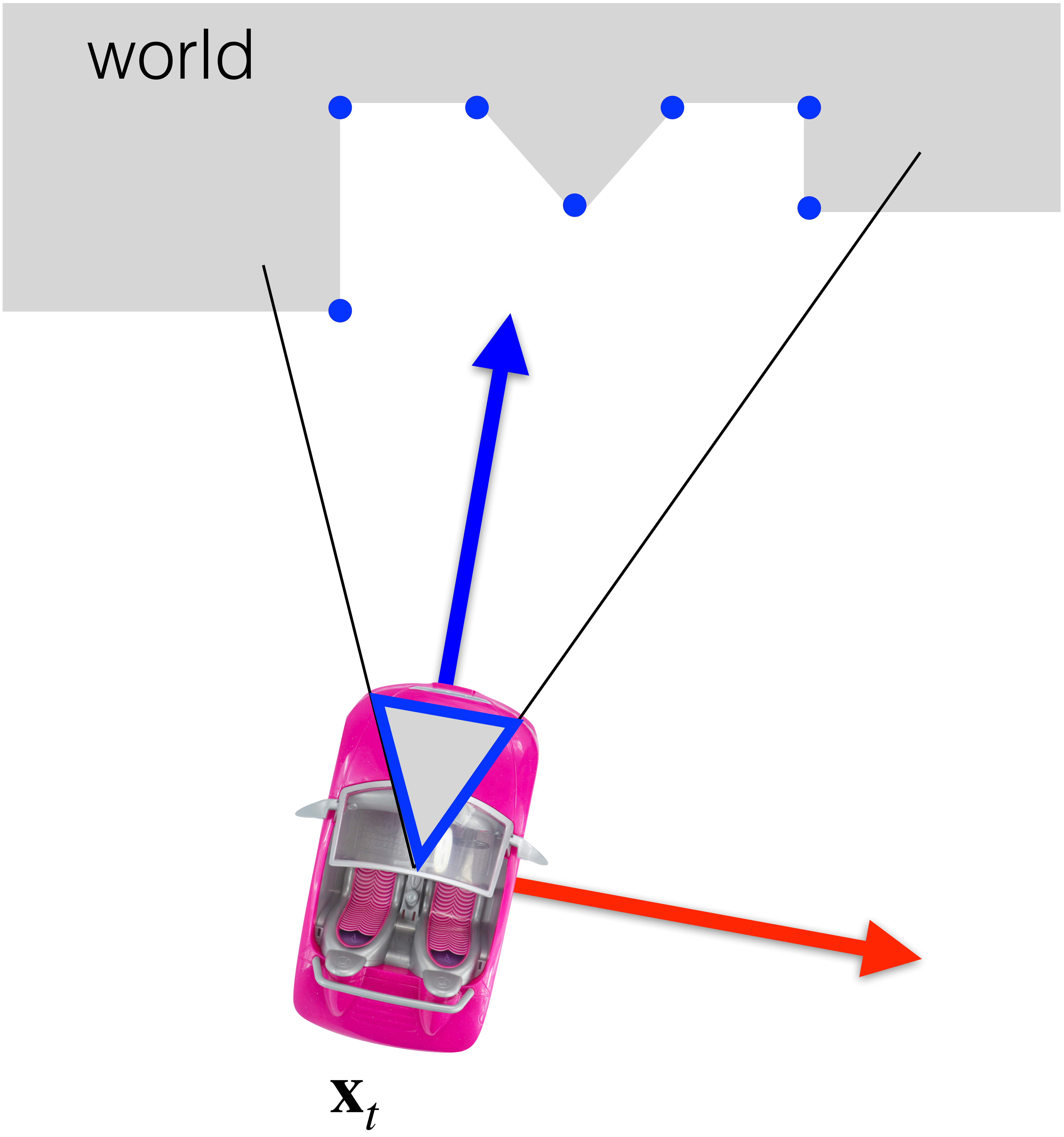
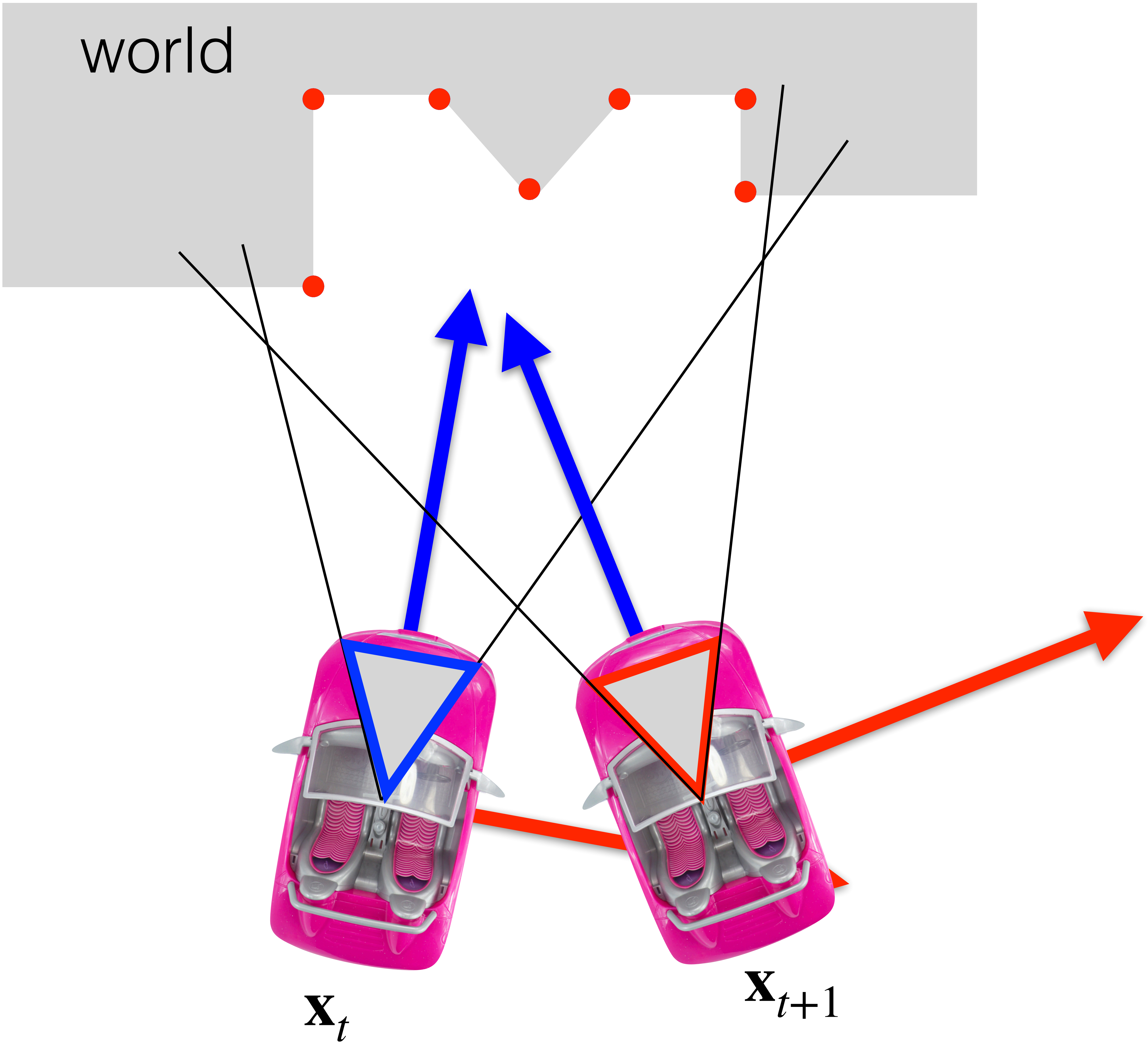
Transformation frames and lidar

