Localization: MLE estimate

Karel Zimmermann

Prerequisites: Bayes theorem

A … have disease $p(A) = 0.001$ Prior probability of having disease 0.1%

$p(\overline{A}) = 0.999$ Prior probability of being healthy 99.9%

B … positive test $p(B|A) = 1$ Everyone who have disease tests always positive

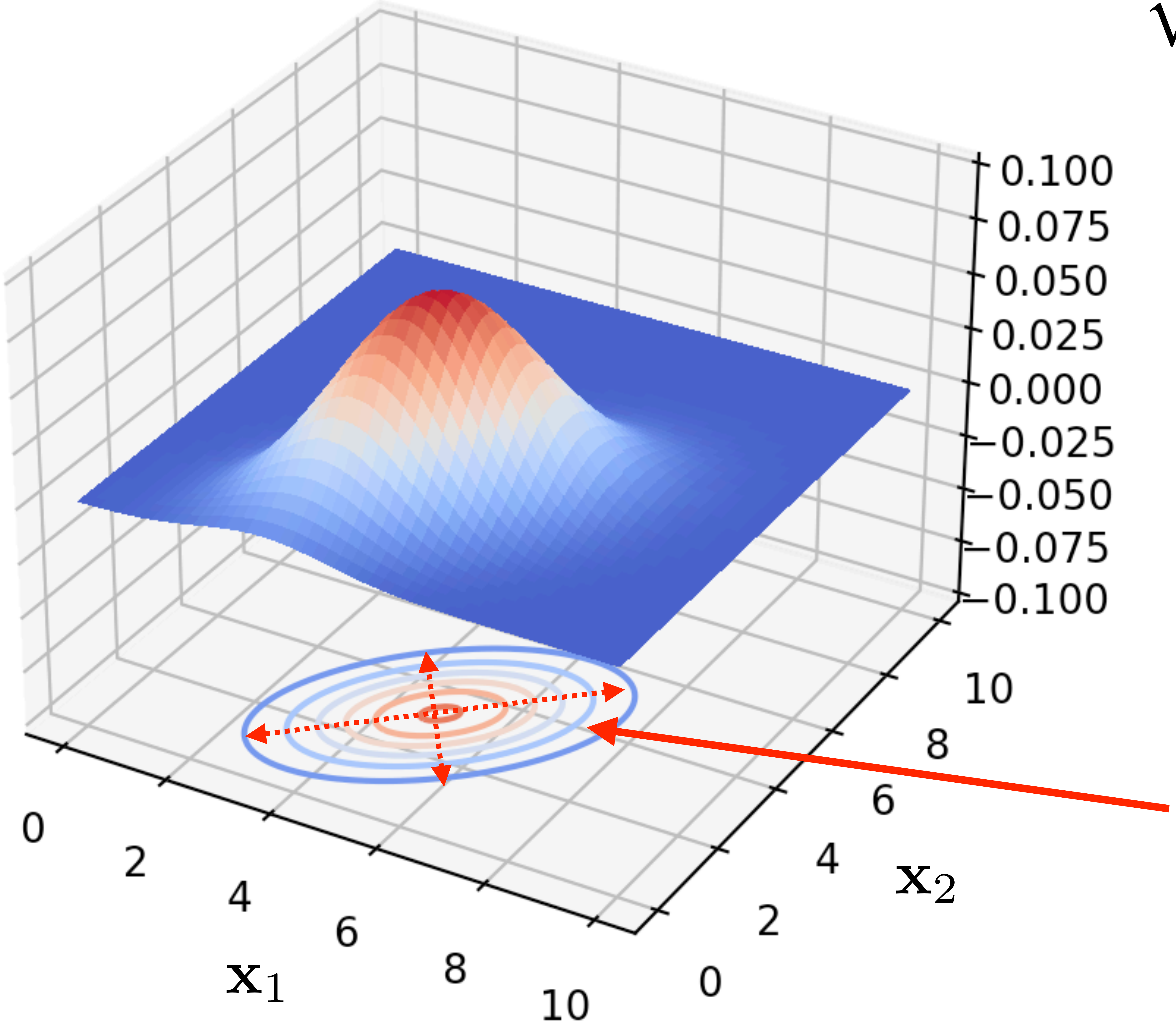
$p(B|\overline{A}) = 0.05$ There are 5% of healthy people, with positive test

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|\overline{A})p(\overline{A})} = \frac{1 * 0.001}{1 * 0.001 + 0.999 * 0.05} \approx 1.9\,\%$$

Only 18% of doctors+students from Harvard Medical school answered correctly.

Prerequisites: Multivariate gaussian

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\exp\Big(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\Big)}{\sqrt{(2\pi)^n \det(\boldsymbol{\Sigma})}}$$



$\mathbf{x} \in \mathcal{R}^n$ … real n-dimensional random column vector

$\boldsymbol{\mu} \in \mathcal{R}^n$ … real n-dimensional mean

$\boldsymbol{\Sigma} \in \mathcal{R}^{n \times n}$ … symmetric positive definite covariance matrix

eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$ determine ellipse axes

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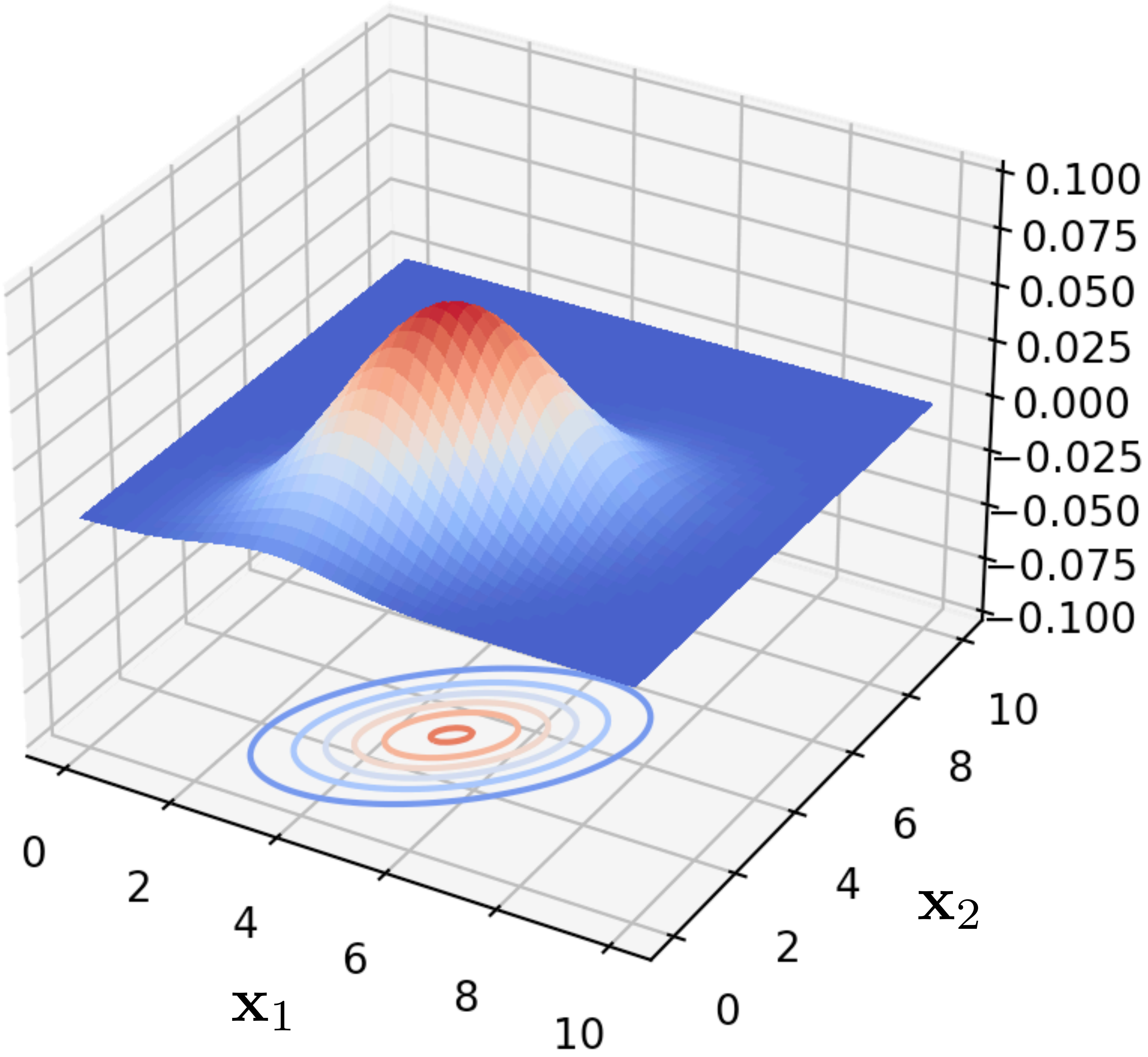

Gaussian distributions are closed under:

- Affine transformation
- Chain rule
- Conditioning
- Marginalization

Prerequisites: Multivariate gaussian

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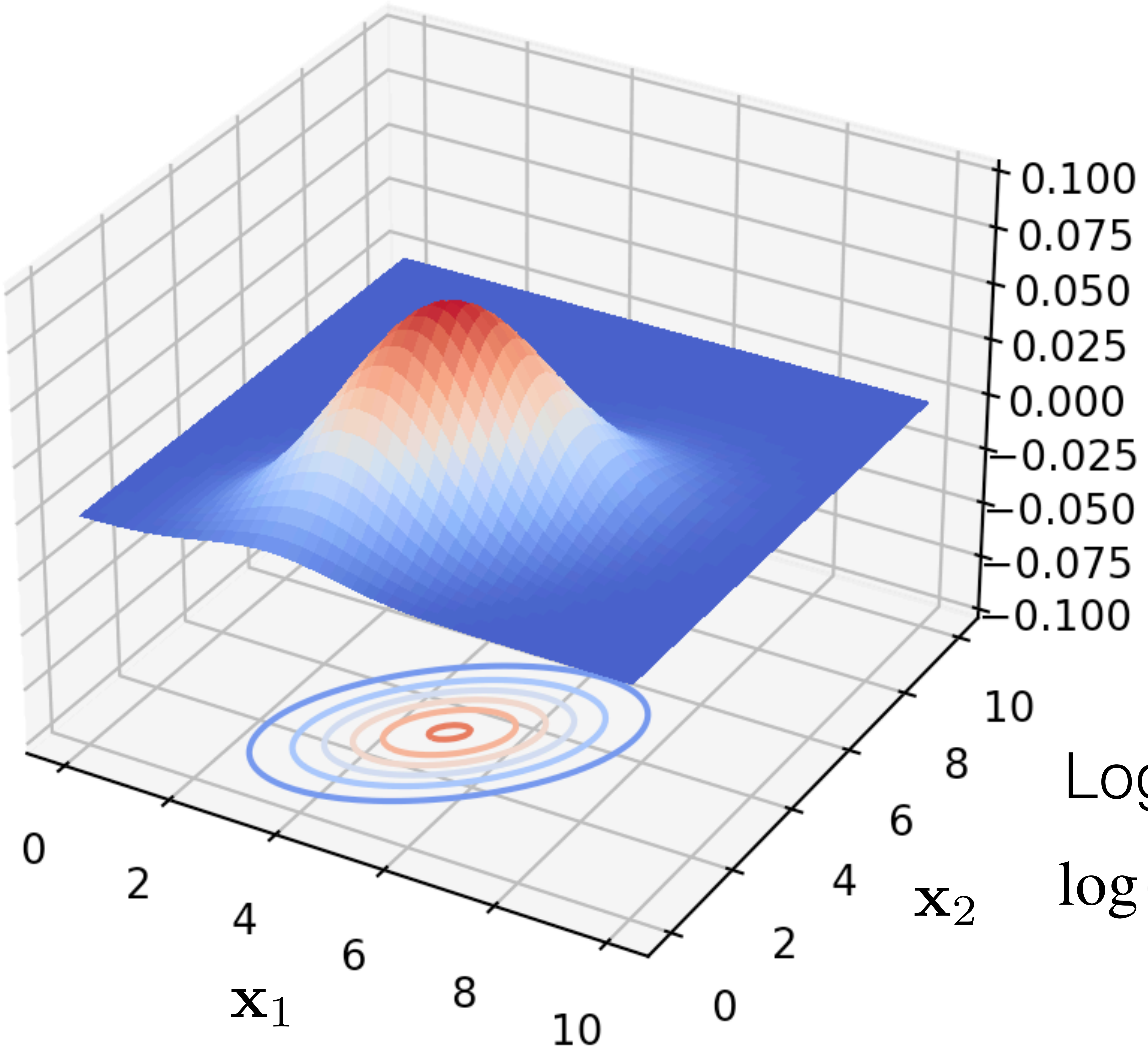


- Chain rule

Product of two Gaussians is Gaussian:

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \cdot \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_1^{-1} + \boldsymbol{\Sigma}_2^{-1}$$

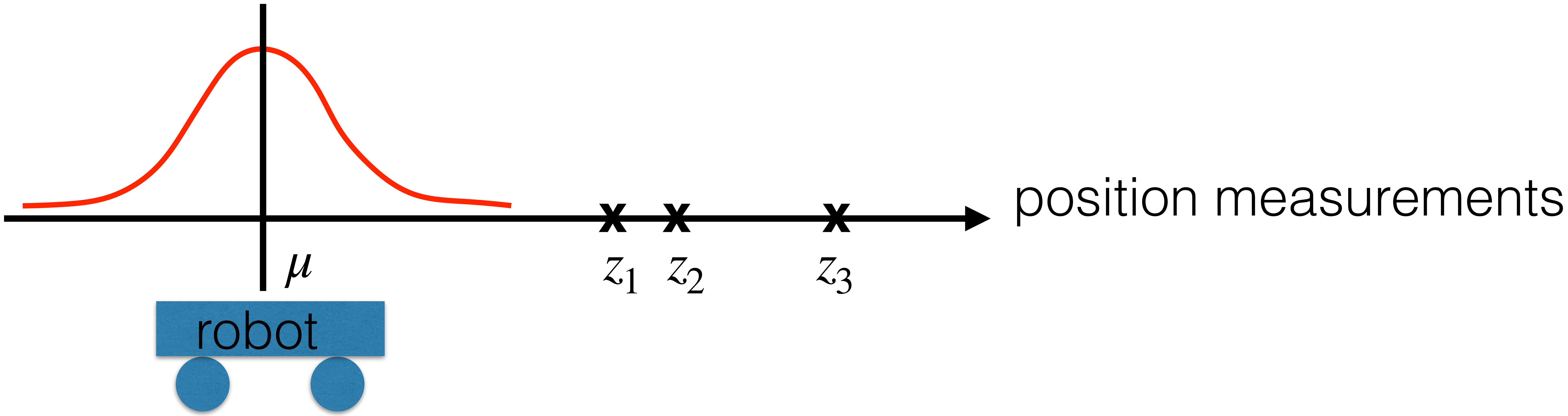
$$\boldsymbol{\mu} = \boldsymbol{\Sigma}(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_2^{-1}\boldsymbol{\mu}_2)$$

Logarithm of Gaussian is quadratic form:

$$\log\left(\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\right) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) + C$$

Motivation example

Measurement probability: $p(z_i \mid \mu) = \mathcal{N}(z_i; \mu, \sigma^2)$



maximizing product of gaussians <=> minimizing the sum of L2 differences

MLE LS

$$\mu^* = \arg\max_{\mu} \left(\prod_i \mathcal{N}(z_i; \mu, \sigma^2) \right) = \arg\max_{\mu} \prod_i K \cdot \exp\left(-\frac{\|z_i - \mu\|_2^2}{\sigma^2} \right) = \arg\min_{\mu} \sum_i \|z_i - \mu\|_2^2$$

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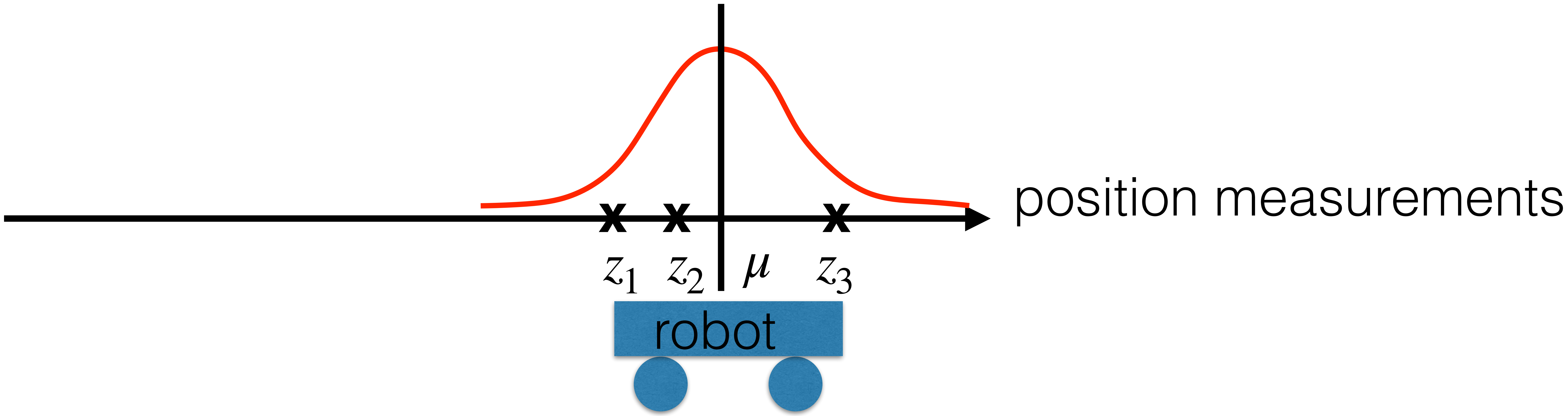

