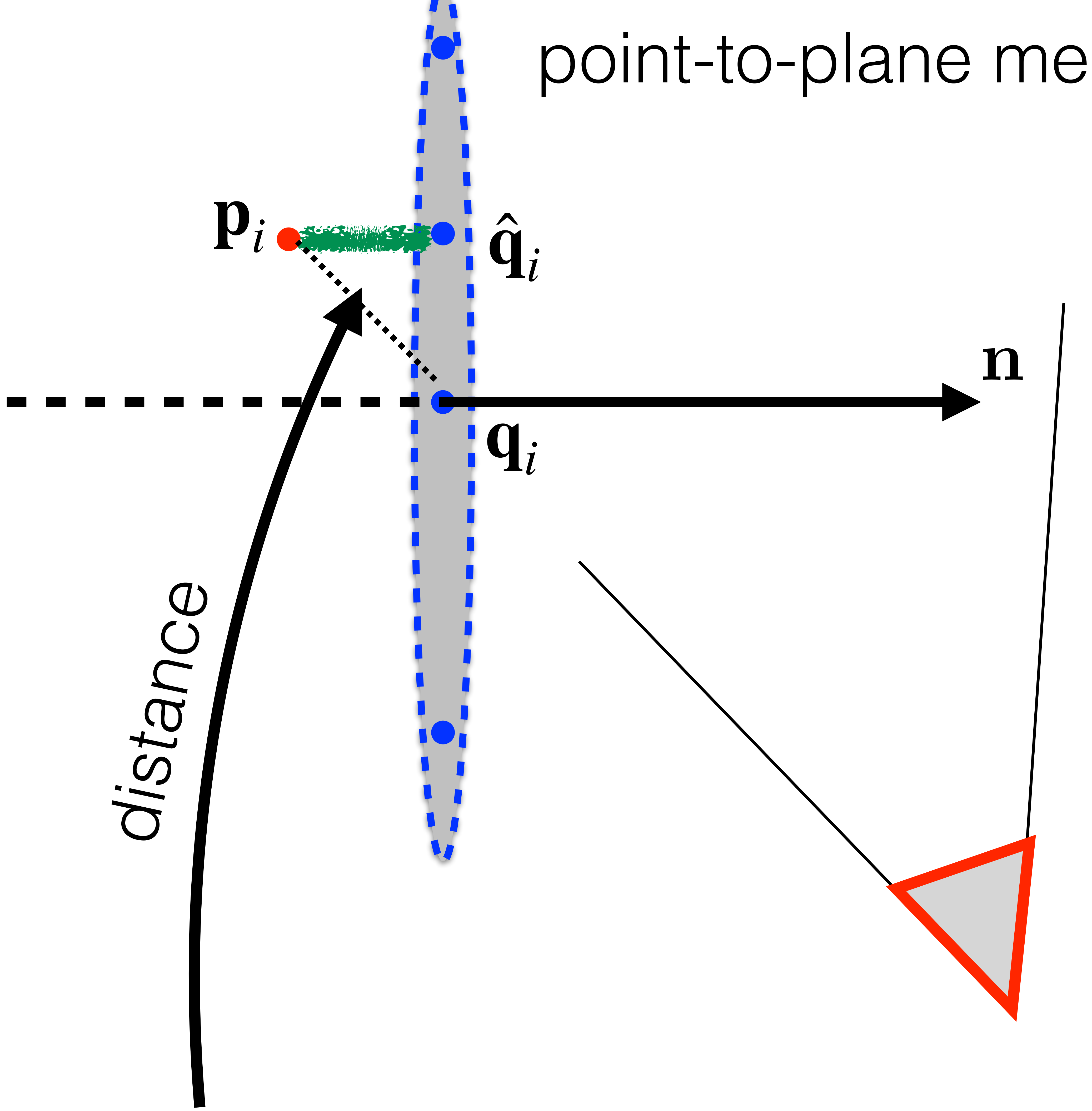
Does not have closed-form solution

point-to-plane measurement probability model

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Avoid gradient optimization => create virtual map point $\hat{\mathbf{q}}_i$