Karel Zimmermann

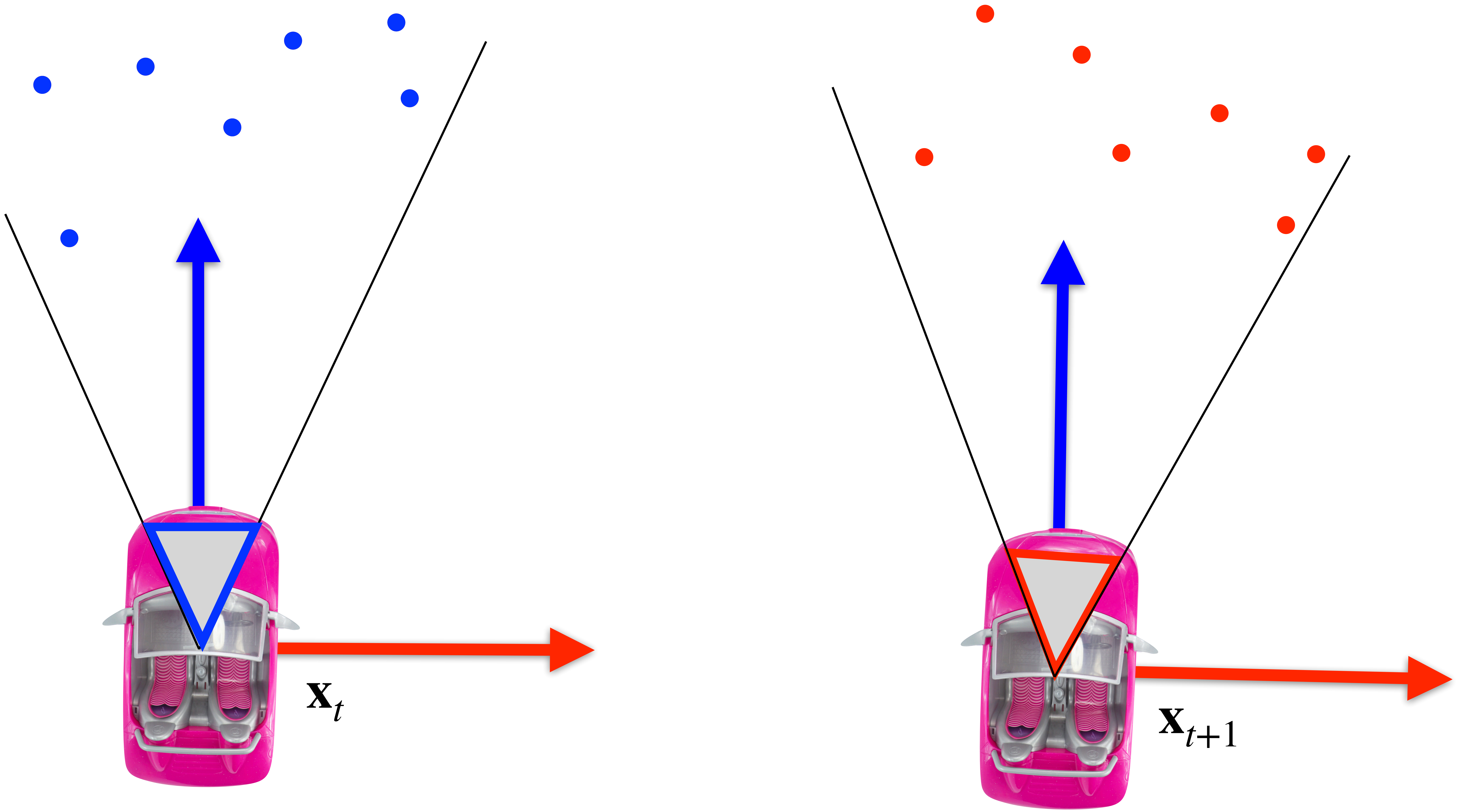
Pose estimation from known correspondences