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LS

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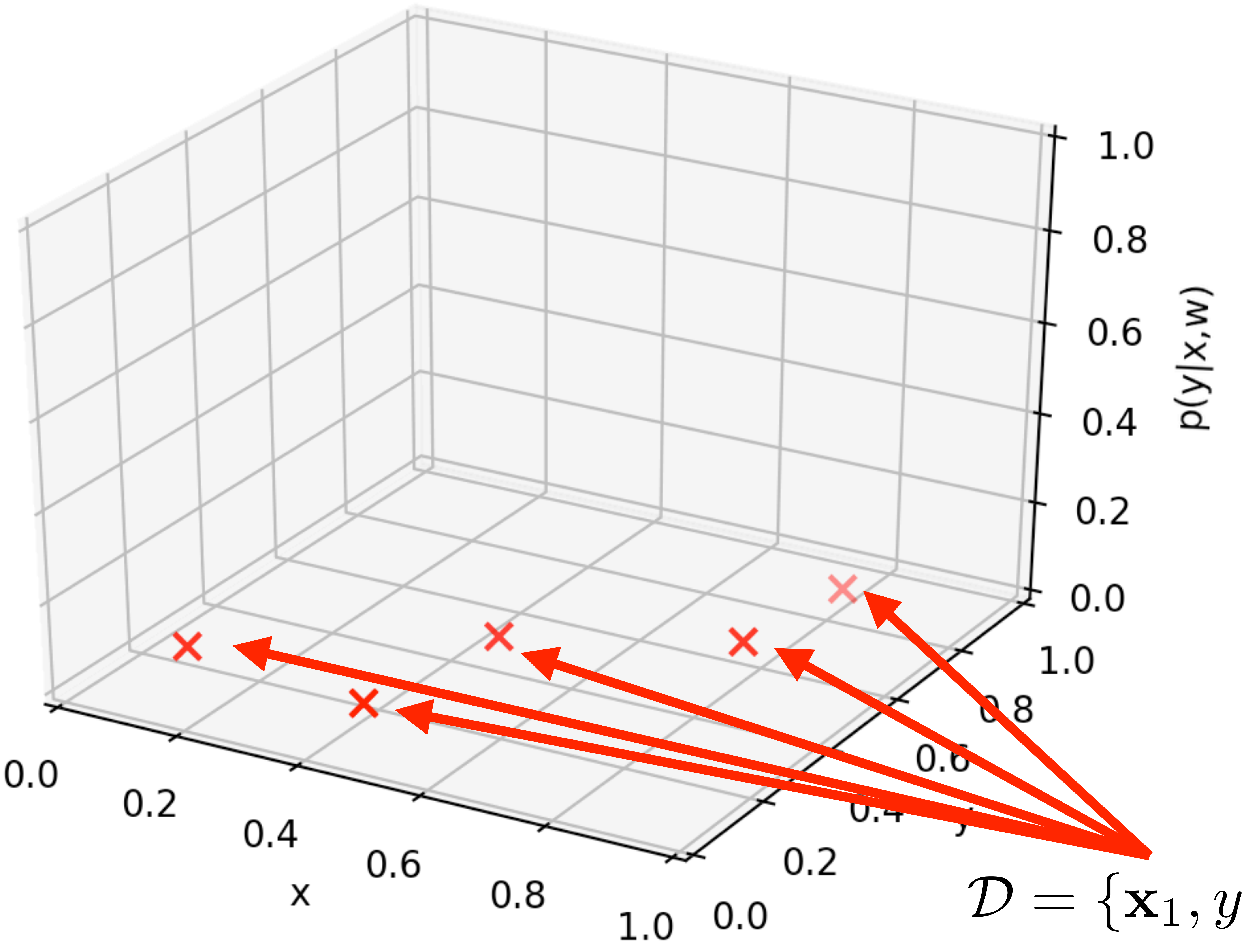
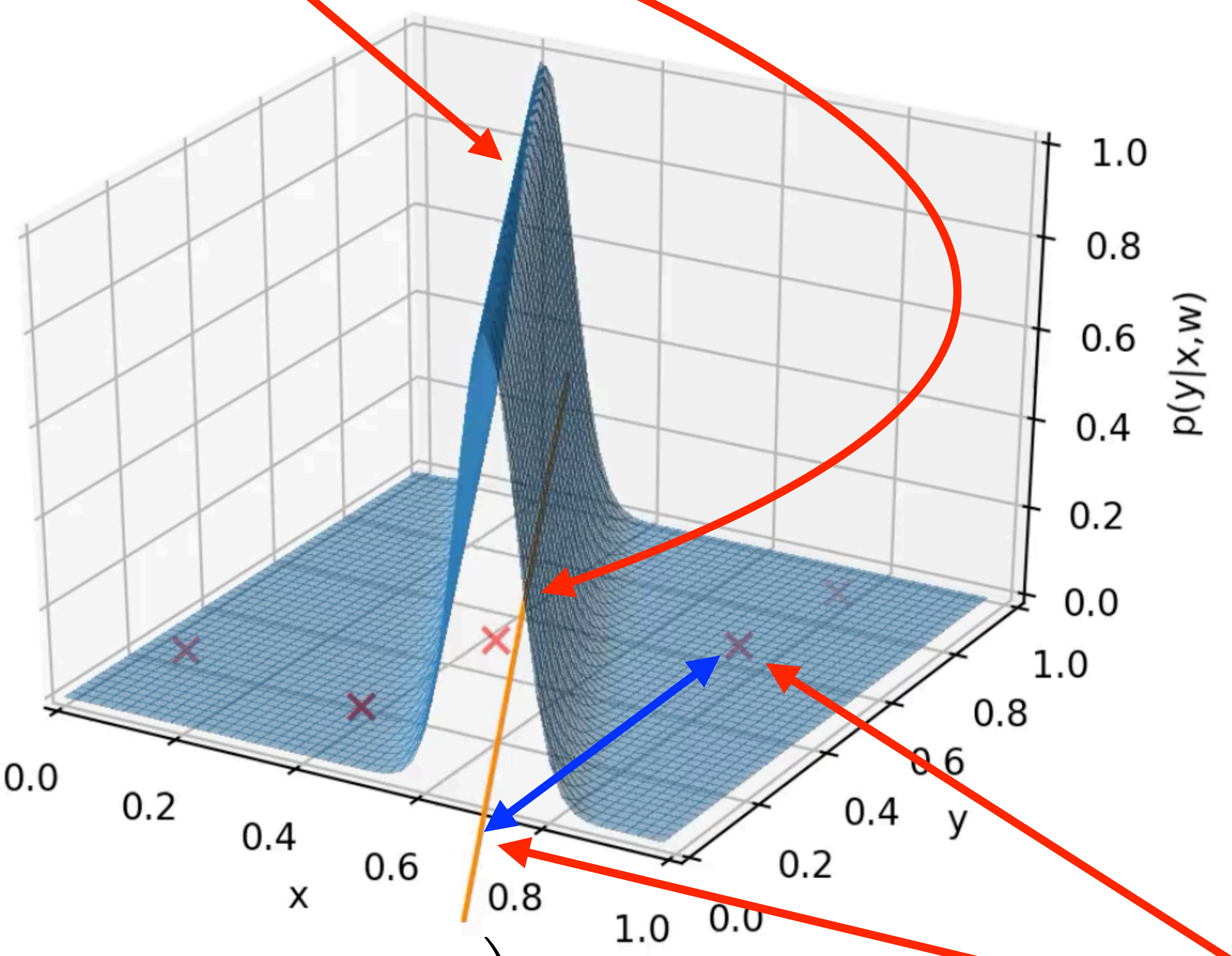
$$= \frac{\sum_i z_i}{N}$$

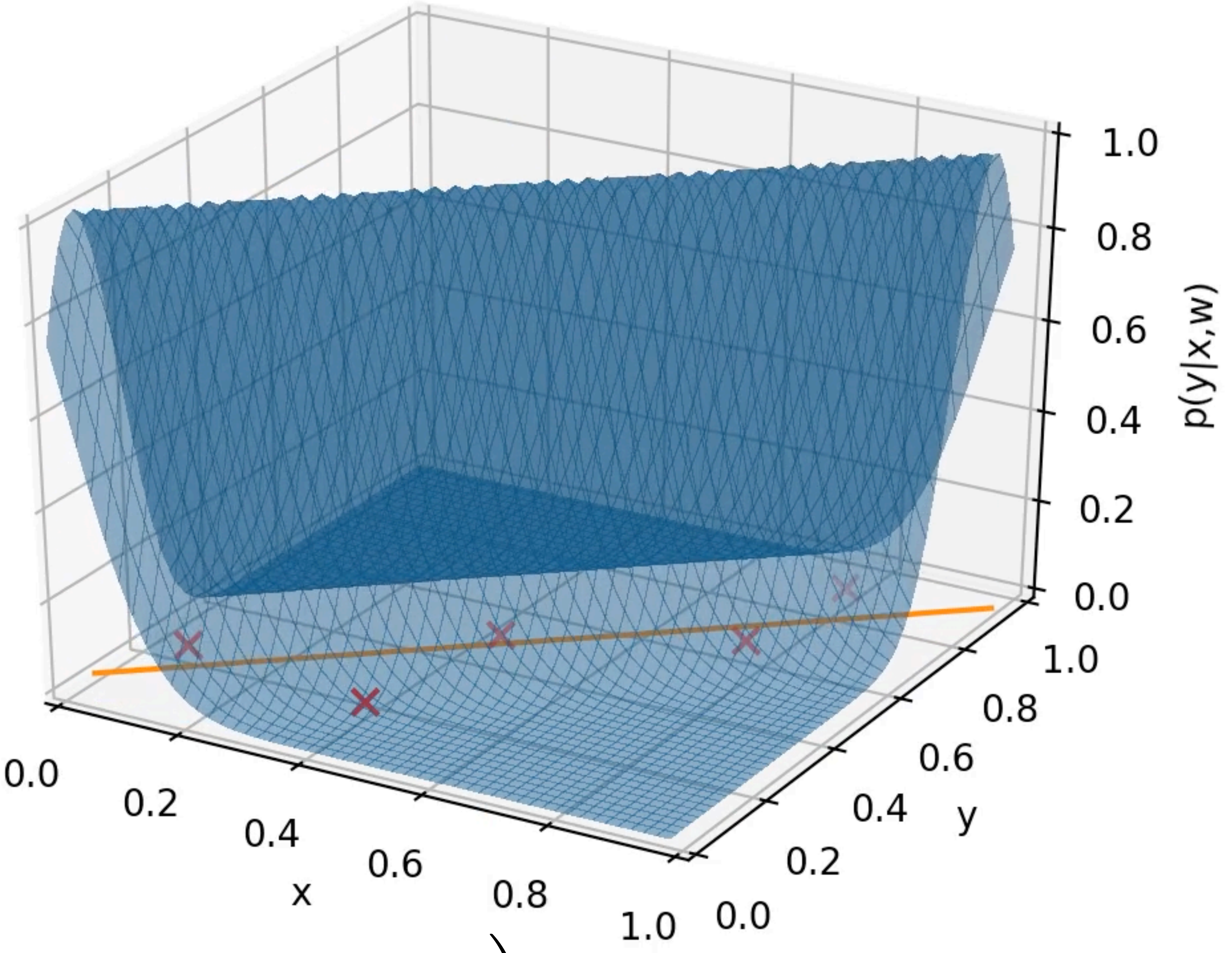


$$\mathcal{D} = \{\mathbf{x}_1, y_1 \ldots \mathbf{x}_N, y_N\}$$

$$\mathcal{N}_y(w_1 x + w_0,\ \sigma^2)$$



$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \left(\prod_i \mathcal{N}_{y_i}(w_1 x_i + w_0, \sigma^2) \right) = \arg\min_{\mathbf{w}} \sum_i (w_1 x_i + w_0 - y_i)^2$$



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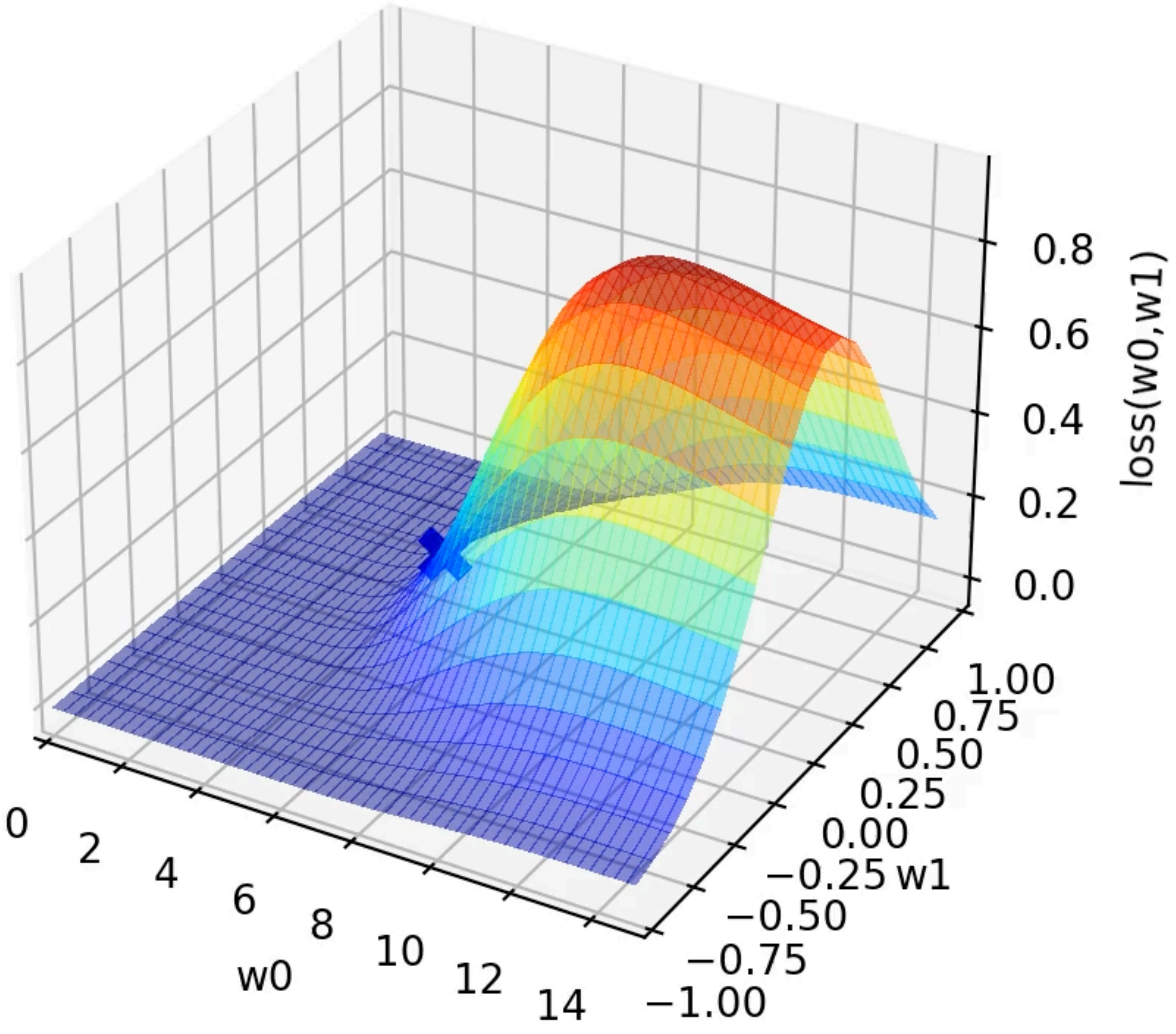
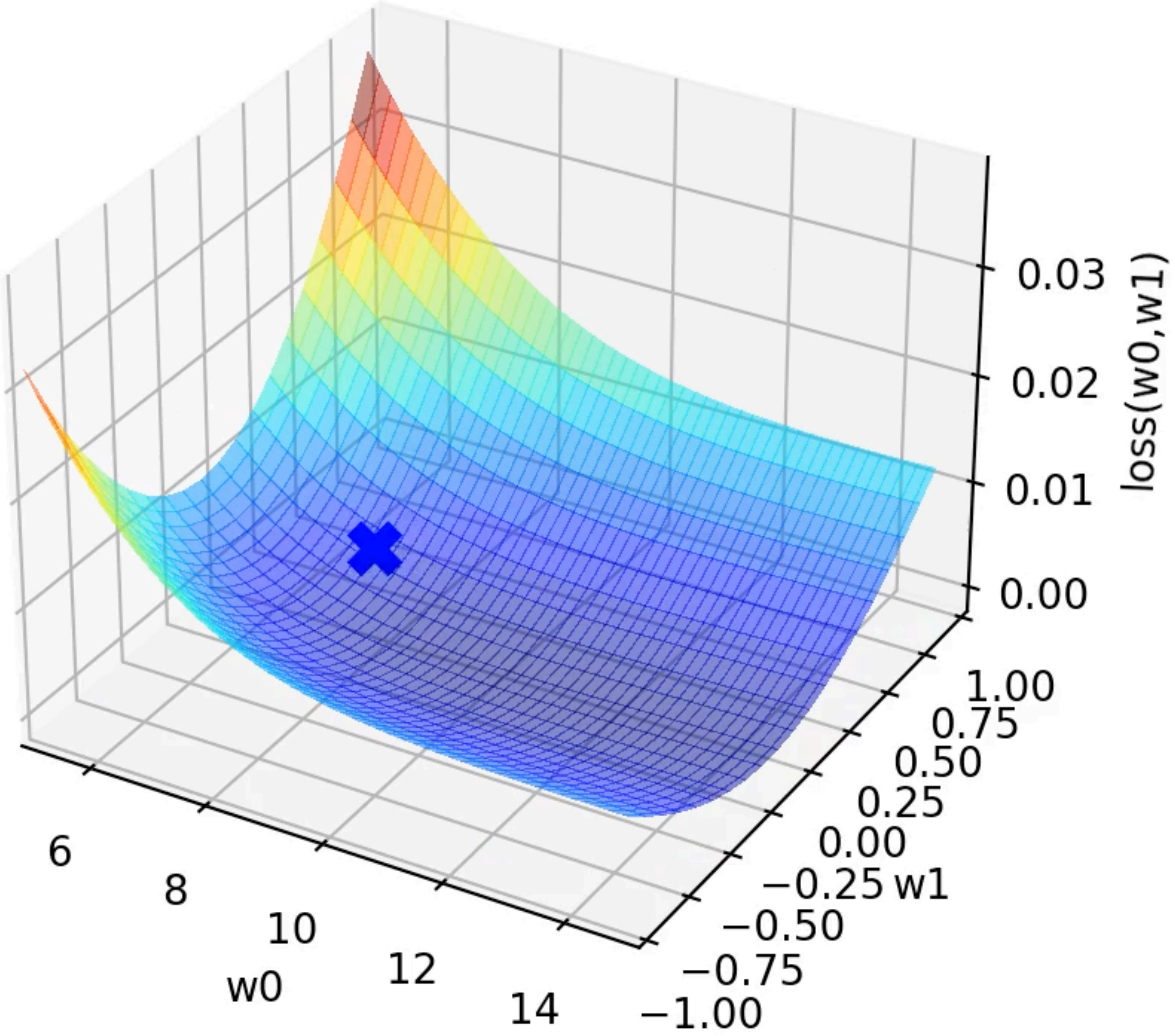