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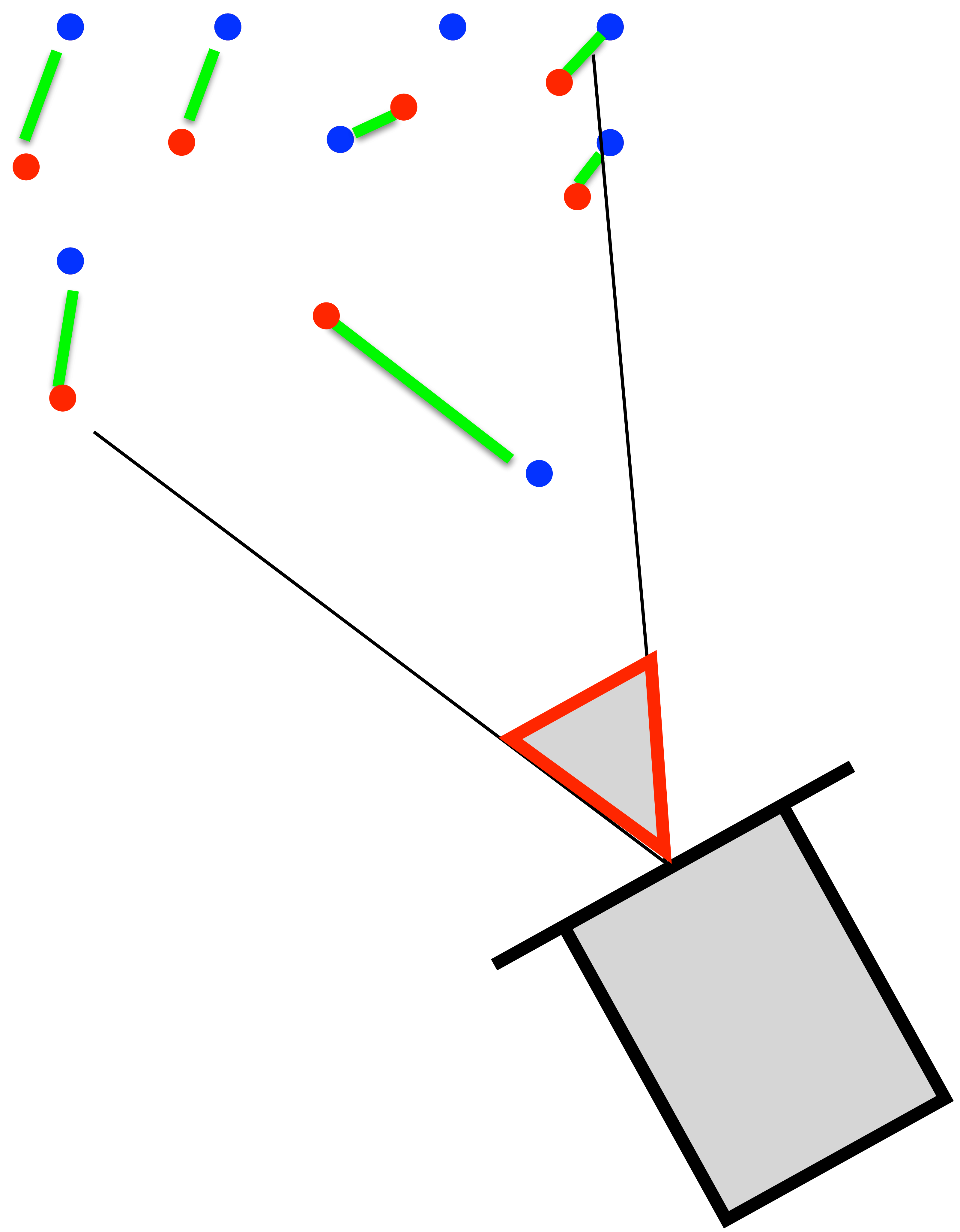
# Alignment quality is determined by the quality of correspondences



Compatibility measure:

- Lidar
  - normals
  - curvature
  - any measure of shape similarity
- Camera
  - colors
  - semantic consistency (car-car)
  - any measure of visual similarity
  - dynamic object suppression

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- Pointcloud map is not suitable for planning
  => better representation (e.g. occupancy grid)
- Normal distribution sensitive to outliers due to L2 norm minimization
  => RANSAC

# Maps

- 2D/3D pointcloud map
- 2D/3D Occupancy grid
- Surfel/feature map
- 2.5D map (hightmap)
- Costmap / Traversability map
- Semantic map
- Topological map
- Functional map

# Occupancy grid

- planning a robot path typically requires to distinguish “unoccupied” (free) space from “unknown” space.
- simplest representation which allows to do this, is occupancy grid.