Pose estimation from known correspondences

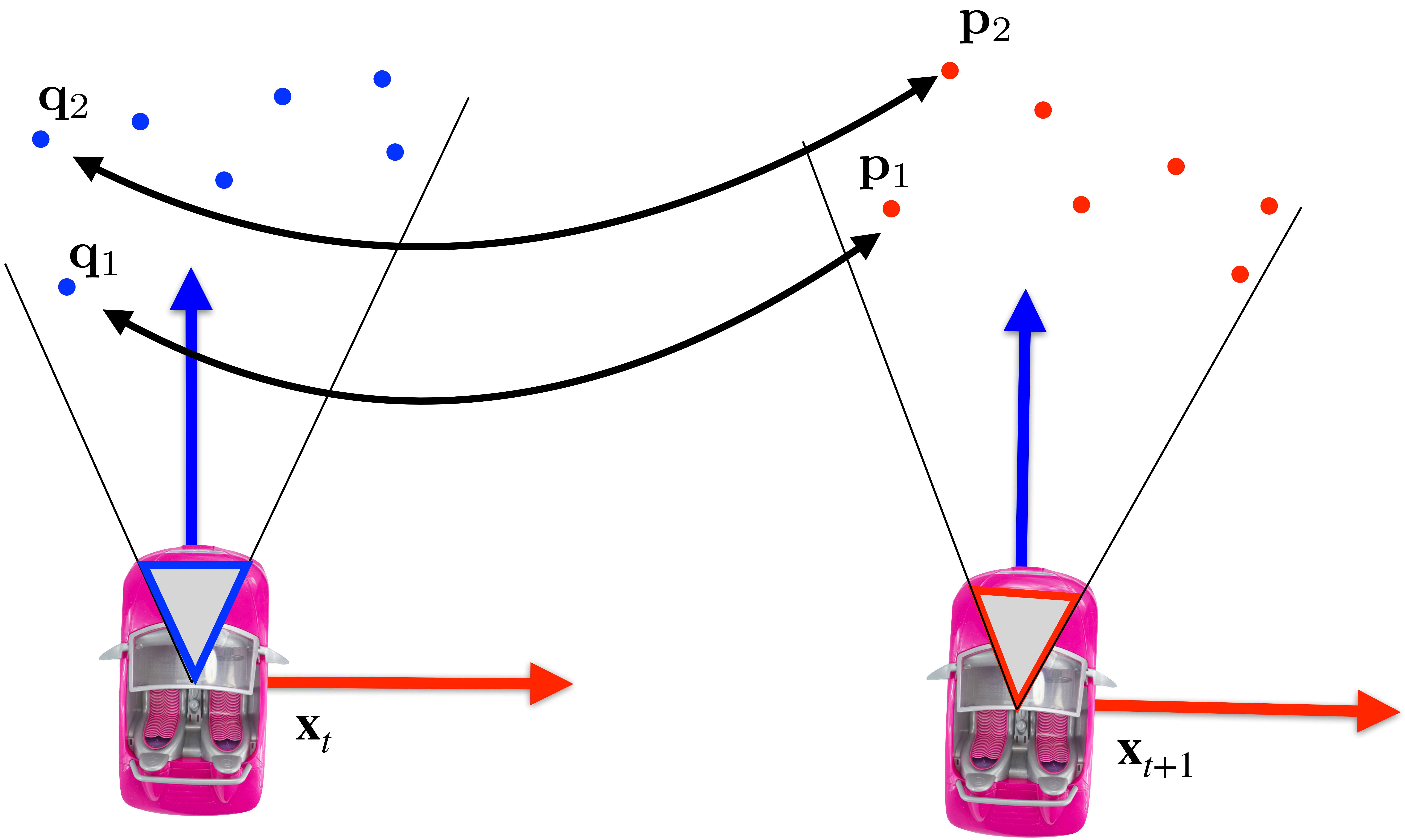


Pose estimation from known correspondences



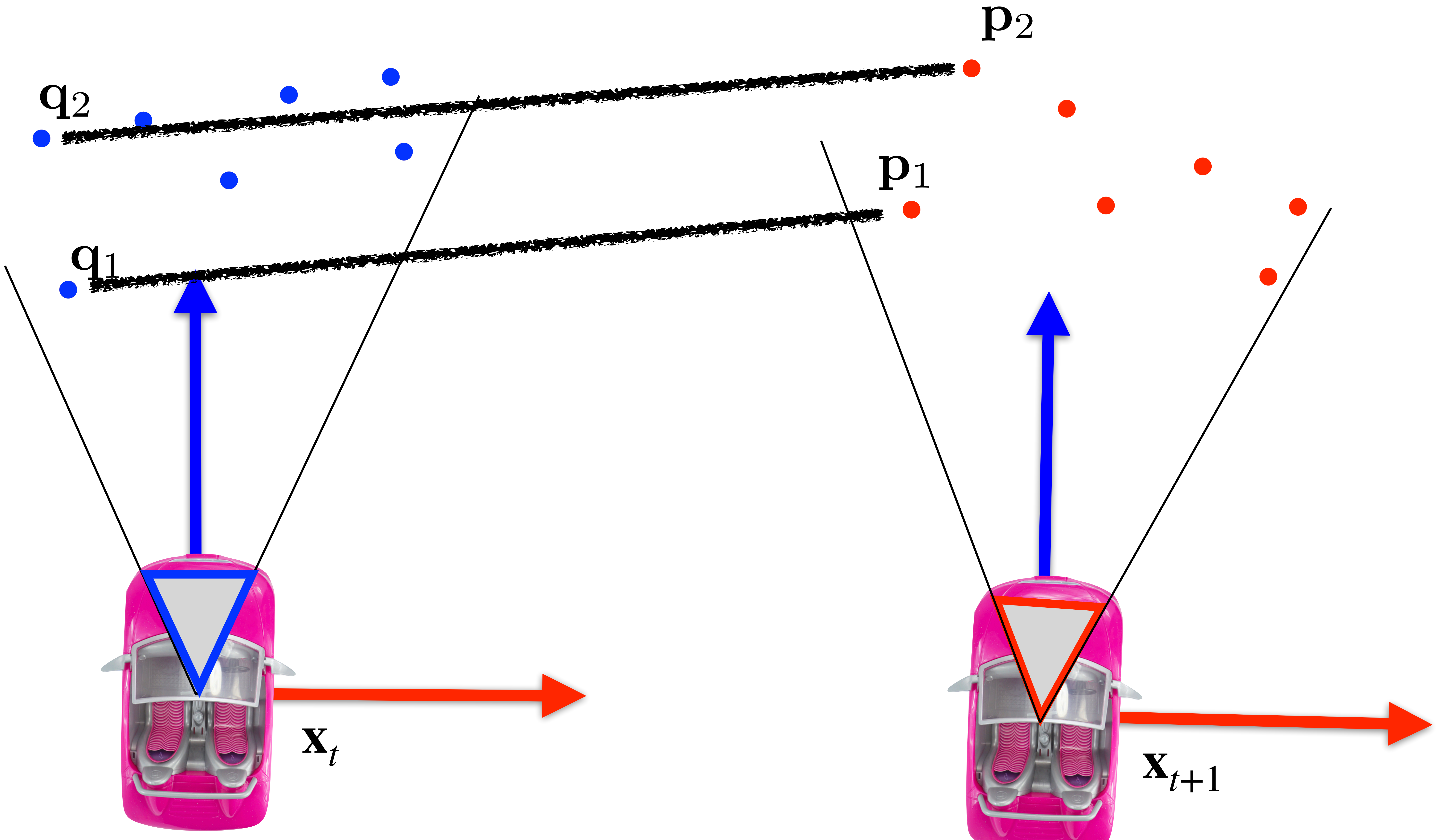
Pose estimation from known correspondences