MLE



LS

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \left(\prod_i \mathcal{N}_{y_i}(w_1 x_i + w_0, \sigma^2) \right) = \arg\min_{\mathbf{w}} \sum_i (w_1 x_i + w_0 - y_i)^2$$

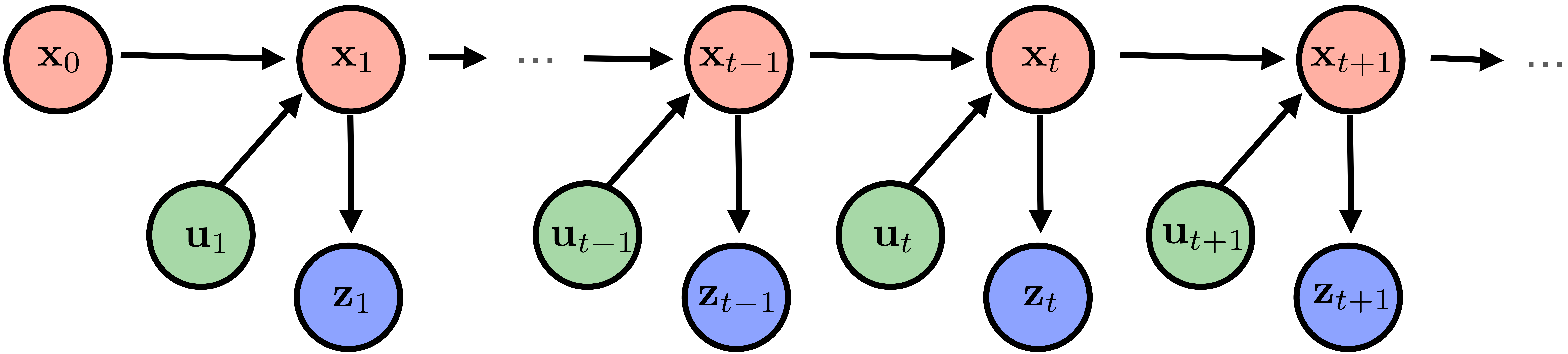
Localization problem definition

States: $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t \in \mathcal{R}^n$ …. 6DOF robot's poses (no map for now)

Actions: $\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathcal{R}^m$ …. generated by external source

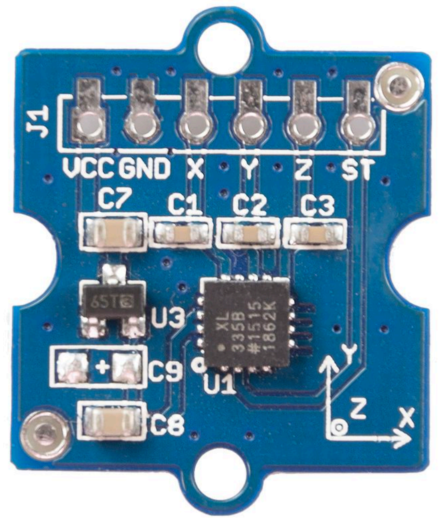
Measurements: $\mathbf{z}_1, \ldots, \mathbf{z}_t \in \mathcal{R}^k$ …. comes from variety of sensors

Goal:
- estimate most probable $\mathbf{x}_0 \ldots \mathbf{x}_t$
- or just $\mathbf{x}_t$
- or full distribution $p(\mathbf{x}_t)$ or even $p(\mathbf{x}_0 \ldots \mathbf{x}_t)$

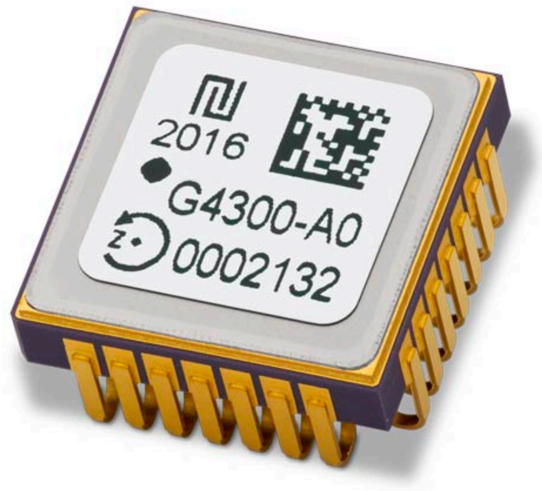


Sensors for localisation (odometry)

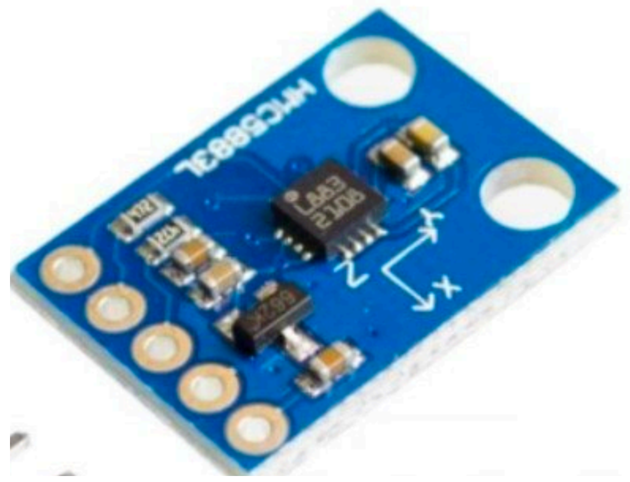
Motor encoders (wheel/joint position/velocity)



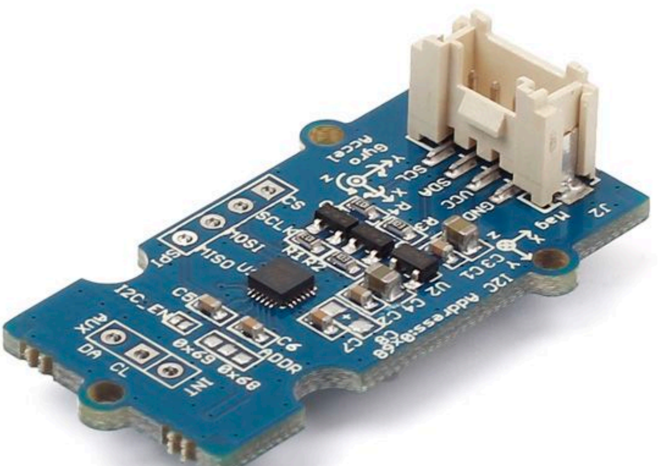
Accelerometer (linear acceleration)



Gyroscope (angular velocity)



Magnetometer (angle to magnetic north)



IMU: Accelerometer+Gyroscope+Magnetometer (9DOF measurements

Sensors for localisation (exteroceptive)

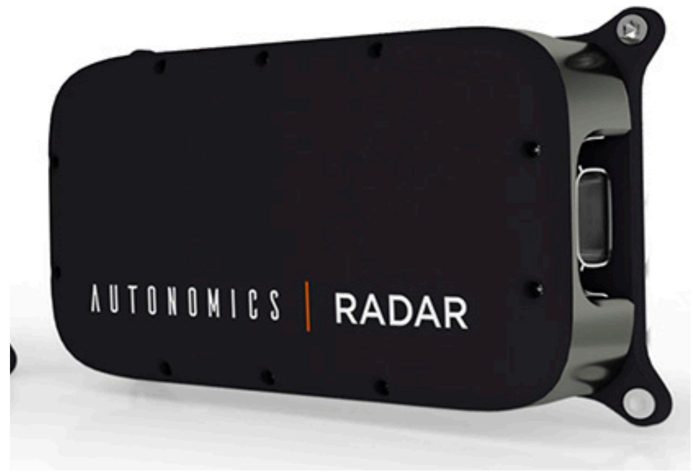
Camera (RGB images - spectral responses projected on image plane)

Stereo camera

RGBD camera (kinect, real sense, …

Lidar

Sonar



Radar

Satelite navigation (GPS/GNSS)

SONARDYNE beacons

UWB

Sensor measurements

- Noise characteristic (GPS vs camera for localisation)
- Operates in its own coordinate frame
- Spatiotemporal (and spectral) resolution
  (i.e. number of pixels/channels in image, number of measurements per second)
- Absolute/relative measurements wrt a reference coordinate frame
  (e.g. GPS/IMU) and integrating the relative measurements does not work!

Consequence: Need a reasonable probabilistic approach that fuses all measurements in order to estimate the most probable pose(s)

Localisation problem definition

Today only 1D/2D translations (no rotations)

States: $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_t \in \mathcal{R}^n$ .... ~~6DOF~~ robot's poses (no map for now)

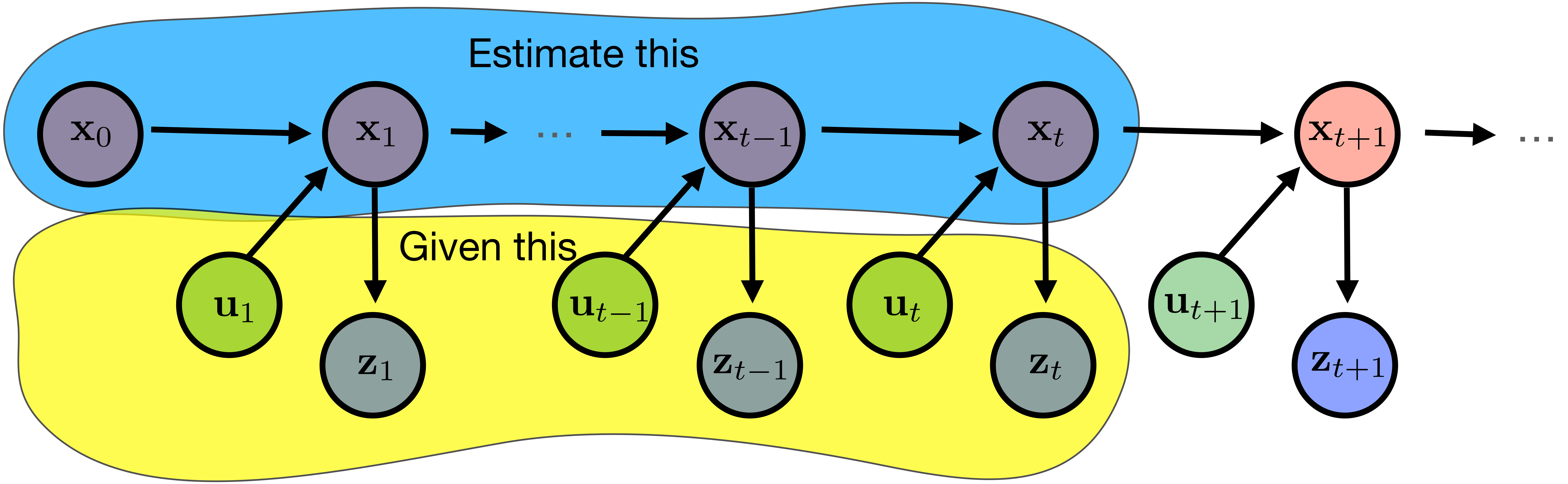
Actions: $\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathcal{R}^m$ .... generated by external source

Measurements: $\mathbf{z}_1, \ldots, \mathbf{z}_t \in \mathcal{R}^k$ .... comes from variety of sensors

Unknown

MAP: $\mathbf{x}^* = \arg\max_{\mathbf{x}} p(\mathbf{x} \,|\, \mathbf{z}, \mathbf{u}) = \arg\max_{\mathbf{x}_0 \ldots \mathbf{x}_t} p(\mathbf{x}_0 \ldots \mathbf{x}_t \,|\, \mathbf{z}_1 \ldots \mathbf{z}_t, \mathbf{u}_1 \ldots \mathbf{u}_t)$

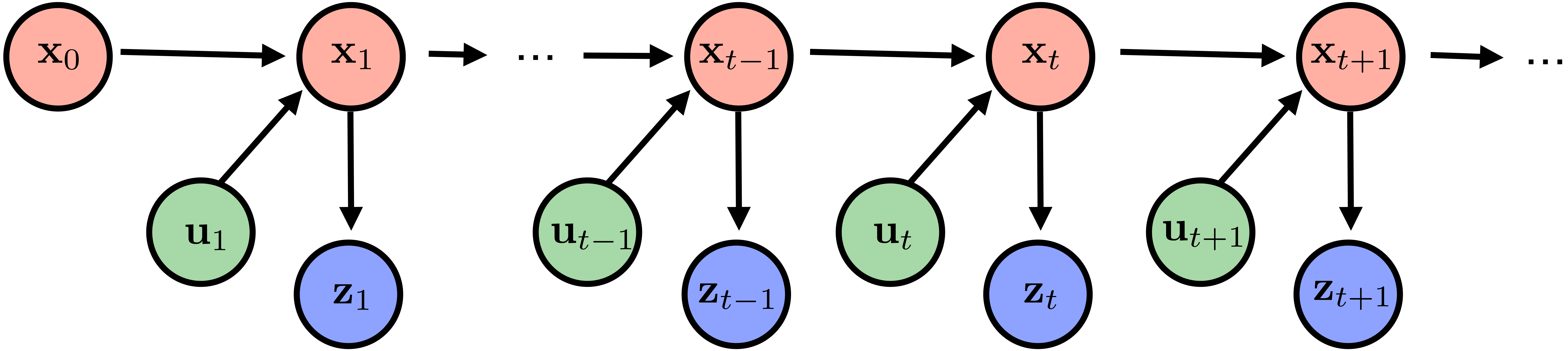
1. Construct p(x|z)
2. Optimize poses



Problem definition

Can we simplify it? $p(\mathbf{x}_0 \ldots \mathbf{x}_t \,|\, \mathbf{z}_1 \ldots \mathbf{z}_t, \mathbf{u}_1 \ldots \mathbf{u}_t)$

Def: A and B are *conditionally independent* given C iff $p(A|B,C) = p(A|C)$



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Def: A and B are *conditionally independent* given C iff $p(A|B,C) = p(A|C)$

state-transition probability: $p(\mathbf{x}_{t+1} \,|\, \mathbf{x}_t, \mathbf{u}_{t+1}) = p(\mathbf{x}_{t+1} \,|\, \mathbf{x}_t, \mathbf{u}_{t+1}, \mathbf{x}_{0:t-1}, \mathbf{z}_{1:t}, \mathbf{u}_{1:t})$

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Can we simplify it? $p(\mathbf{x}_0 \ldots \mathbf{x}_t \,|\, \mathbf{z}_1 \ldots \mathbf{z}_t, \mathbf{u}_1 \ldots \mathbf{u}_t)$

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measurement probability: $p(\mathbf{z}_t \,|\, \mathbf{x}_t) = p(\mathbf{z}_t \,|\, \mathbf{x}_t, \mathbf{x}_{0:t-1}, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t})$

Problem definition

Can we simplify it? $p(\mathbf{x}_0 \ldots \mathbf{x}_t \mid \mathbf{z}_1 \ldots \mathbf{z}_t, \mathbf{u}_1 \ldots \mathbf{u}_t)$

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$p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{x}_{t-1}) = p(\mathbf{z}_t \mid \mathbf{x}_t, \mathbf{x}_{t-1}, \mathbf{x}_{0:t-2}, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t})$

Single time instance only

+

Absolute pose measurement (e.g. GPS)

Localisation from **GPS** measurements only in **single time instance**

Assume only gps measurement in time t is known

MAP: $\mathbf{x}^* = \arg\max_{\mathbf{x}} p(\mathbf{x} \mid \mathbf{z}, \mathbf{u}) = \arg\max_{\mathbf{x}_0 \dots \mathbf{x}_t} p(\mathbf{x}_0 \dots \mathbf{x}_t \mid \mathbf{z}_1 \dots \mathbf{z}_t, \mathbf{u}_1 \dots \mathbf{u}_t) = \arg\max_{\mathbf{x}_t} p(\mathbf{x}_t \mid \mathbf{z}_t^{GPS})$