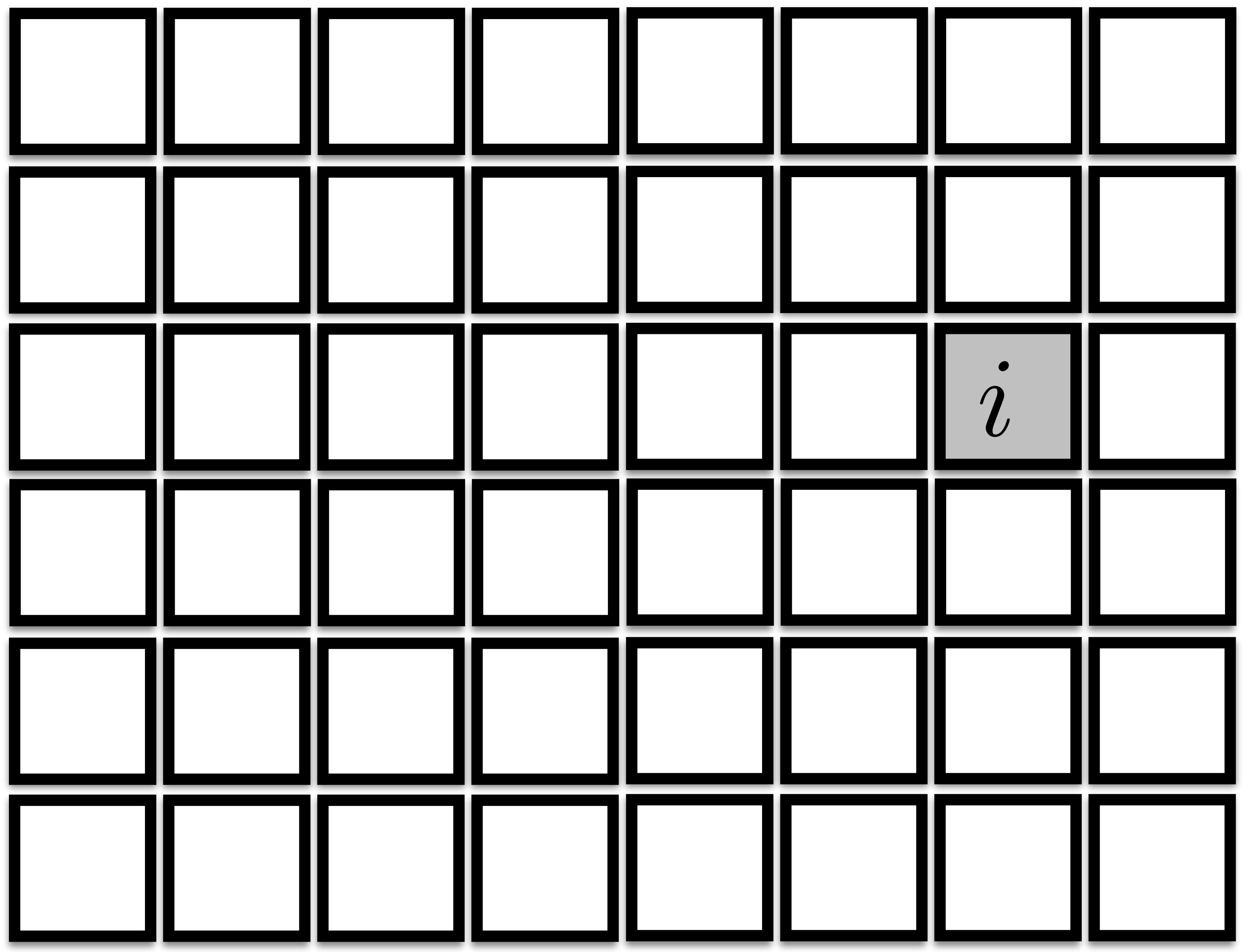
■ …. occupied (+1)