Assume 2D-2D correspondences known



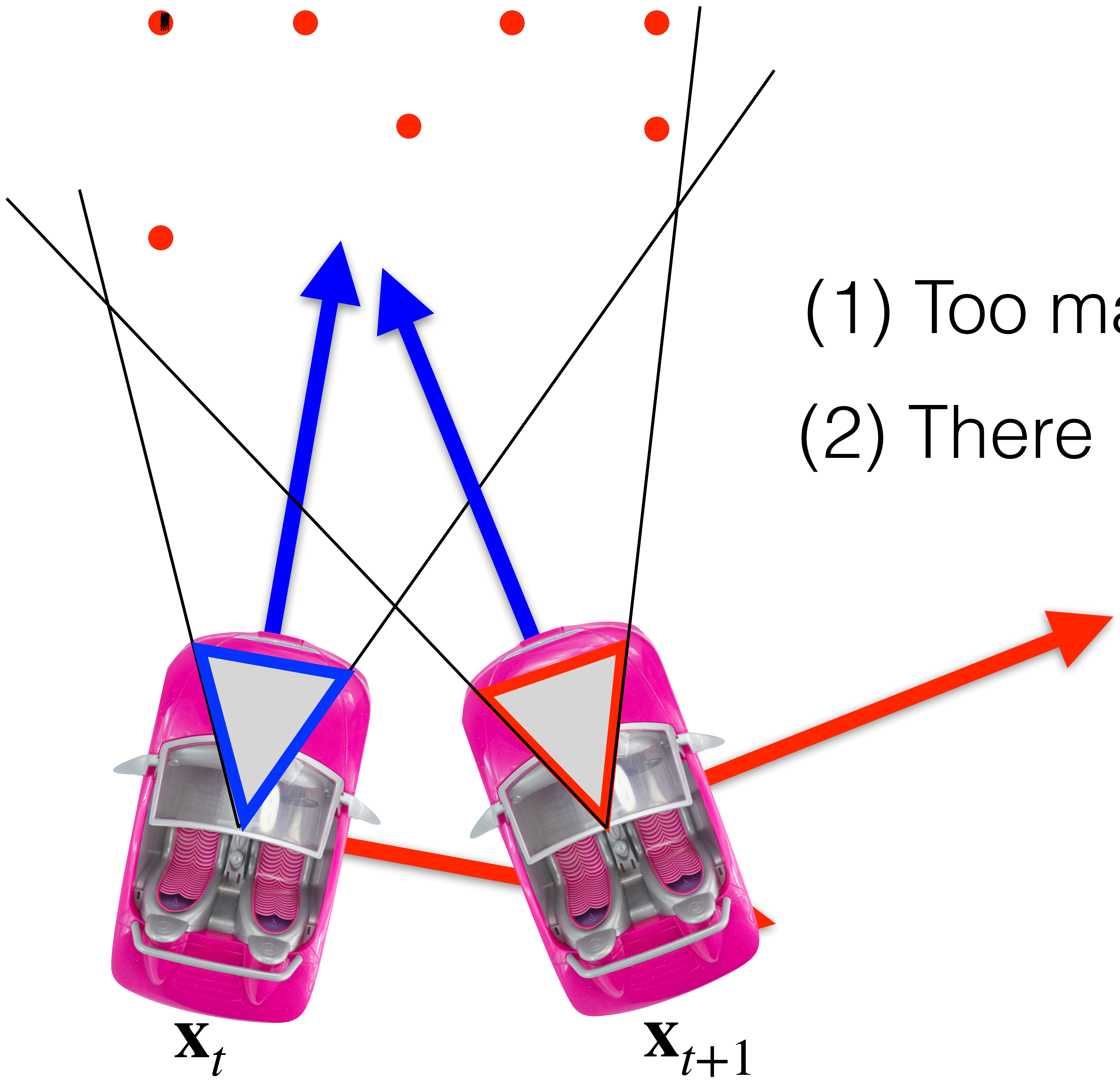
Pose estimation from known correspondences

$$\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \sum_i \|T(\mathbf{p}_i, \mathbf{x}_{t+1}) - T(\mathbf{q}_i, \mathbf{x}_t)\|^2$$