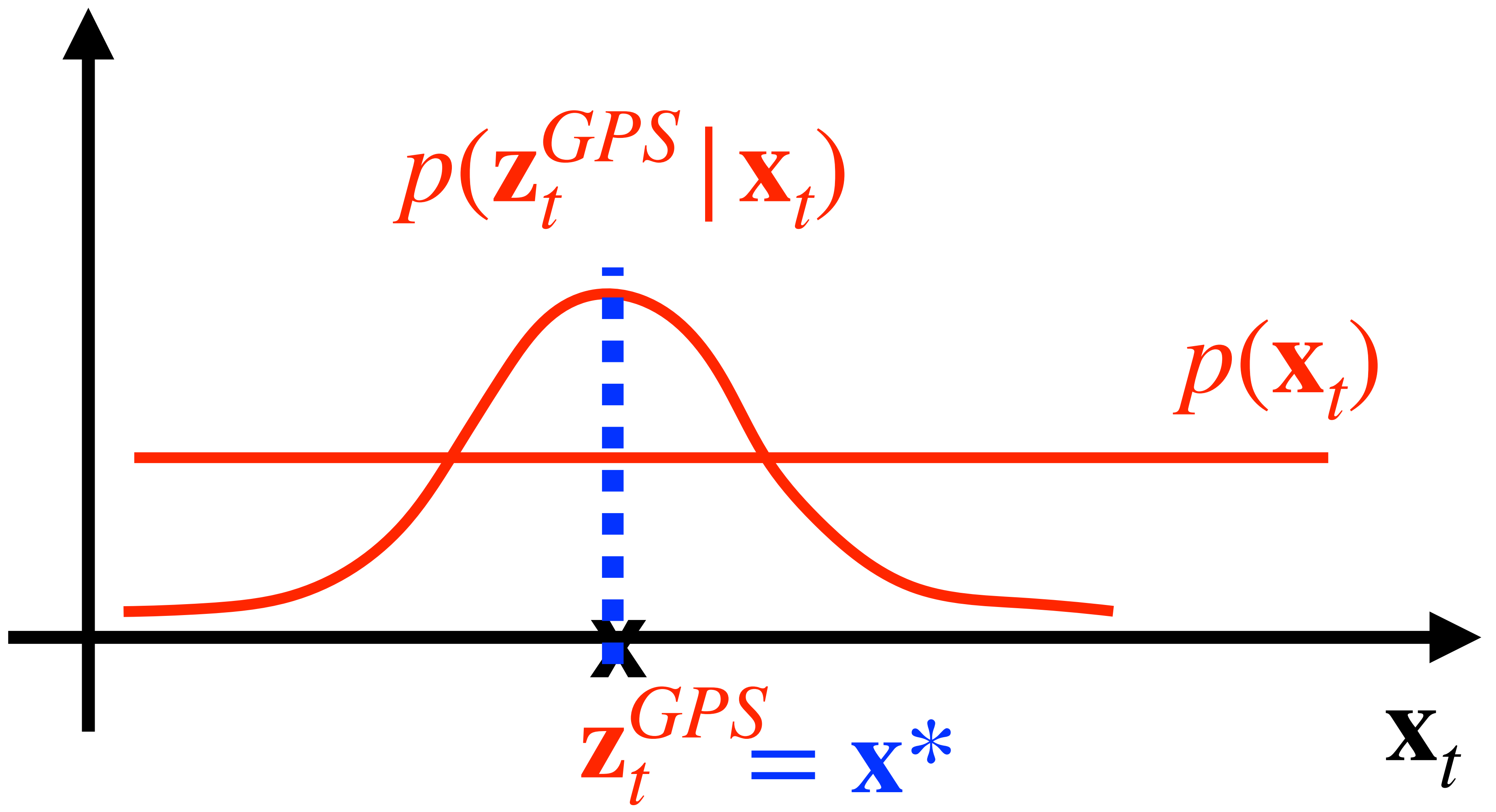
Bayes theorem — Uniform prior — Normal likelihood

$$= \arg\max_{\mathbf{x}_t} \frac{p(\mathbf{z}_t^{GPS} \mid \mathbf{x}_t)\, p(\mathbf{x}_t)}{p(\mathbf{z}_t^{GPS})} = \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS} \mid \mathbf{x}_t) = \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})$$

Measurements $\mathbf{z}_t^{GPS}$ are normally distributed around the true position $\mathbf{x}_t$

$$= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{x}_t; \mathbf{z}_t^{GPS}, \Sigma_t^{GPS})$$

True positions $\mathbf{x}_t$ are normally distributed around measurement $\mathbf{z}_t^{GPS}$



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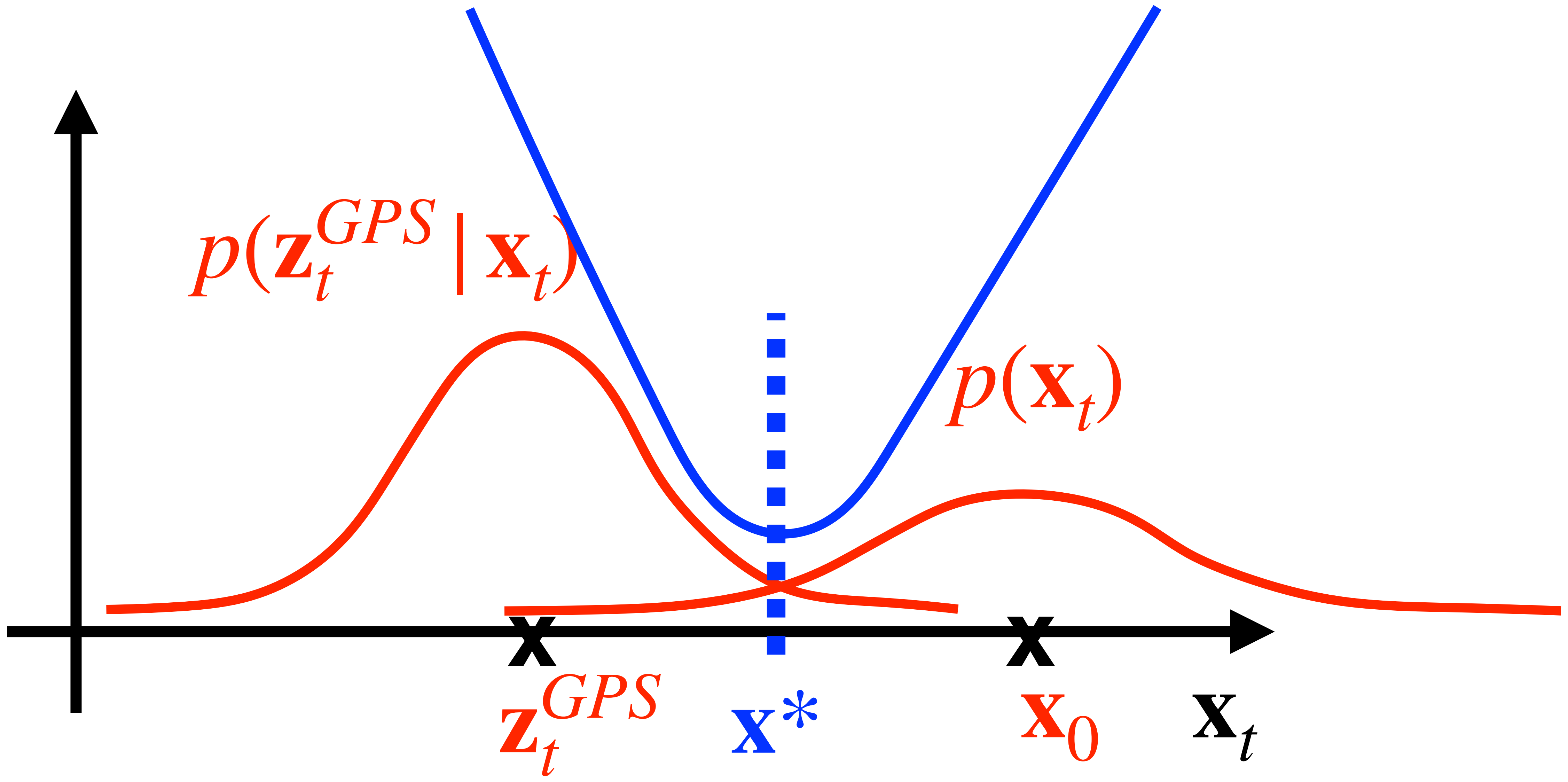
Bayes theorem

Normal prior and likelihood

$$= \arg\max_{\mathbf{x}_t} \frac{p(\mathbf{z}_t^{GPS} \mid \mathbf{x}_t)\, p(\mathbf{x}_t)}{p(\mathbf{z}_t^{GPS})} \quad = \arg\max_{\mathbf{x}_t} p(\mathbf{z}_t^{GPS} \mid \mathbf{x}_t) p(\mathbf{x}_t)$$

$$= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS}) \mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \quad = \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^2 \frac{1}{\Sigma_t^{GPS}} + (\mathbf{x}_t - \mathbf{x}_0)^2 \frac{1}{\Sigma_0}$$

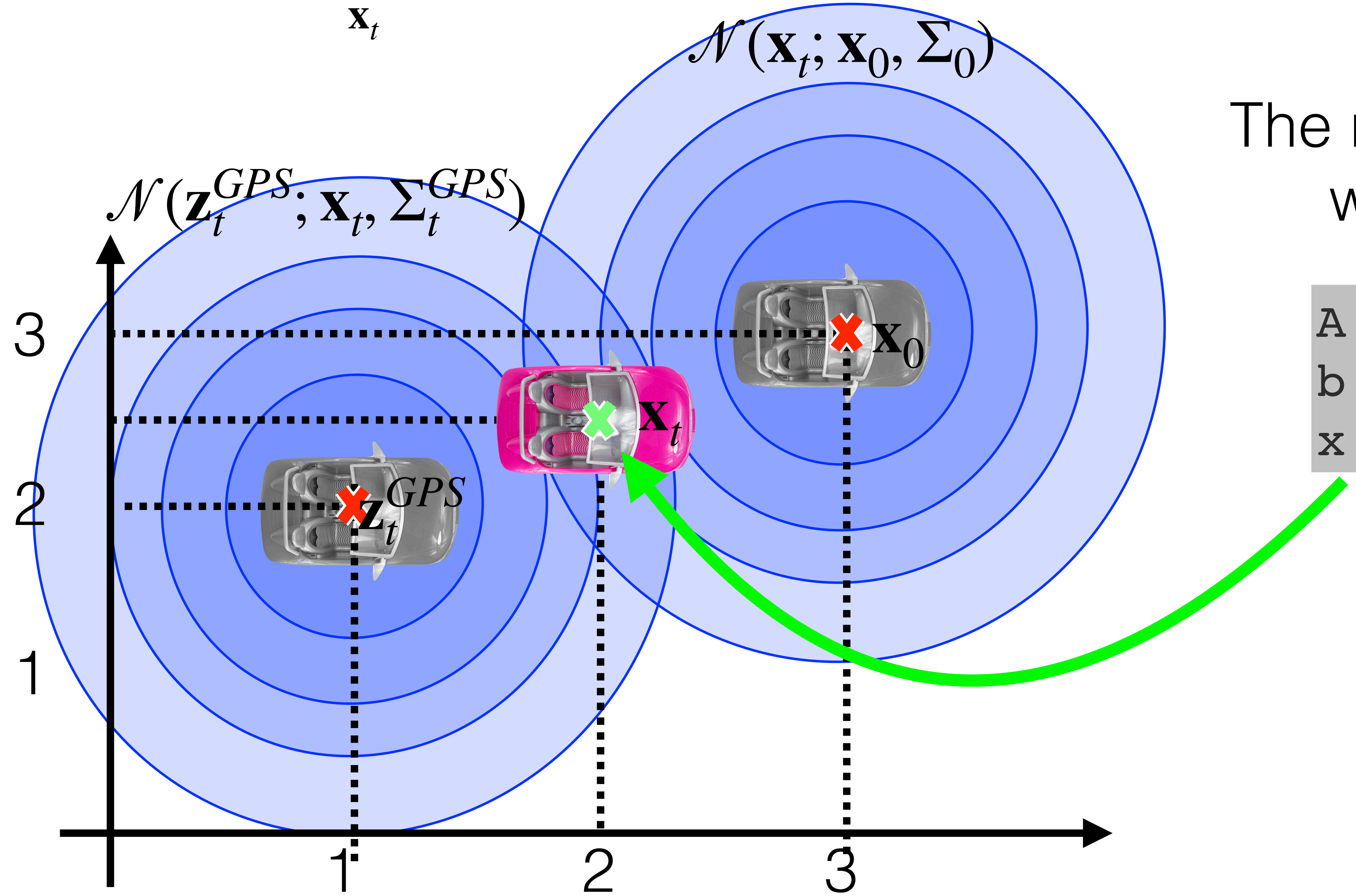


Example: 2D Localisation from **GPS** measurements only in **single time instance**

$$\begin{aligned}\mathbf{x}^\star &= \arg\max_{\mathbf{x}_t} \mathcal{N}(\mathbf{z}_t^{GPS}; \mathbf{x}_t, \Sigma_t^{GPS})\mathcal{N}(\mathbf{x}_t; \mathbf{x}_0, \Sigma_0) \\ &= \arg\min_{\mathbf{x}_t} \ \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2 + \|\mathbf{x}_t - \mathbf{x}_0\|^2 \\ &= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top(\mathbf{x}_t - \mathbf{x}_0)\end{aligned}$$

$$\mathbf{z}_t^{GPS} = [1,2]^\top, \ \ \mathbf{x}_0 = [3,3]^\top$$

$$\Sigma_t^{GPS} = \Sigma_0 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$



The result is linear least squares with closed-form solution

```
A = [1,0; 0,1; 1,0; 0,1]
b = [1; 2; 3; 3]
x = pinv(A)*b
```

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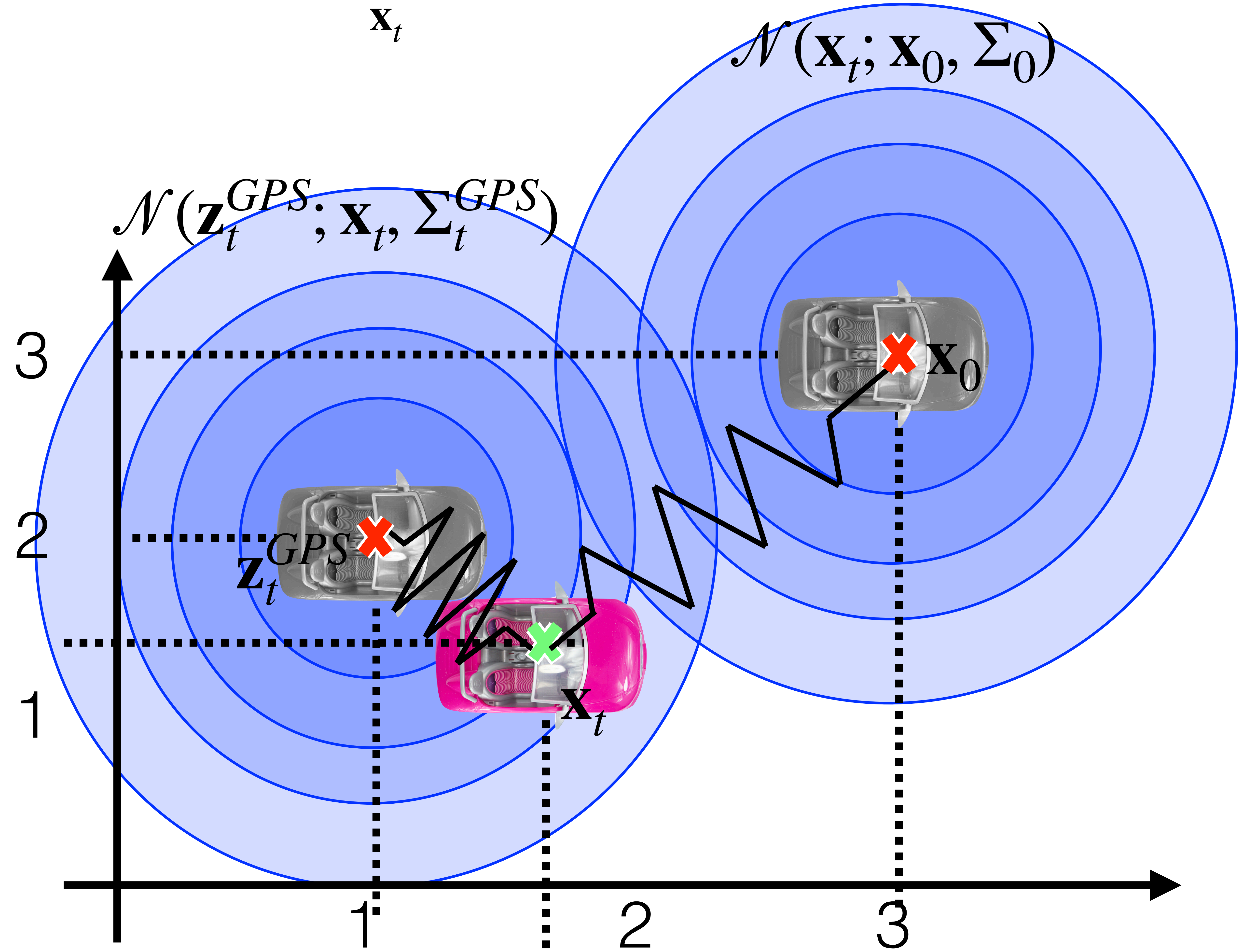

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Equilibrium of mechanical machine (i.e. state with minimum energy)

$$F = k\mathbf{x} \Rightarrow E = \frac{1}{2}k\mathbf{x}^2$$

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$$= \arg\min_{\mathbf{x}_t} (\mathbf{x}_t - \mathbf{z}_t^{GPS})^\top(\mathbf{x}_t - \mathbf{z}_t^{GPS}) + (\mathbf{x}_t - \mathbf{x}_0)^\top(\mathbf{x}_t - \mathbf{x}_0)$$