■ …. unknown (0)

■ …. unoccupied (-1)

# Occupancy grid

- We model only the probability $p(o_i|\mathbf{z}_{1:t})$ that cell $i$ is occupied given measurements $\mathbf{z}_{1:t}$

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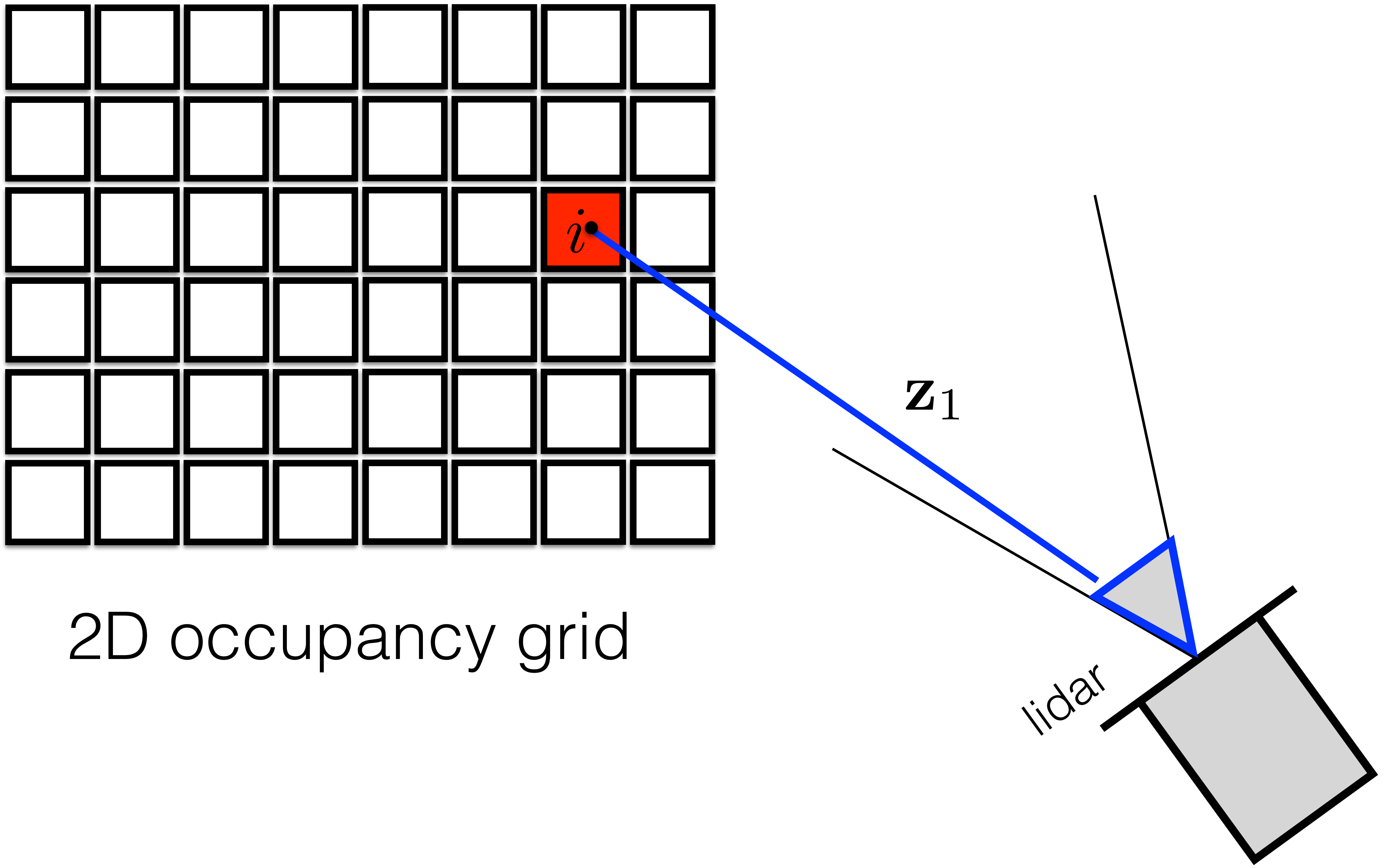