Pose estimation from known correspondences

$$\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \sum_i \|T(\mathbf{p}_i, \mathbf{x}_{t+1}) - T(\mathbf{q}_i, \mathbf{x}_t)\|^2$$

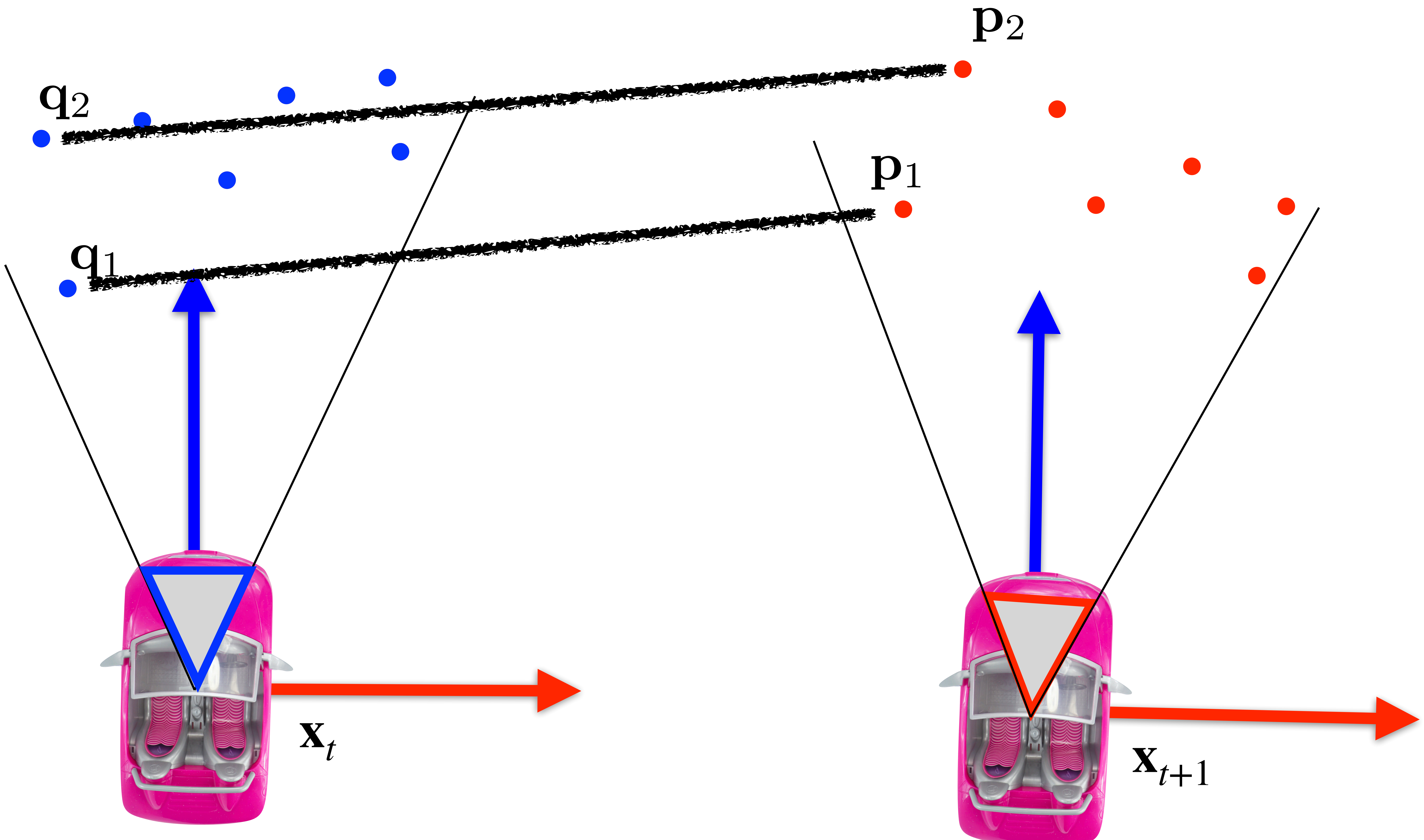


(1) Too many residuals !!!