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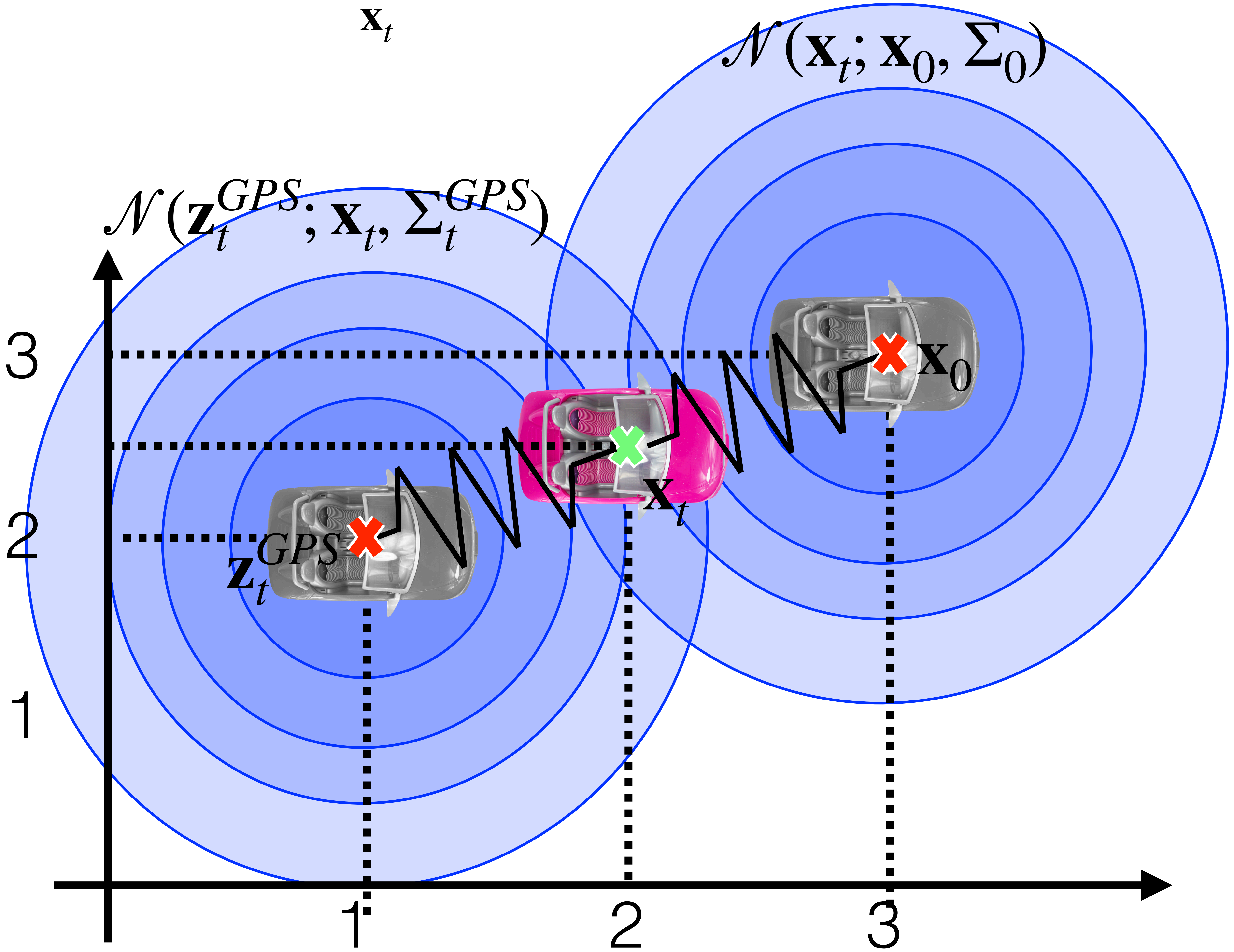

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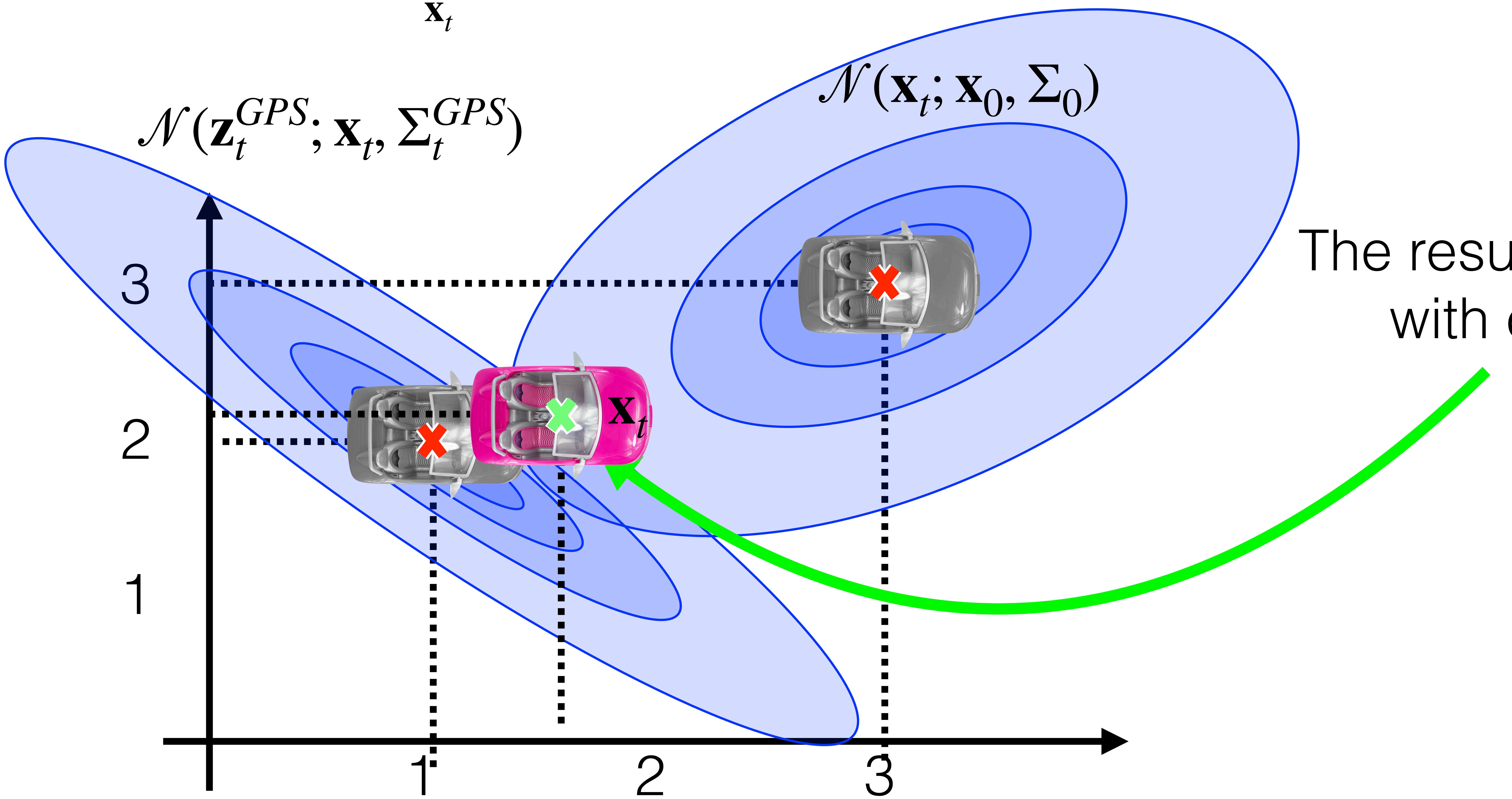
$\mathbf{z}_t^{GPS} = [1,2]^\top, \;\; \mathbf{x}_0 = [3,3]^\top$

Σ_t^{GPS}, Σ_0



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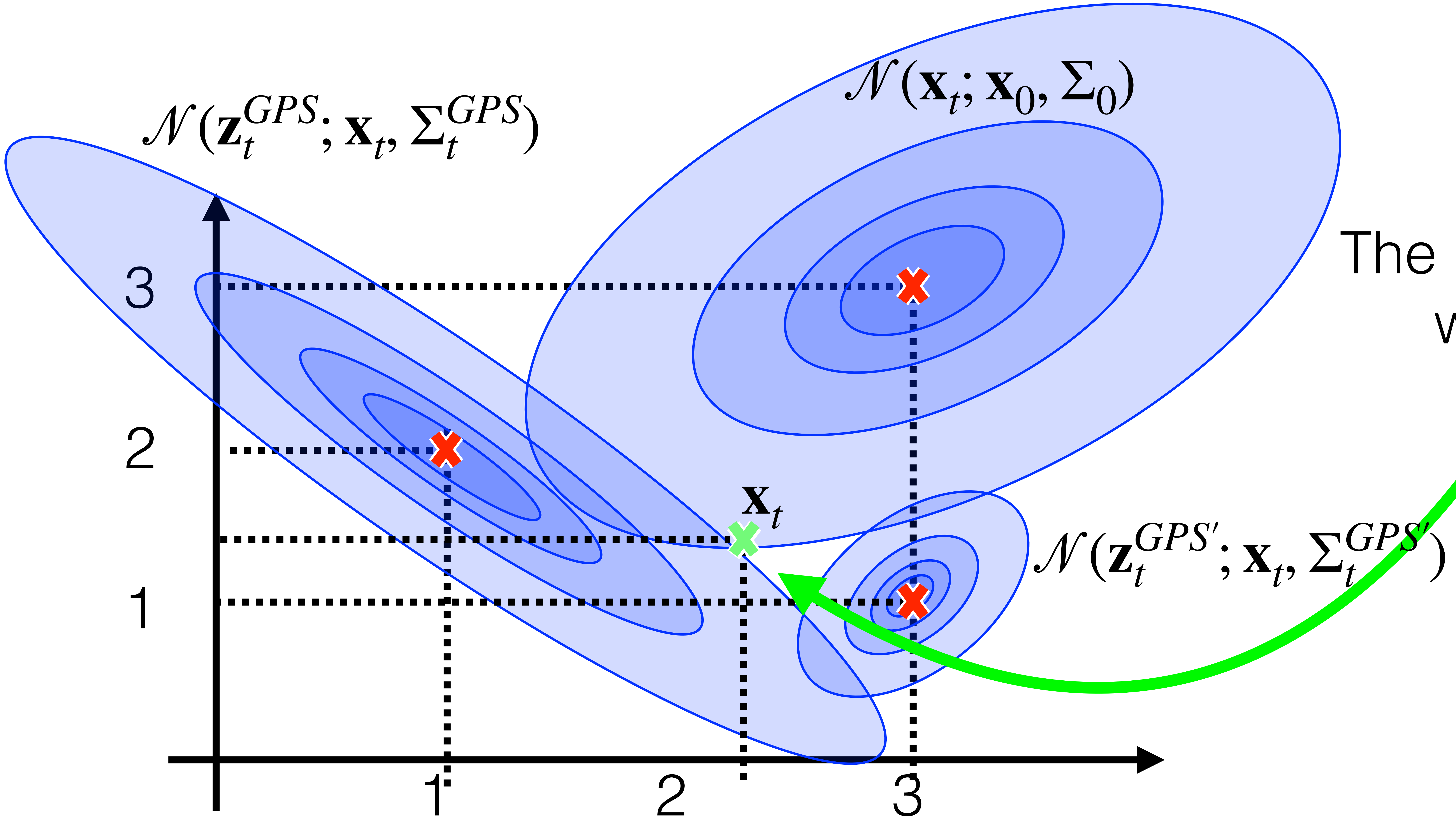
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_t} \; \|\mathbf{x}_t - \mathbf{z}_t^{GPS}\|^2_{\Sigma_t^{GPS}} + \|\mathbf{x}_t - \mathbf{z}_t^{GPS'}\|^2_{\Sigma_t^{GPS'}} + \|\mathbf{x}_t - \mathbf{x}_0\|^2_{\Sigma_0}$$



The result is linear least squares with closed-form solution

Multiple time instances

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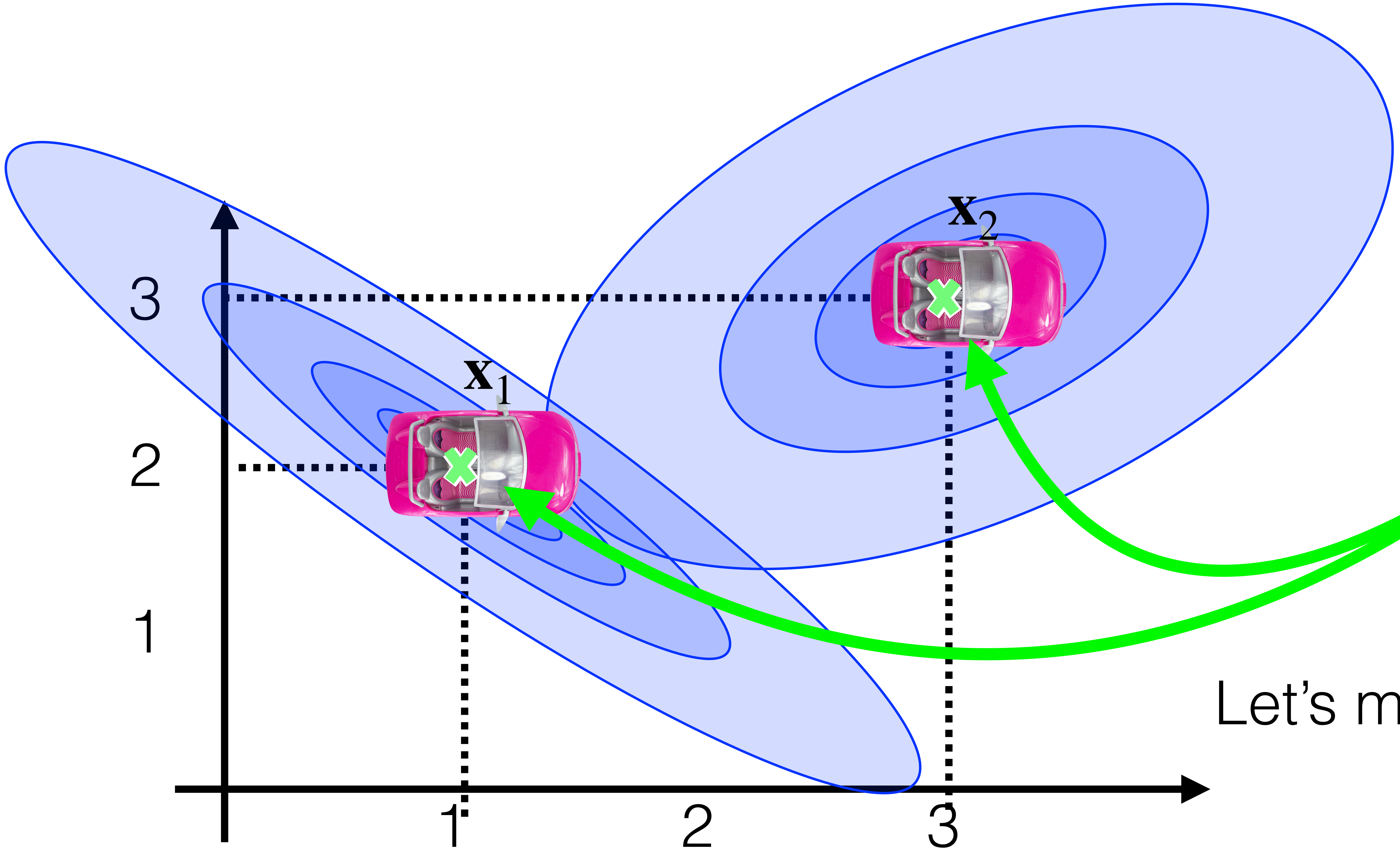
Absolute pose measurement (e.g. GPS)

Localisation in **multiple time instances**

$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_1,\mathbf{x}_2} \mathcal{N}(\mathbf{z}_1^{GPS}; \mathbf{x}_1, \Sigma_1^{GPS})\mathcal{N}(\mathbf{z}_2^{GPS}; \mathbf{x}_2, \Sigma_2^{GPS})$$
$$= \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}$$

$\mathbf{z}_1^{GPS} = [1,2]^\top$, $\mathbf{z}_2^{GPS} = [3,3]^\top$

$\Sigma_1^{GPS}, \Sigma_2^{GPS}$



The solution splits into two trivial subproblems (no relation between poses is modeled)

Let's model a relation between poses !

Multiple time instances

+

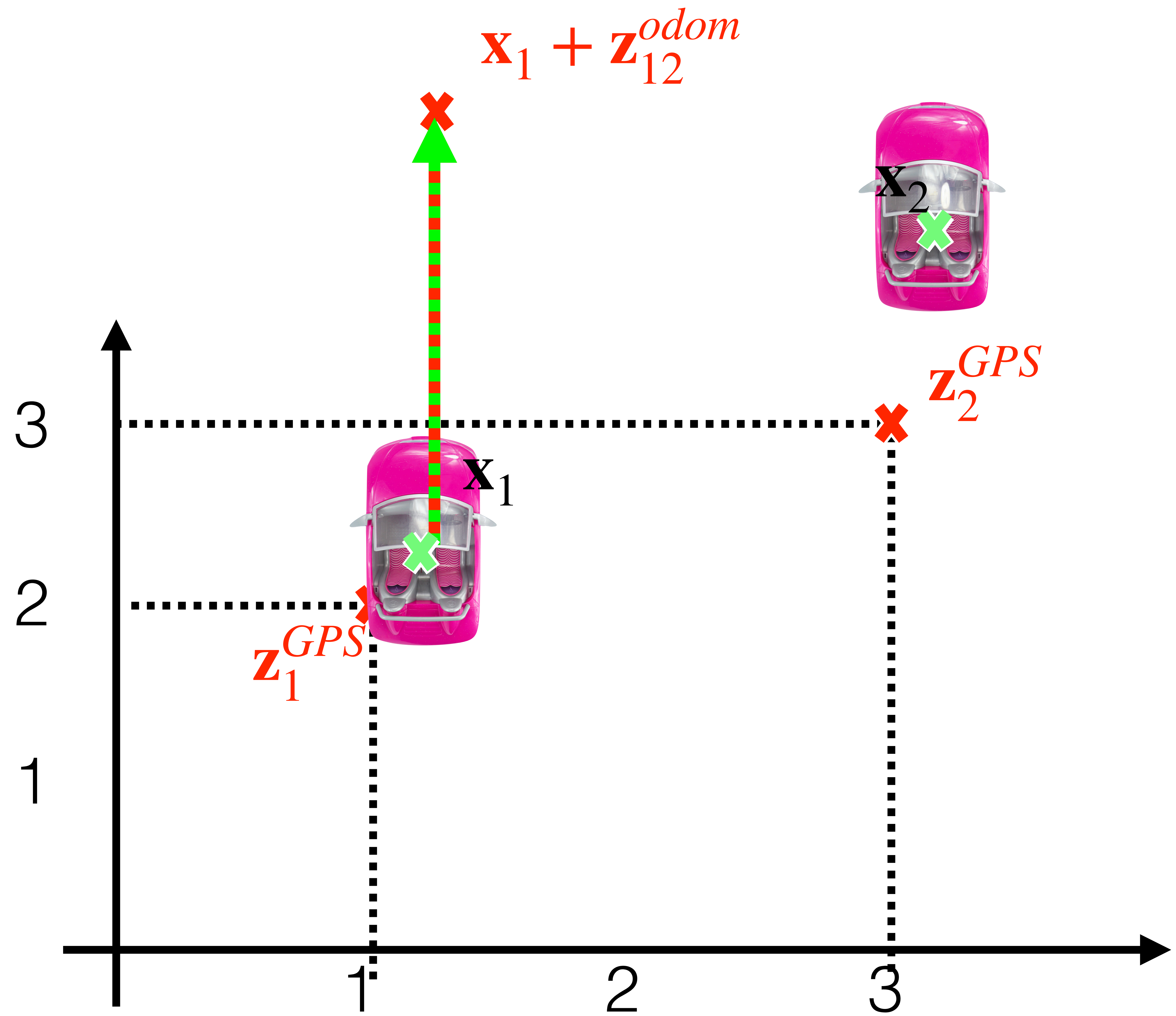
Absolute pose measurement (e.g. GPS)