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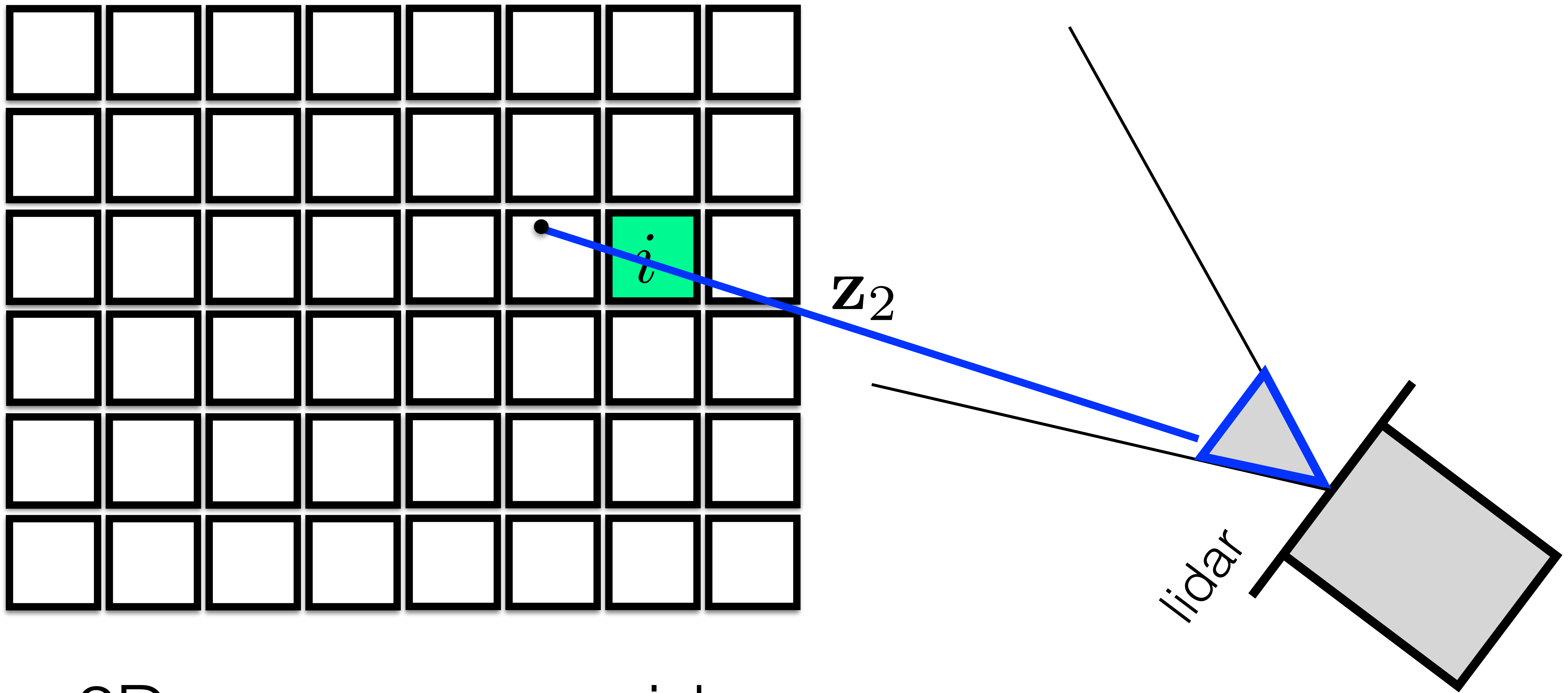