(2) There is closed-form solution

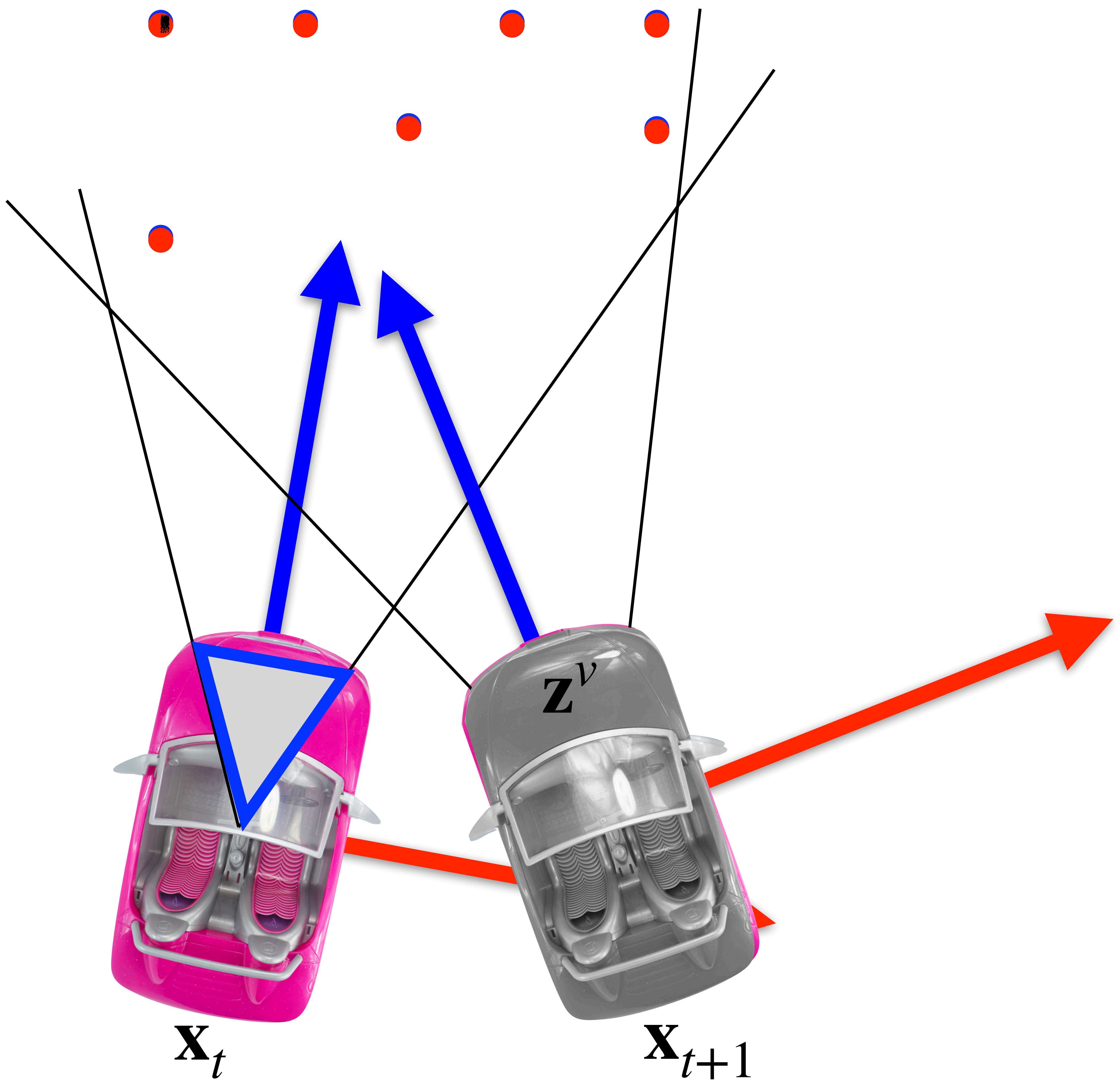
Pose estimation from known correspondences

Virtual odometry measurement: $$\mathbf{z}^v = \arg\min_{x,y,\theta} \sum_i \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{p}_i + \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{q}_i \right\|^2$$



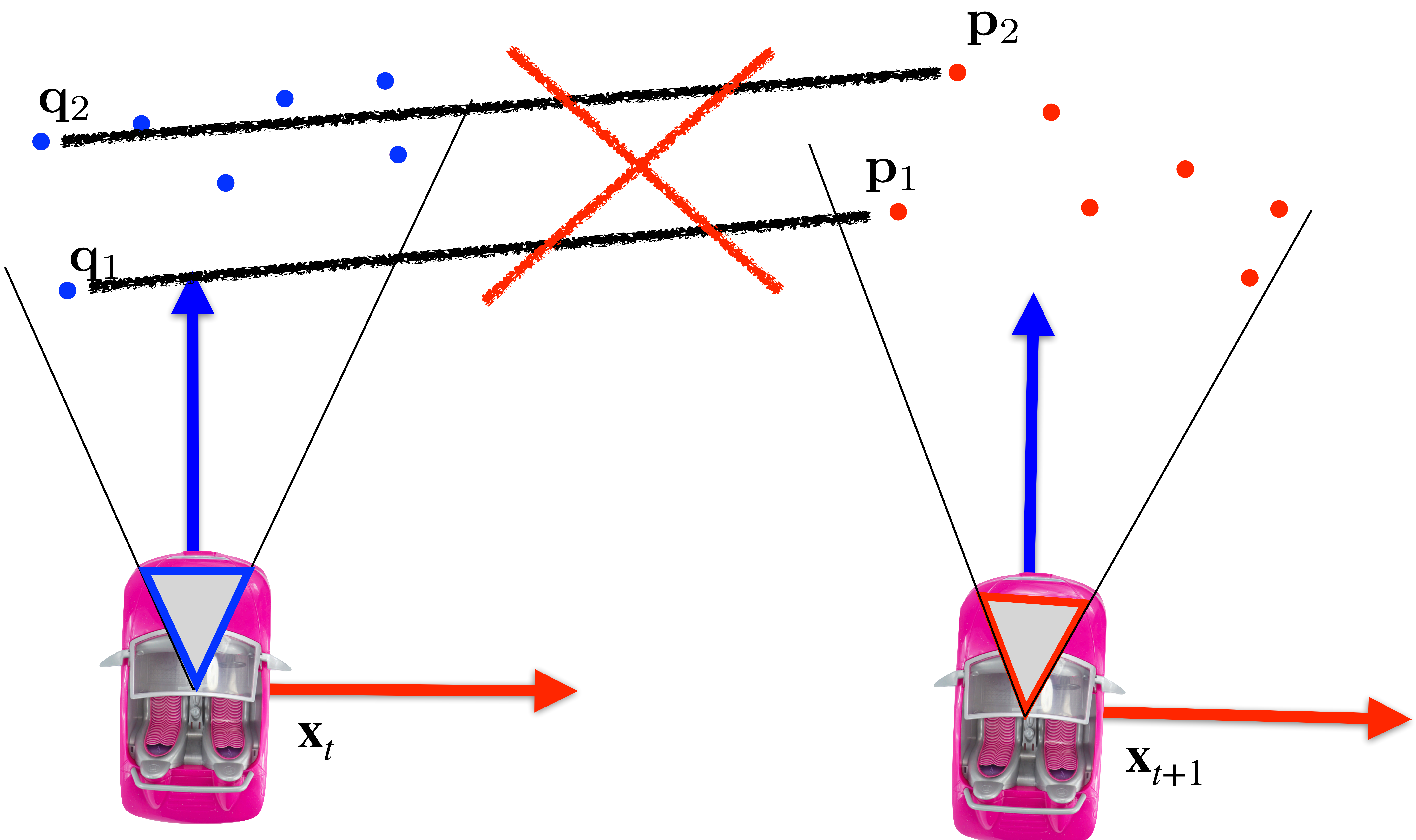
Pose estimation from known correspondences