+

Relative pose measurement (e.g.odometry from wheels/IMU/camera/lidar)

2D Localisation in **multiple time instances** from **GPS+odom**

$\mathbf{x}_1^\star, \mathbf{x}_2^\star \;\; =???$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

Assume only two absolute gps measurements and one relative odom. measurement

$$\mathbf{x}^* = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{x}_0\ldots\mathbf{x}_t \mid \mathbf{z}_1\ldots\mathbf{z}_t, \mathbf{u}_1\ldots\mathbf{u}_t) = \arg\max_{\mathbf{x}_1,\mathbf{x}_2} p(\mathbf{x}_1, \mathbf{x}_2 \mid \mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom})$$

Bayes theorem

$$= \arg\max_{\mathbf{x}_1,\mathbf{x}_2} \frac{p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom} \mid \mathbf{x}_1, \mathbf{x}_2)\, p(\mathbf{x}_1, \mathbf{x}_2)}{p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom})}$$

Uniform prior

$$= \arg\max_{\mathbf{x}_t} p(\mathbf{z}_1^{GPS}, \mathbf{z}_2^{GPS}, \mathbf{z}_{12}^{odom} \mid \mathbf{x}_1, \mathbf{x}_2)$$

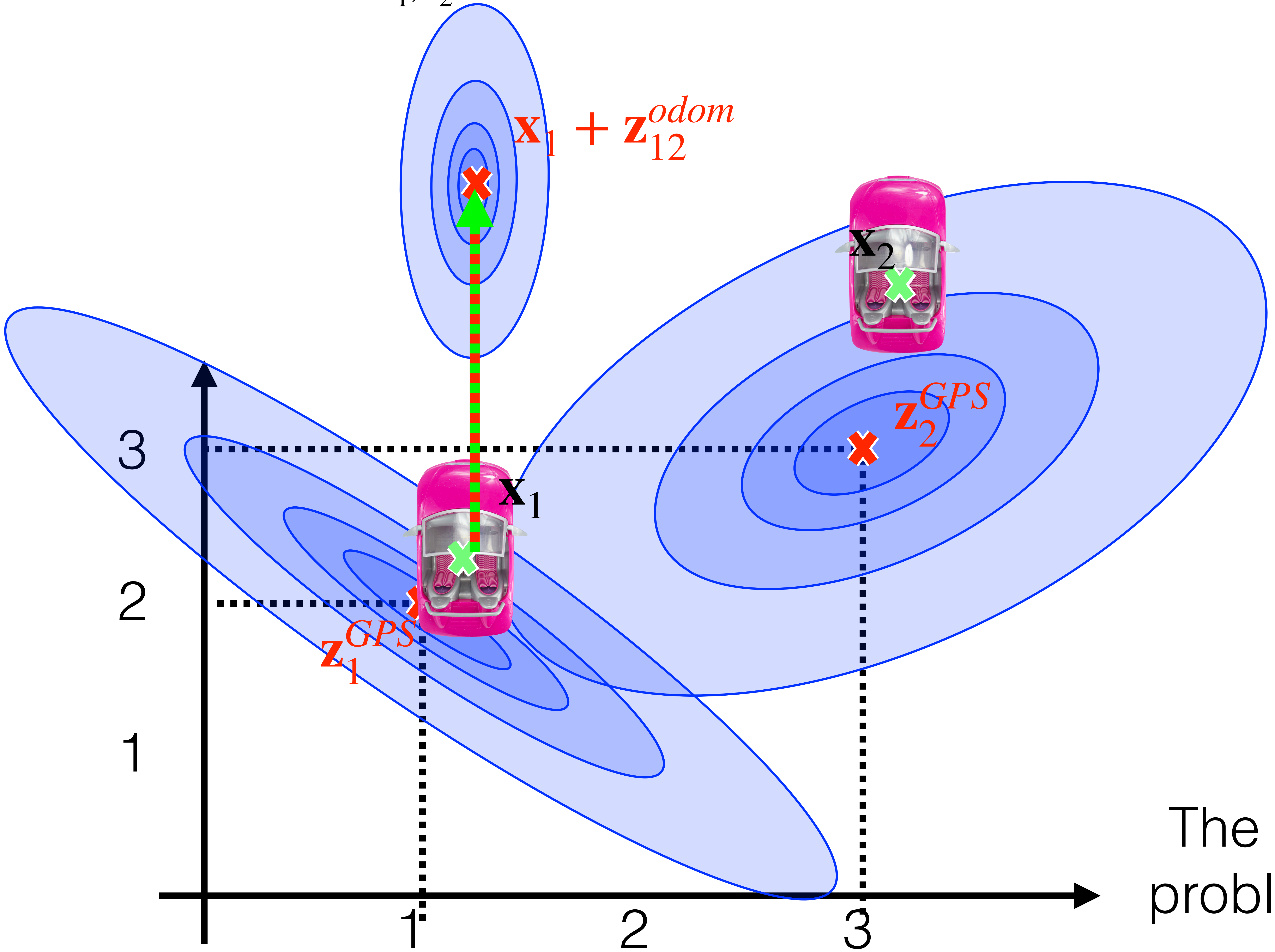
Normal likelihood

$$= \arg\max_{\mathbf{x}_1,\mathbf{x}_2} \mathcal{N}(\mathbf{z}_1^{GPS}; \mathbf{x}_1, \Sigma_1^{GPS})\mathcal{N}(\mathbf{z}_2^{GPS}; \mathbf{x}_2, \Sigma_2^{GPS})\mathcal{N}(\mathbf{z}_{12}^{odom}; \mathbf{x}_2 - \mathbf{x}_1, \Sigma_{12}^{odom})$$

$$= \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \ \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$

2D Localisation in **multiple time instances** from **GPS+odom**

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$

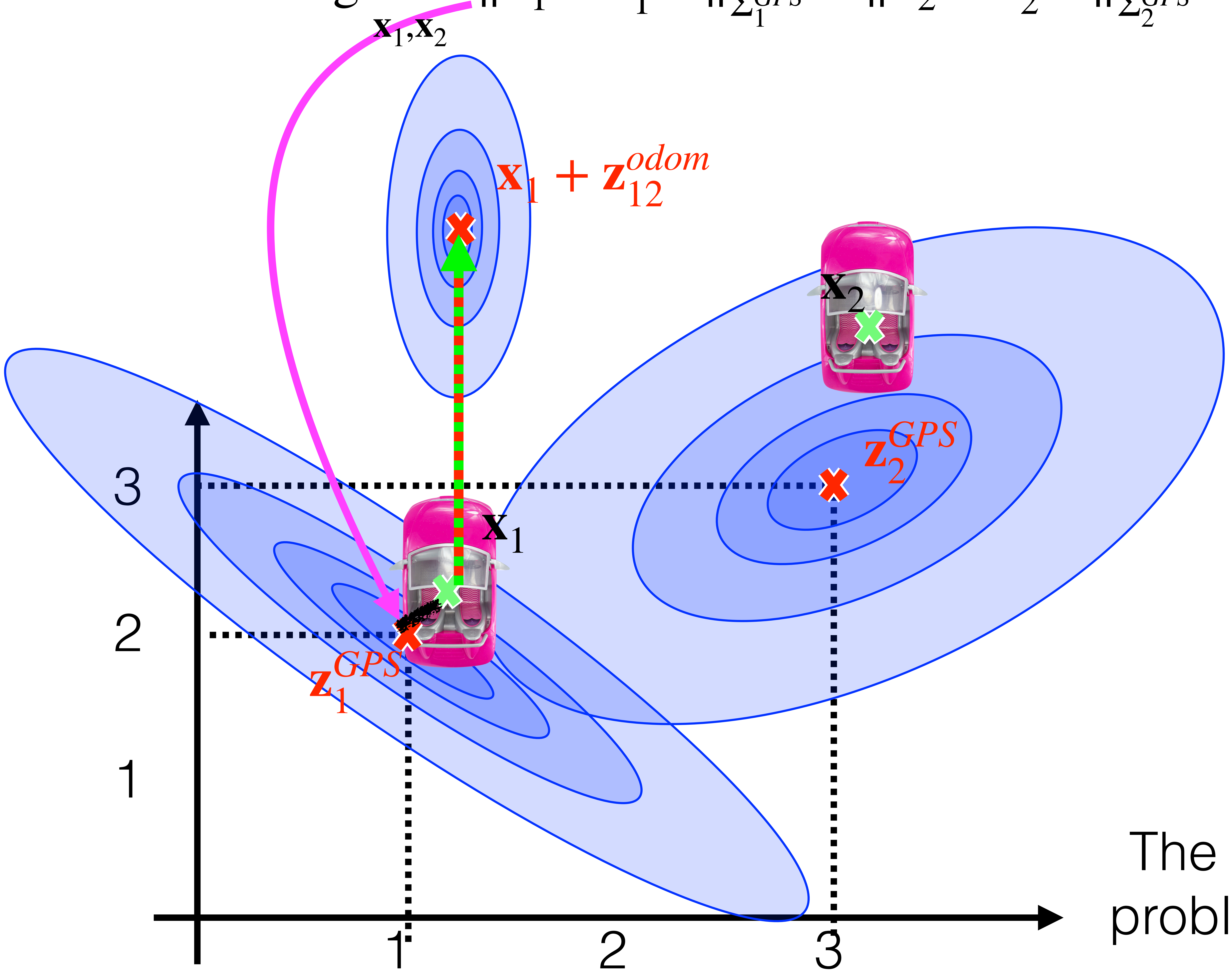


The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

unary

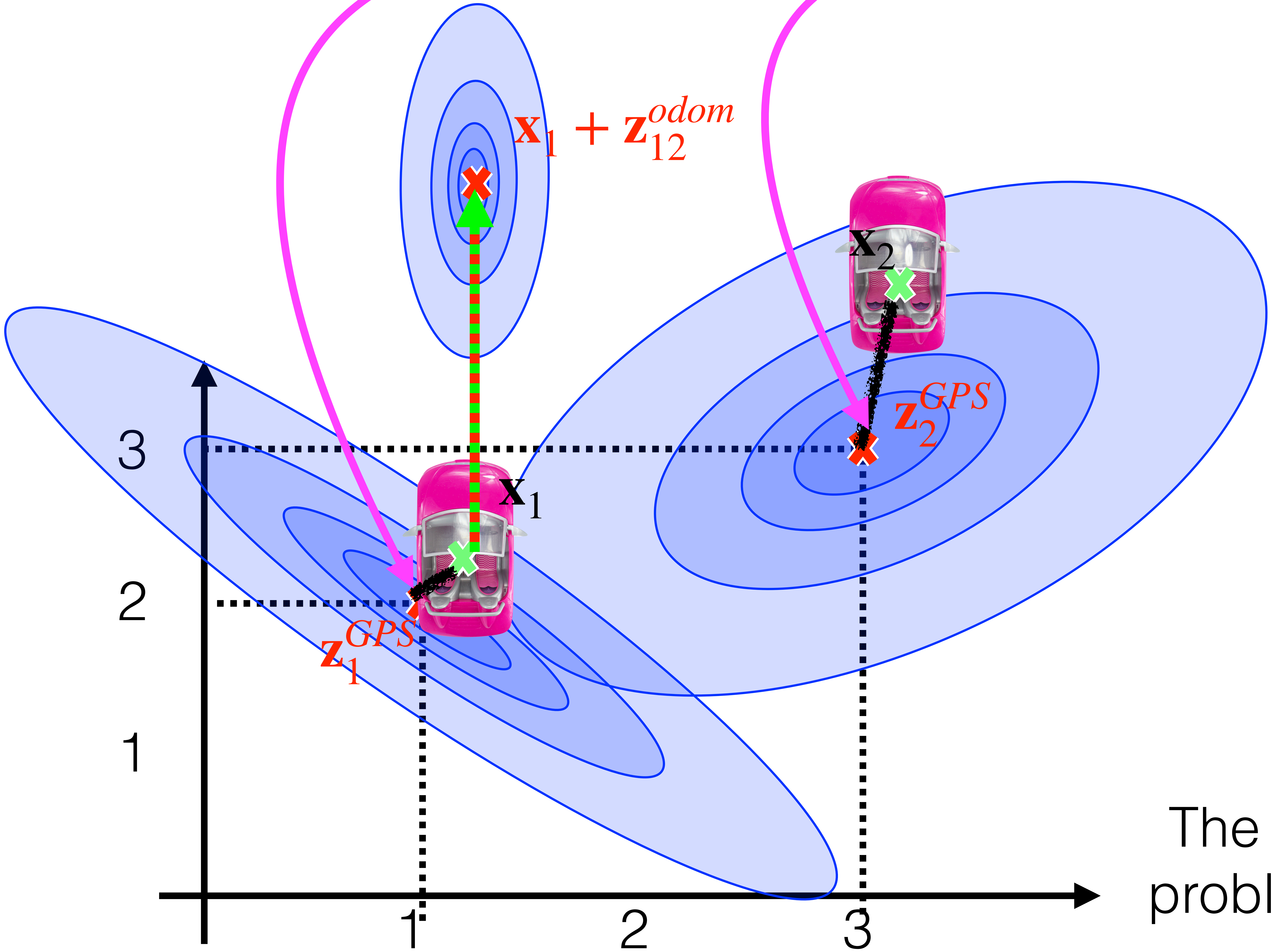
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \underbrace{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}}_{\text{unary}} + \underbrace{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}}_{\text{unary}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$