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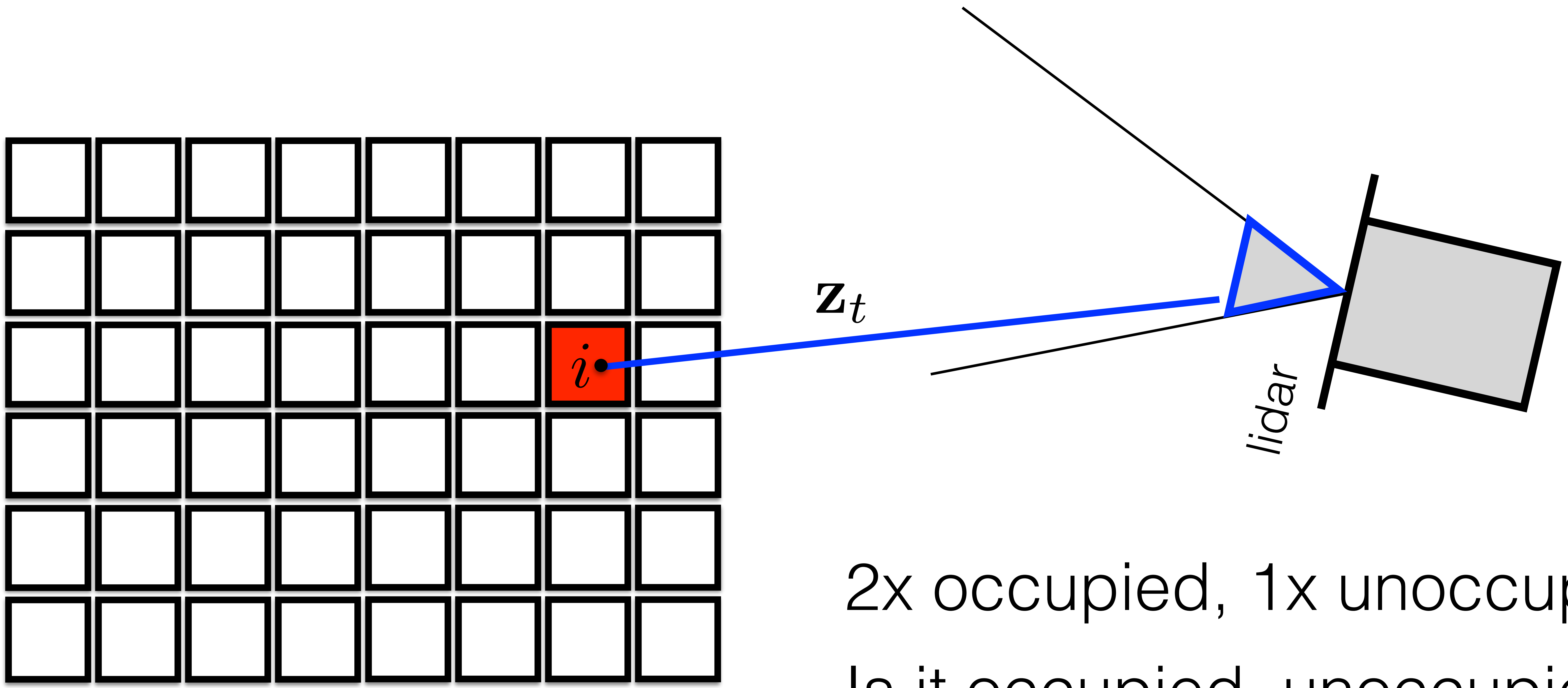


2D occupancy grid

2x occupied, 1x unoccupied

Is it occupied, unoccupied or unknown?

Simple hack: $b = 2 - 1$

$b \leq \theta_L$ …. unoccupied

$\theta_L < b < \theta_H$ …. unknown

$b \geq \theta_H$ …. occupied

There exists a probabilistic justification of this hack!

# ICP SLAM - known issues

- Converges to a local minima with unknown region of convergence
=> good initialization needed
- Measurement probability model fails in degenerate environments
(e.g. forward motion in long hallways, or rotation in circular room)
=> other models (point2plane) and sensors (camera) often introduced.
- Pointcloud map is not suitable for planning
=> better representation (e.g. occupancy grid)
- **Next:** Normal distribution sensitive to outliers due to L2 norm minimization
=> RANSAC