Virtual odometry measurement: $\mathbf{z}^v = \arg\min_{x,y,\theta} \sum_i \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{p}_i + \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{q}_i \right\|^2$



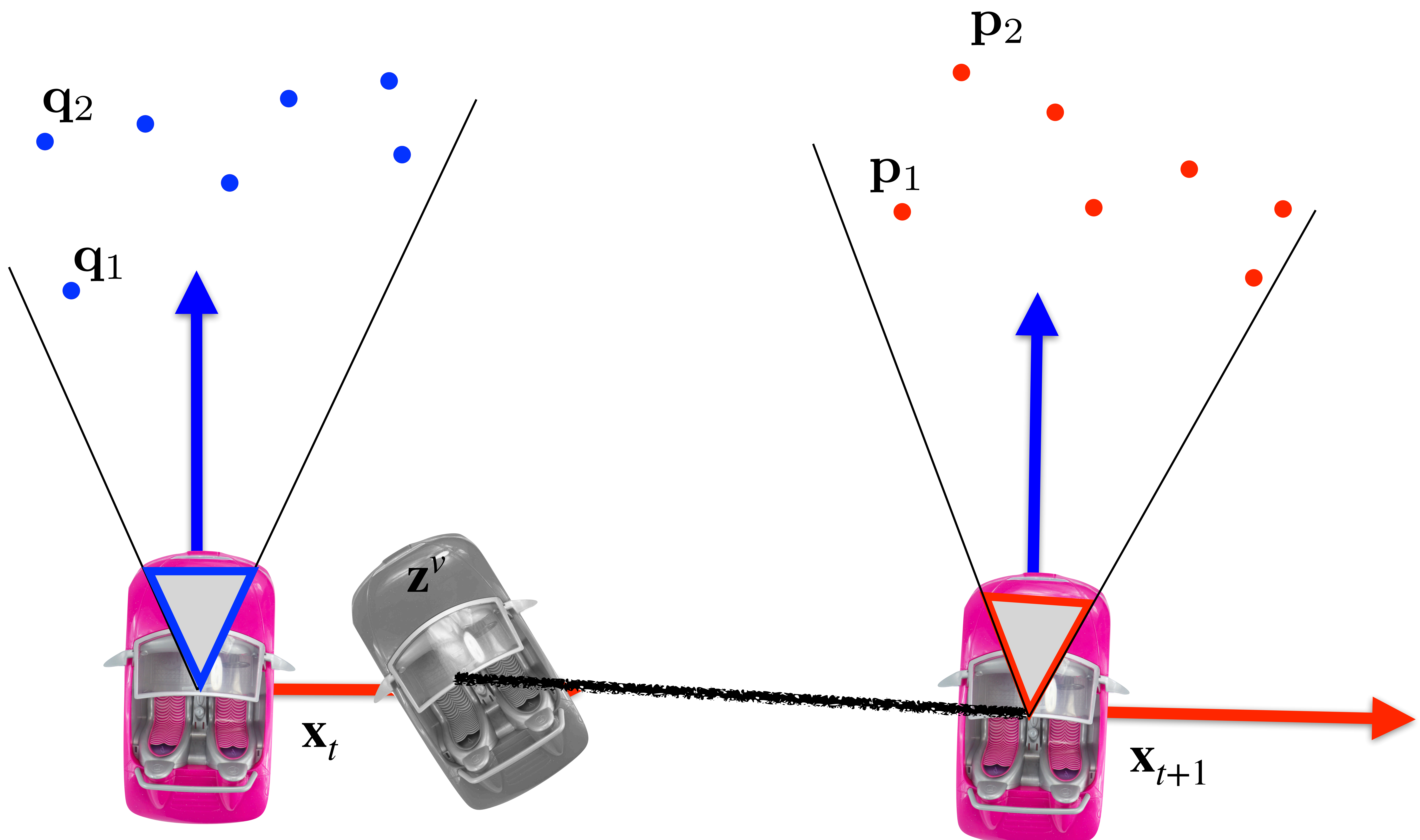
Pose estimation from known correspondences

Apply only single odometry factor: $\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} ||T(\mathbf{z}^v, \mathbf{x}_t) - \mathbf{x}_{t+1})||^2$