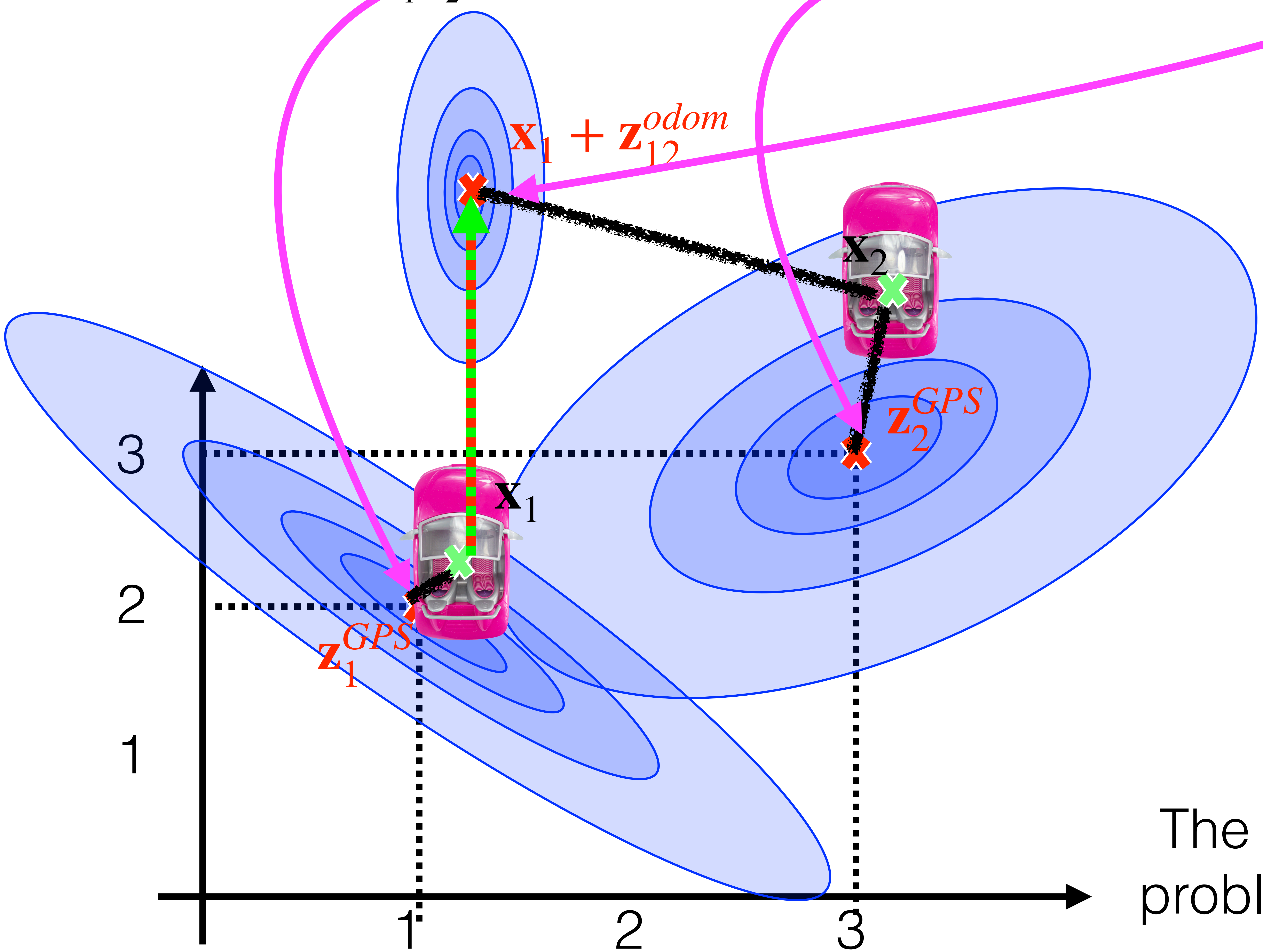


The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

unary — unary — pair-wise

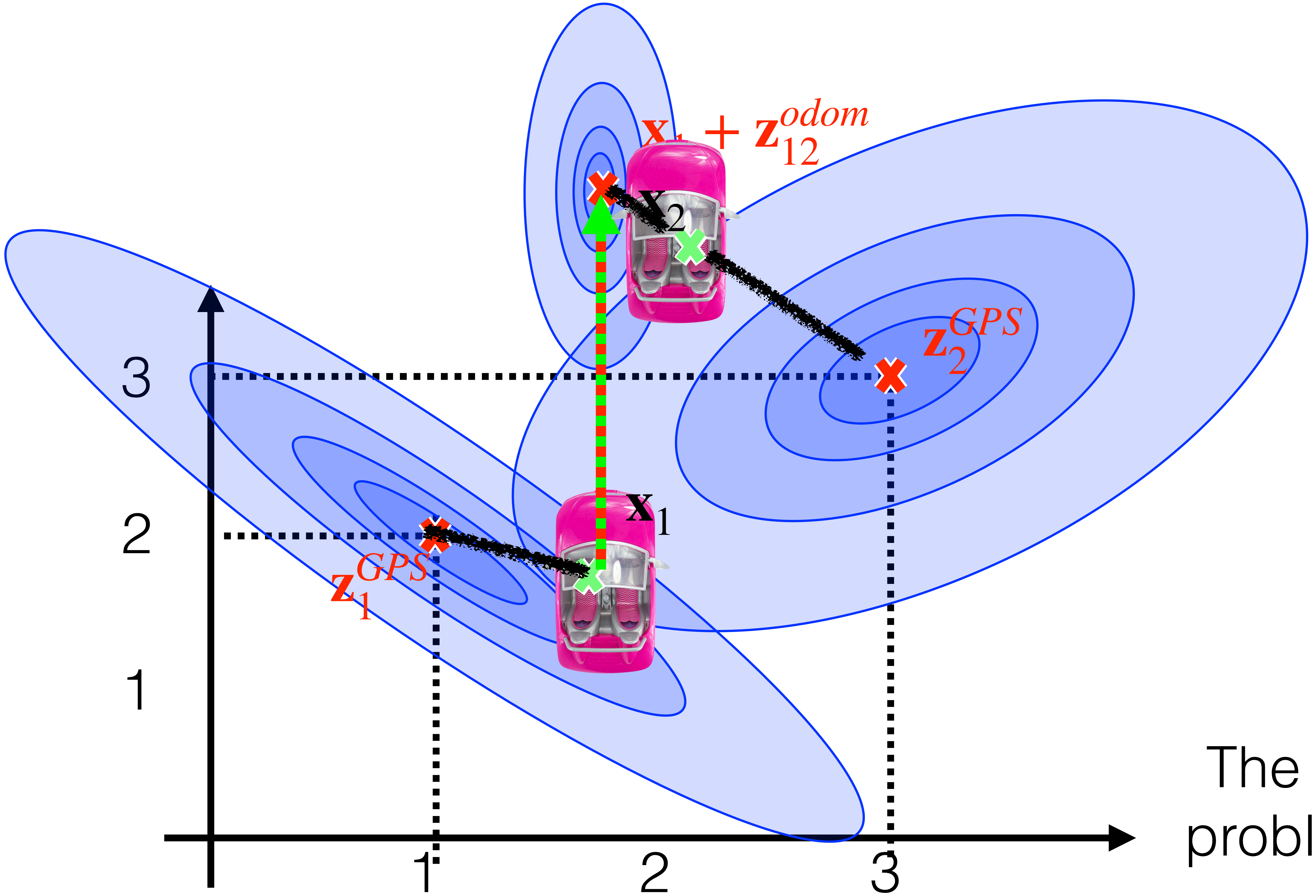
$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \underbrace{\|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}}}_{\text{unary}} + \underbrace{\|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}}}_{\text{unary}} + \underbrace{\|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}}_{\text{pair-wise}}$$

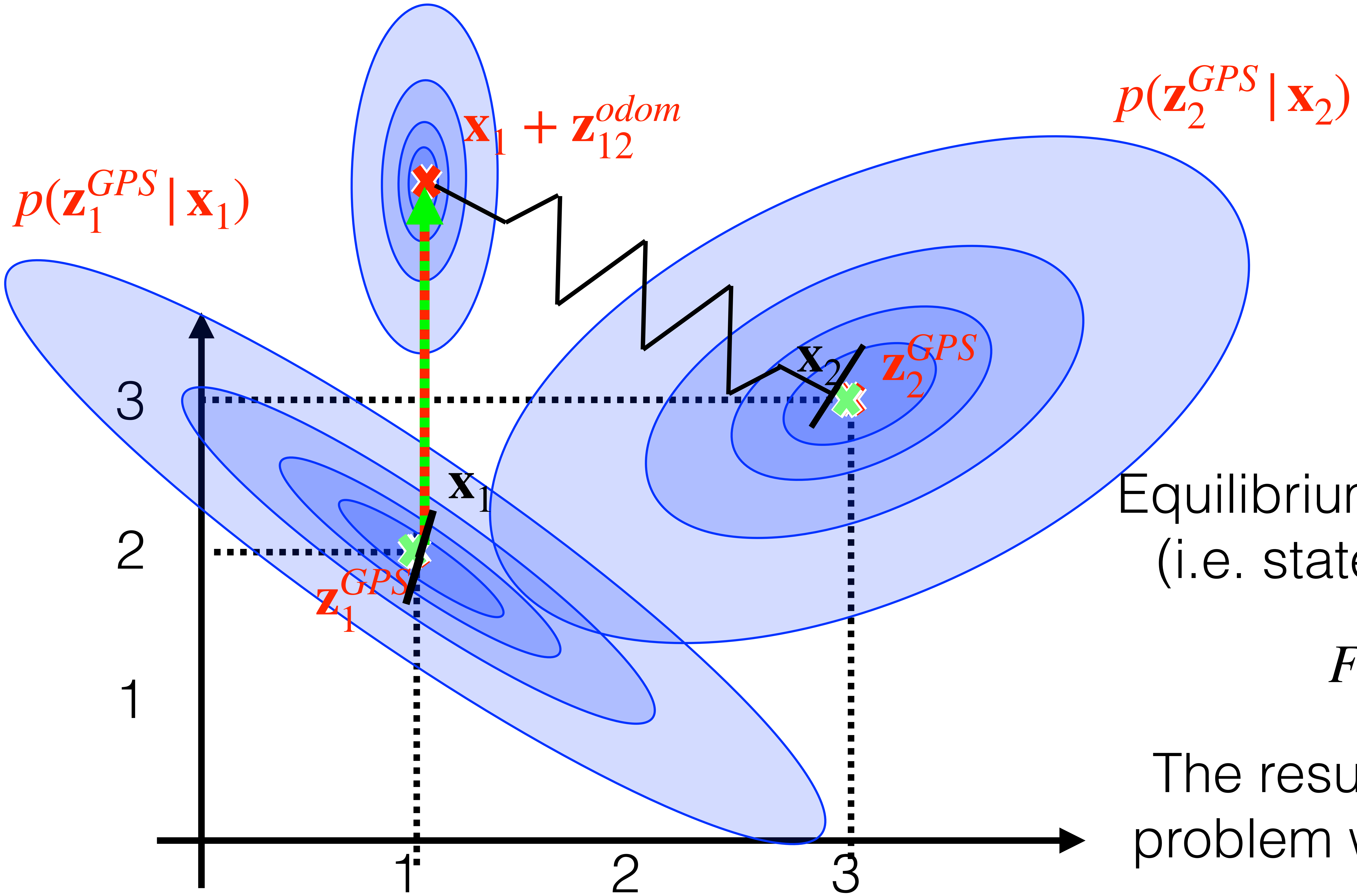


The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

unary — unary — pair-wise

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



Equilibrium of mechanical machine (i.e. state with minimum energy)

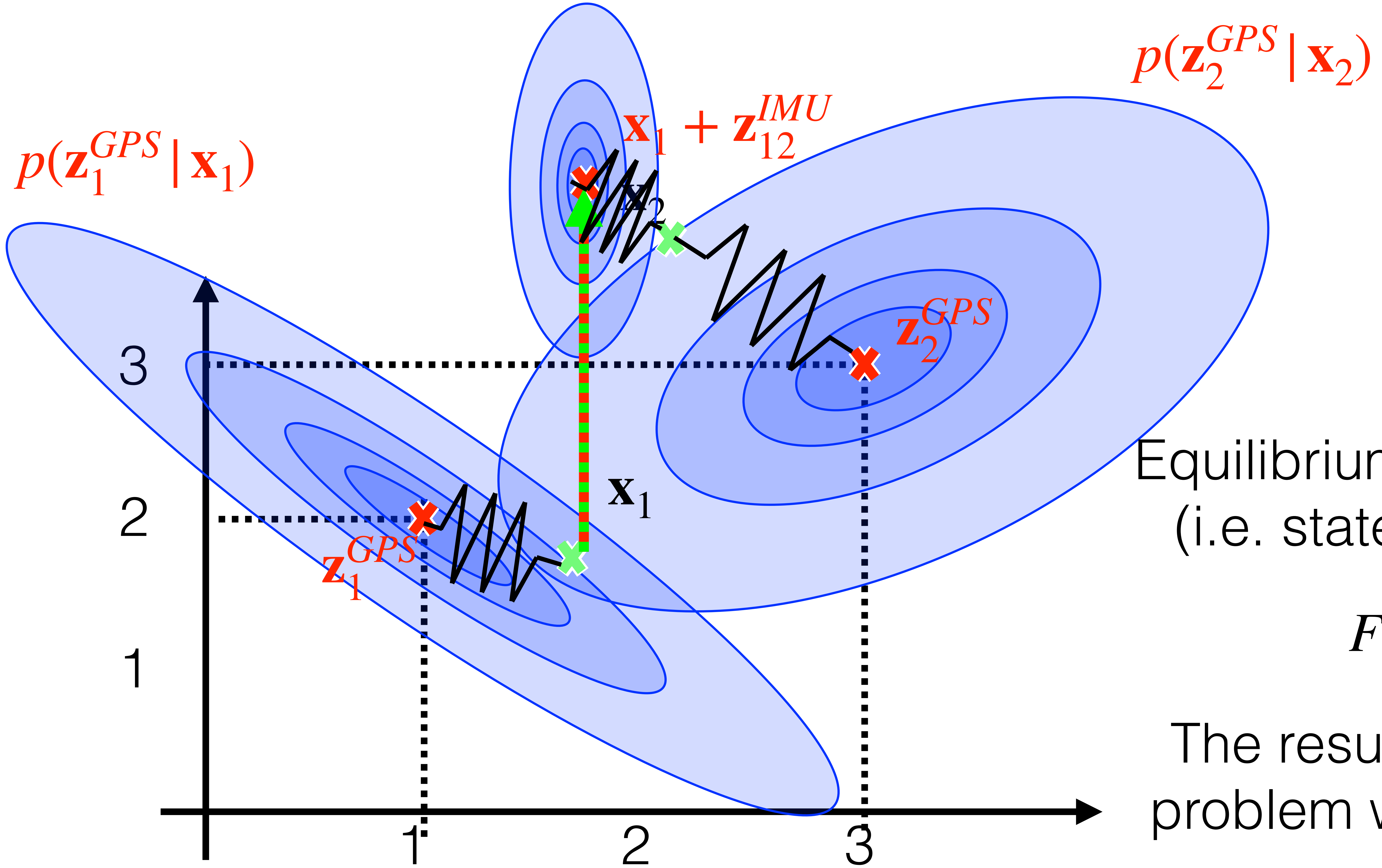
$$F = k\mathbf{x} \Rightarrow E = \frac{1}{2}k\mathbf{x}^2$$

The result is linear least squares problem with closed-form solution

2D Localisation in **multiple time instances** from **GPS+odom**

unary, unary, pair-wise

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_1,\mathbf{x}_2} \|\mathbf{x}_1 - \mathbf{z}_1^{GPS}\|^2_{\Sigma_1^{GPS}} + \|\mathbf{x}_2 - \mathbf{z}_2^{GPS}\|^2_{\Sigma_2^{GPS}} + \|\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{z}_{12}^{odom}\|^2_{\Sigma_{12}^{odom}}$$



Equilibrium of mechanical machine (i.e. state with minimum energy)

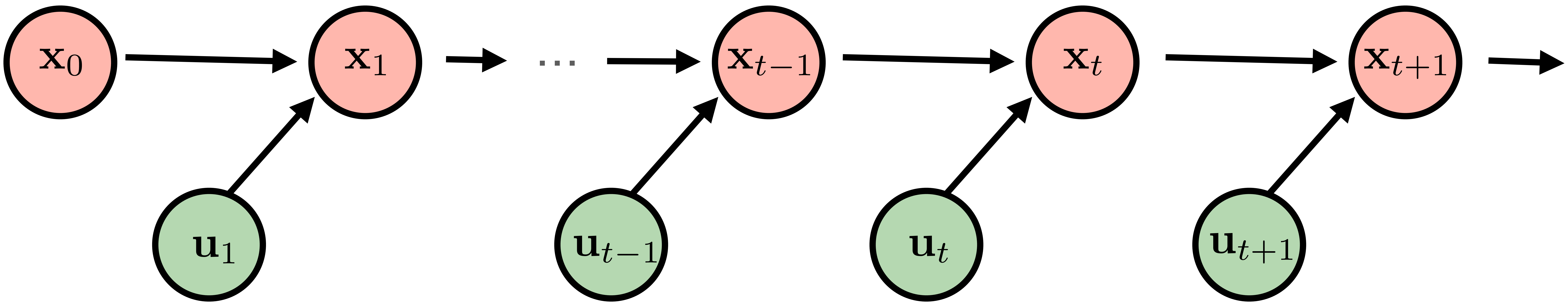
$$F = k\mathbf{x} \Rightarrow E = \frac{1}{2}k\mathbf{x}^2$$

The result is linear least squares problem with closed-form solution

Motion model

Localisation in **multiple time instances** from **actions and motion model**

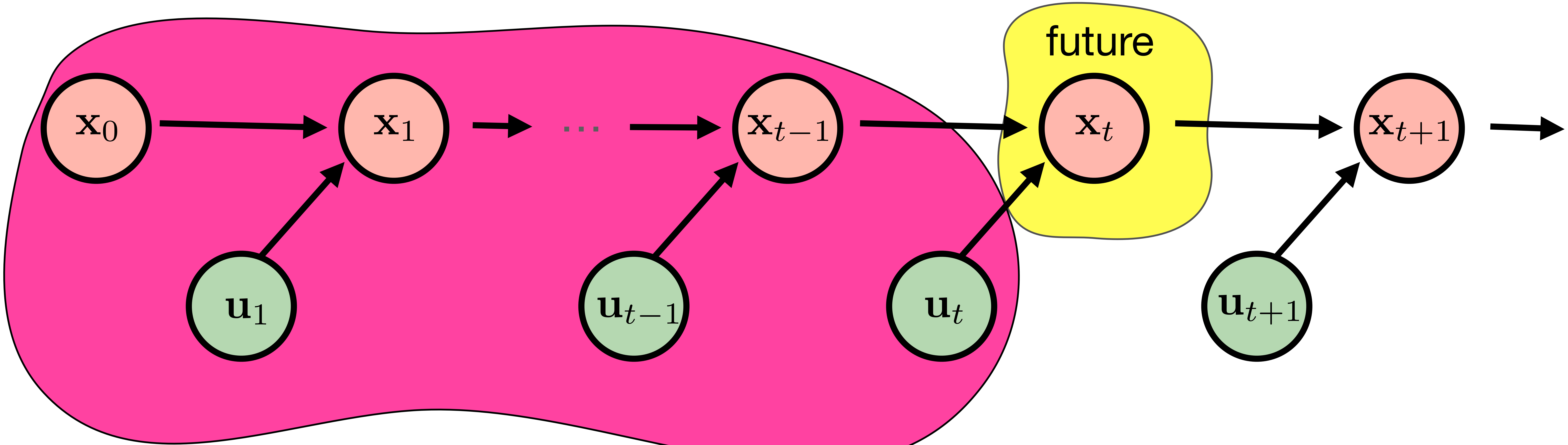
Actions: $\mathbf{u}_1 \ldots \mathbf{u_t}$ (generated by external source)



Localisation in **multiple time instances** from **actions and motion model**

Actions: $\mathbf{u}_1 \ldots \mathbf{u}_\mathbf{t}$ (generated by external source)

State-transition probability: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \ldots, \mathbf{x}_0, \mathbf{u}_{t-1}, \ldots, \mathbf{u}_1, \mathbf{z}_{t-1}, \ldots, \mathbf{z}_1)$



Localisation in **multiple time instances** from **actions and motion model**

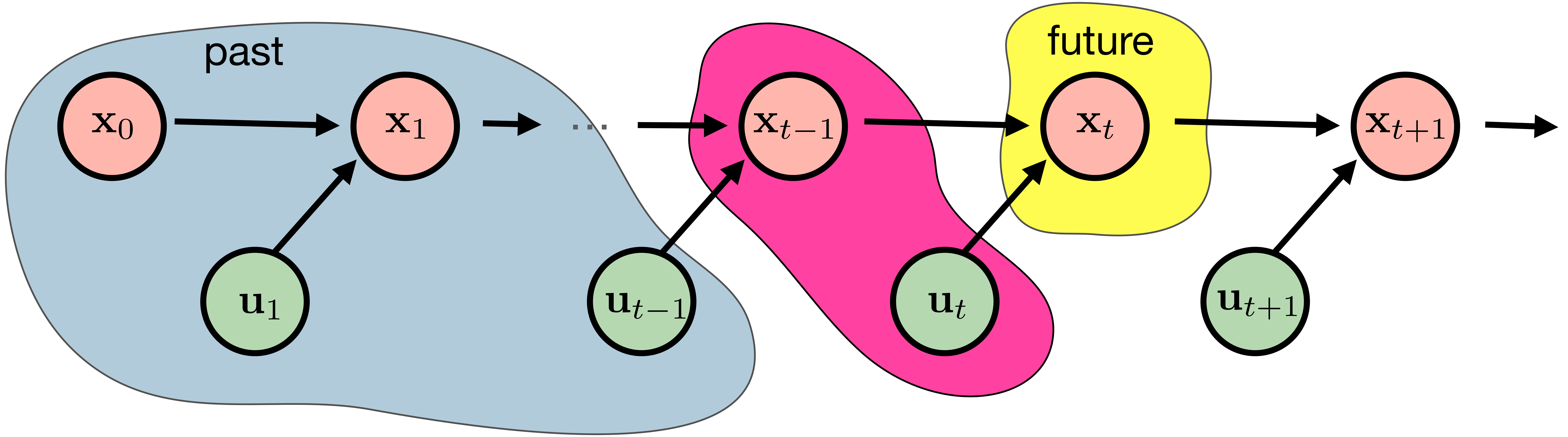
Actions: $\mathbf{u}_1 \dots \mathbf{u_t}$ (generated by external source)

State-transition probability: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \dots, \mathbf{x}_0, \mathbf{u}_{t-1}, \dots, \mathbf{u}_1, \mathbf{z}_{t-1}, \dots, \mathbf{z}_1)$

Markov assumption: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t, \mathbf{x}_{t-2}, \dots, \mathbf{x}_0, \mathbf{u}_{t-1}, \dots, \mathbf{u}_1, \mathbf{z}_{t-1}, \dots, \mathbf{z}_1)$

Motion model: $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \epsilon_{\text{noise}}$ (prior about robot's behaviour)

Example: $p(\mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{x}_t; g(\mathbf{x}_{t-1}, \mathbf{u}_t), \Sigma_t^g)$ e.g. linear $\mathcal{N}(\mathbf{x}_t; \mathbf{x}_{t-1} + \mathbf{u}_t, \Sigma_t^g)$



Putting measurements and motion model together

Localisation from **actions + GPS + IMU**

Bayes theorem

$$\mathbf{x}^* = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{x}_0\ldots\mathbf{x}_t \mid \mathbf{z}_1\ldots\mathbf{z}_t, \mathbf{u}_1\ldots\mathbf{u}_t) = \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} \frac{p(\mathbf{z}_1\ldots\mathbf{z}_t \mid \mathbf{x}_0\ldots\mathbf{x}_t, \cancel{\mathbf{u}_1\ldots\mathbf{u}_t})\, p(\mathbf{x}_0\ldots\mathbf{x}_t \mid \mathbf{u}_1\ldots\mathbf{u}_t)}{\cancel{p(\mathbf{z}_0\ldots\mathbf{z}_t \mid \mathbf{u}_1\ldots\mathbf{u}_t)}}$$

Conditional independence of z on u given x

$$= \arg\max_{\mathbf{x}_0\ldots\mathbf{x}_t} p(\mathbf{z}_1\ldots\mathbf{z}_t \mid \mathbf{x}_0\ldots\mathbf{x}_t)\; p(\mathbf{x}_0\ldots\mathbf{x}_t \mid \mathbf{u}_1\ldots\mathbf{u}_t)$$

Normal likelihoods

$$= \arg\max_{\mathbf{x}_0,\ldots\mathbf{x}_t} \prod_i^t \mathcal{N}(\mathbf{z}_i^{GPS}; \mathbf{x}_i, \Sigma_i^{GPS}) \prod_i^t \mathcal{N}(\mathbf{z}_i^{odom}; \mathbf{x}_i - \mathbf{x}_{i-1}, \Sigma_i^{odom}) \prod_i^t \mathcal{N}(\mathbf{x}_i; g(\mathbf{x}_{i-1}, \mathbf{u}_i), \Sigma_i^g)$$

$$= \arg\min_{\mathbf{x}_0,\ldots\mathbf{x}_t} \sum_{i=1}^t \|\mathbf{x}_i - \mathbf{z}_i^{GPS}\|^2_{\Sigma_i^{GPS}} + \sum_{i=1}^t \|\mathbf{x}_i - \mathbf{x}_{i-1} - \mathbf{z}_i^{odom}\|^2_{\Sigma_i^{odom}} + \sum_{i=1}^t \|\mathbf{x}_i - g(\mathbf{x}_{i-1}, \mathbf{u}_i)\|^2_{\Sigma_i^g}$$

$$= \arg\min_{\mathbf{x}_0,\ldots\mathbf{x}_t} \sum_j f_j(\mathbf{x}, \mathbf{z})^2$$

Non-linear least squares => GN/LN from OPT

`scipy.optimize.least_squares(fun, x0, jac)`

Factor graph

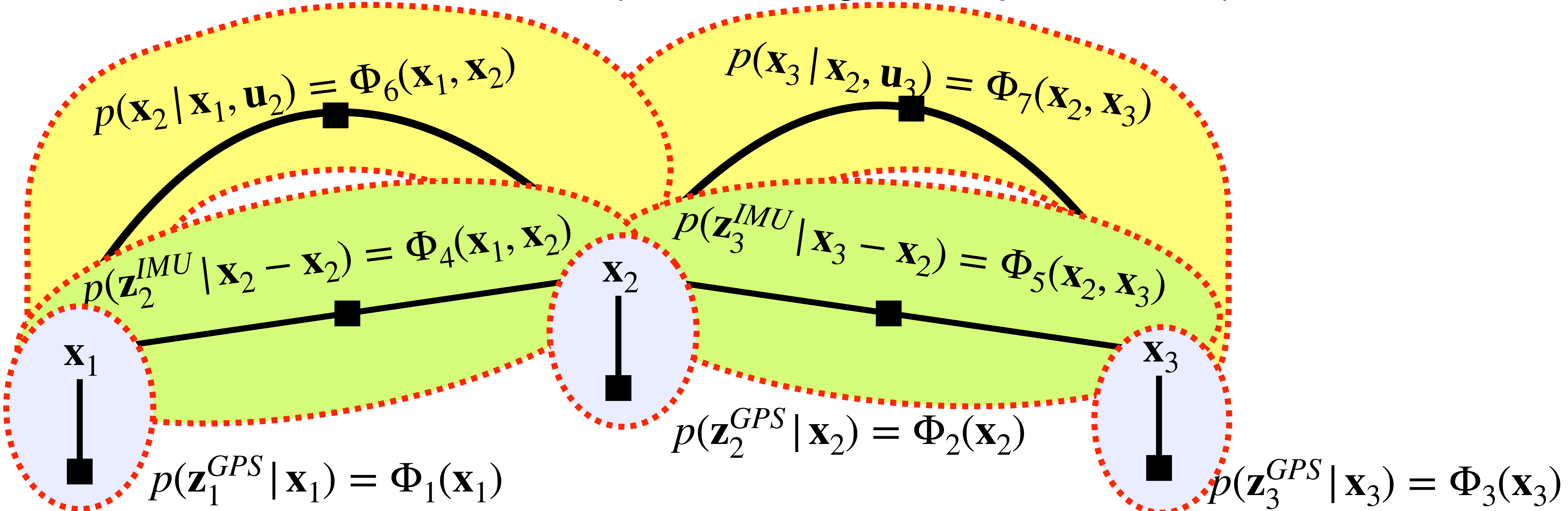
Factor graph

- Design choices (Markov assumption, cond. independence) yielded sparse model

$$\mathbf{x}^\star = \arg\max_{\mathbf{x}_0,\dots\mathbf{x}_t} \underbrace{\prod_i p(\mathbf{z}_i^{GPS}\,|\,\mathbf{x}_i)}_{\text{unary}} \cdot \underbrace{\prod_i p(\mathbf{z}_i^{odom}\,|\,\mathbf{x}_i, \mathbf{x}_{i-1})}_{\text{pair-wise}} \cdot \underbrace{\prod_i p(\mathbf{x}_i\,|\,\mathbf{x}_{i-1}, \mathbf{u}_i)}_{\text{pair-wise}}$$

$$\mathbf{x}^\star = \arg\min_{\mathbf{x}_0,\dots\mathbf{x}_t} \sum_i \|\mathbf{x}_i - \mathbf{z}_i^{GPS}\|^2_{\Sigma_i^{GPS}} + \sum_i \|\mathbf{x}_i - \mathbf{x}_{i-1} - \mathbf{z}_i^{odom}\|^2_{\Sigma_i^{odom}} + \sum_i \|\mathbf{x}_i - g(\mathbf{x}_{i-1}, \mathbf{u}_i)\|^2_{\Sigma_i^{g}}$$

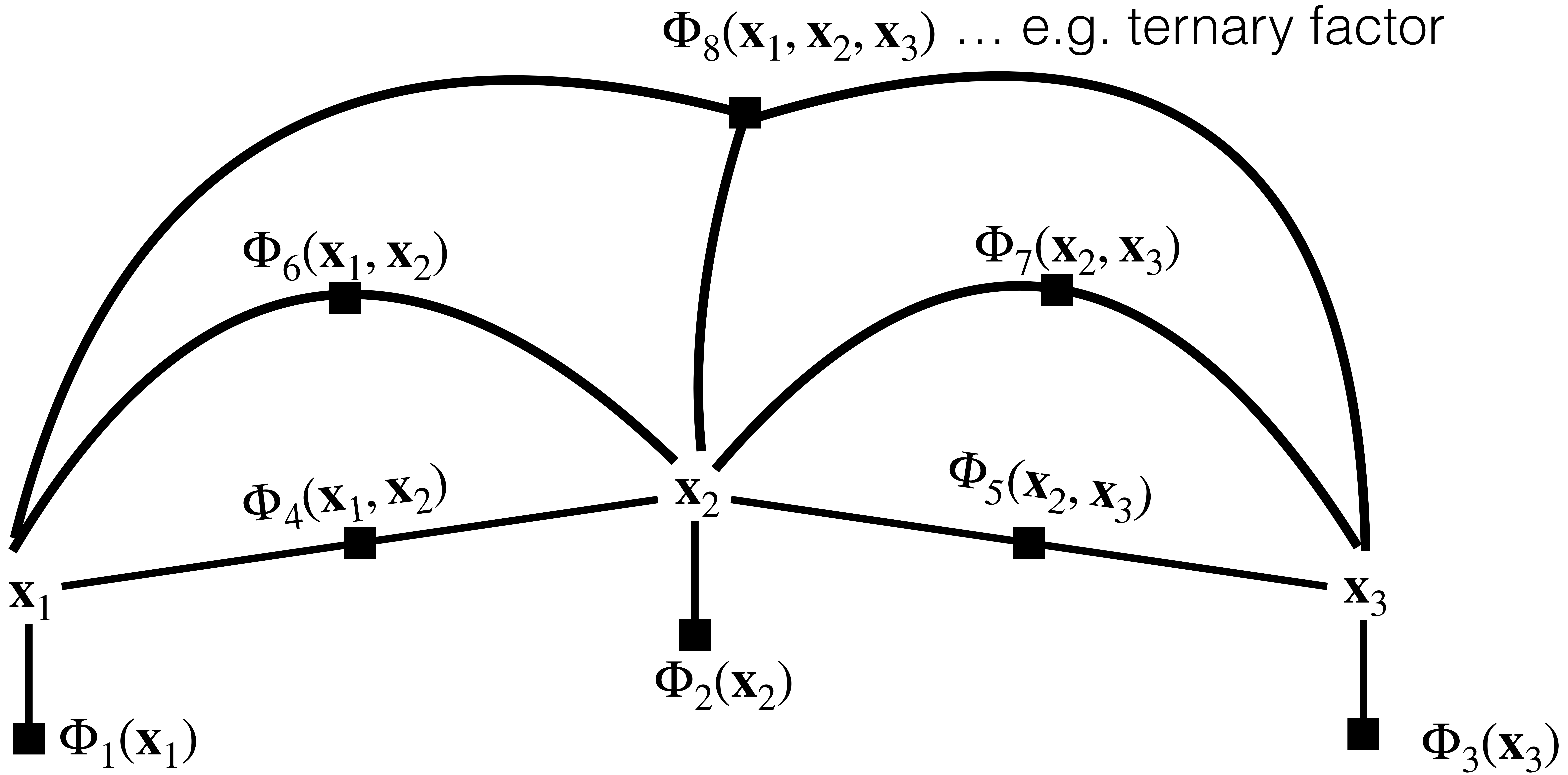
- The structure can be more complicated (e.g. ternary terms, loop closers)



Factor graph

<u>Def:</u> Factor graph is bipartite graph $\mathscr{G} = \{\mathscr{U}, \mathscr{V}, \mathscr{E}\}$ with

- Two types of nodes: factors $\mathbf{\Phi}_i \in \mathscr{U}$ and variables $\mathbf{x}_j \in \mathscr{V}$.
- Edges $\mathbf{e}_{ij} \in \mathscr{E}$ are always between factor nodes and variable nodes.

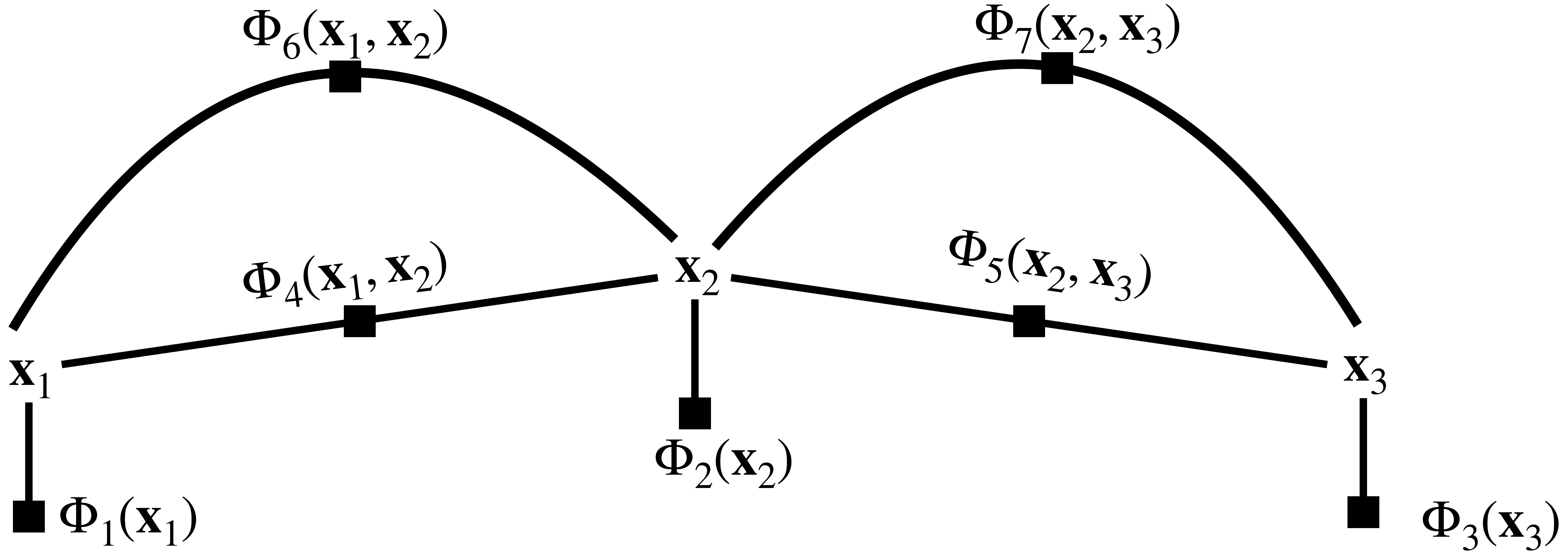


Factor graph

- Convenient visualisation of the problem structure
- Evaluation of unnormalised prob. for given values of variables
- Simple formulation of MAP estimation problem

$$\mathbf{x}_0^\star \ldots \mathbf{x}_t^\star = \arg\max_{\mathbf{x}_0 \ldots \mathbf{x}_t} \prod_i \Phi_i(X_i) = \arg\min_{\mathbf{x}_0 \ldots \mathbf{x}_t} \sum_i -\log\Big(\Phi_i(X_i)\Big)$$

- Optimisation (continuous var. => local gradient opt., discr. var. => graph search)
- Sampling of $p(\mathbf{x}_0 \ldots \mathbf{x}_t)$ (MCMC Gibbs sampling, ancestral sampling for dir. acyclic)



Factor graph

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- Optimisation (continuous var. => local gradient opt., discr. var. => graph search)
- Sampling of $p(\mathbf{x}_0 \ldots \mathbf{x}_t)$ (MCMC Gibbs sampling, ancestral sampling for dir. acyclic)

- Graphical model useful for MAP estimation:
  - SLAM
  - optimal control
  - tracking
  - …

Summary

- **Understand** localisation problem in robotics as MAP estimate of unknown variables
- **Model measurement probability** of simplified IMU and GPS
- **Model state-transition probability** for linear and nonlinear motion models
- **Write down optimisation** criterion in negative log-space for gaussian prob. distr.
- **Solve** underlying opt. problem using least squares / gradient descend algorithm in your favourite optimisation tool (MATLAB, Scipy, Pytorch, Julia, Mosek)
- **Next lecture:** Adds rotation ans solve the optimization in SO2 manifold
- **Optional HW:** $\mathbf{z}_1^{GPS} = [1,2]^\top,\ \mathbf{z}_2^{GPS} = [3,2]^\top,\ \mathbf{z}_3^{GPS} = [3,3]^\top$ Absolute measurements

  $\mathbf{z}_1^{odom} = [1,1]^\top,\ \mathbf{z}_2^{odom} = [1,1]^\top,\ \mathbf{z}_3^{odom} = [1,1]^\top$ Relative measurements

  $\mathbf{u}_1 = [1,1]^\top,\ \mathbf{u}_2 = [1,1]^\top,\ \mathbf{u}_3 = [1,0]^\top,$ Actions

  $\mathbf{x}_0 = [0,0]^\top,\ \mathbf{x}_1 = ?,\ \mathbf{x}_2 = ?,\ \mathbf{x}_3 = ?$ States

  $\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \epsilon_{noise} = \mathbf{x}_{t-1} + \mathbf{u}_t + \epsilon_{noise}$ Motion model

  All probs are 2D Gaussians with $\Sigma_t^{GPS} = \Sigma_t^{odom} = \Sigma_t^{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$