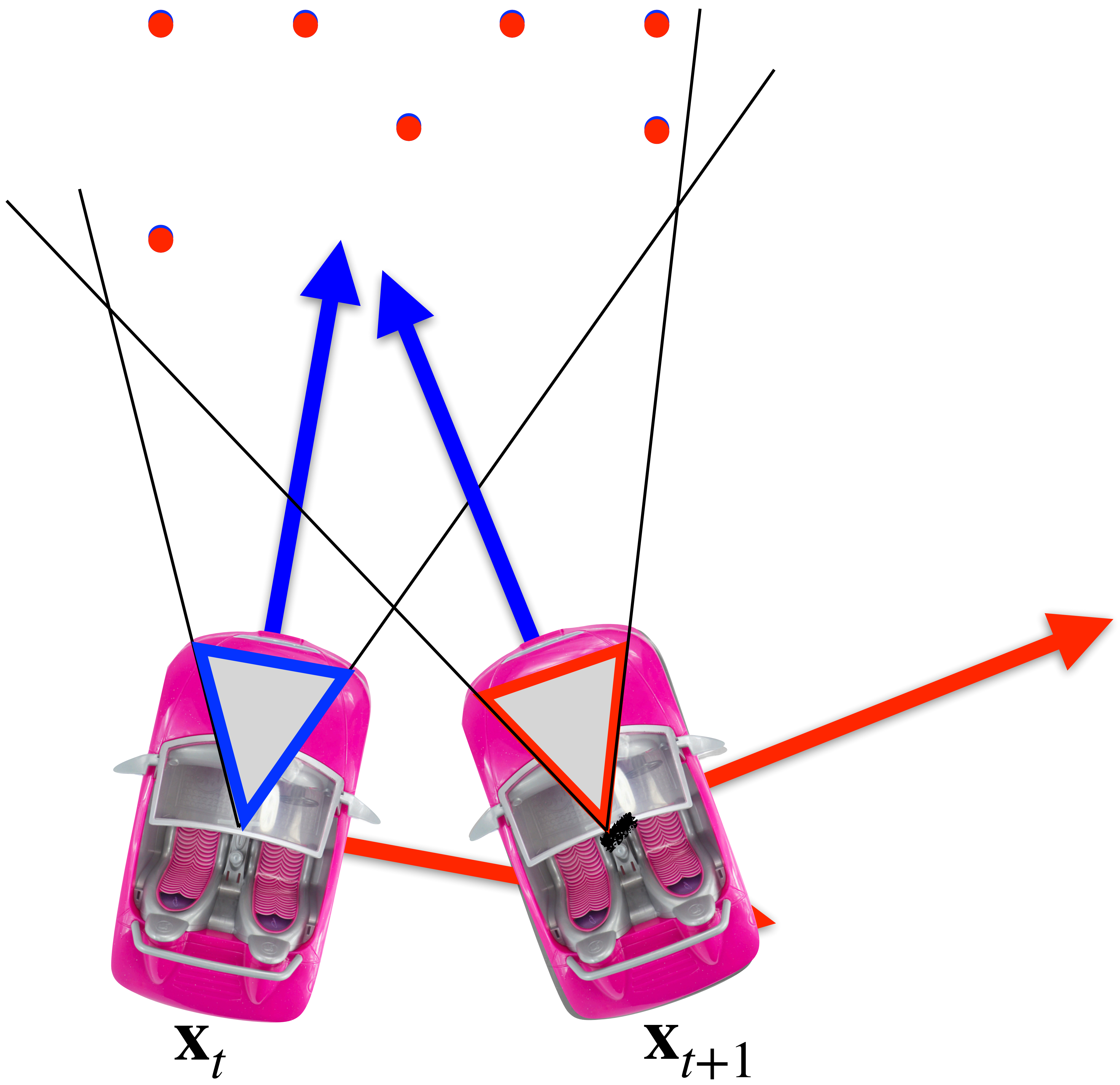
Pose estimation from known correspondences

Apply only single odometry factor: $\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \|T(\mathbf{z}^v, \mathbf{x}_t) - \mathbf{x}_{t+1})\|^2$



Pose estimation from known correspondences

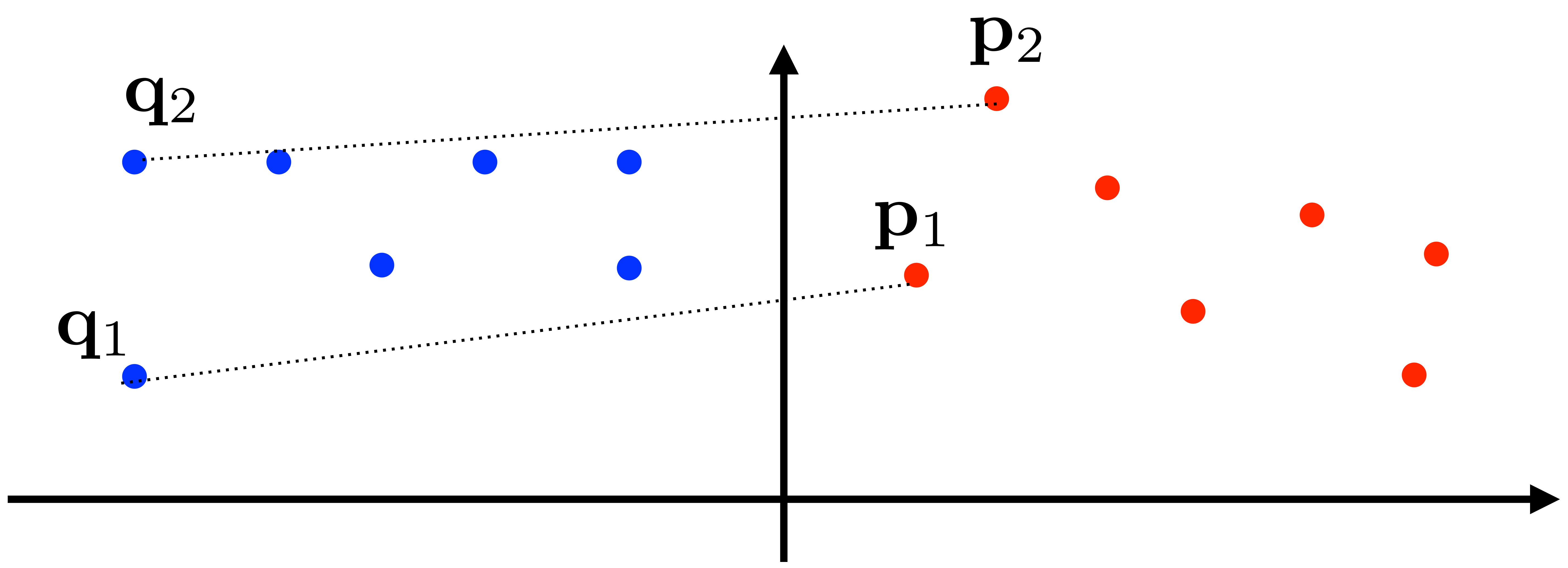
Apply only single odometry factor: $\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \|T(\mathbf{z}^v, \mathbf{x}_t) - \mathbf{x}_{t+1})\|^2$



Absolute orientation problem in SE(2)

$$\mathbf{z}^{v} = \arg\min_{\mathbf{t},\theta} \sum_{i} \left\| \mathbf{R}_{\theta}\mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^{2}$$

Substitution:



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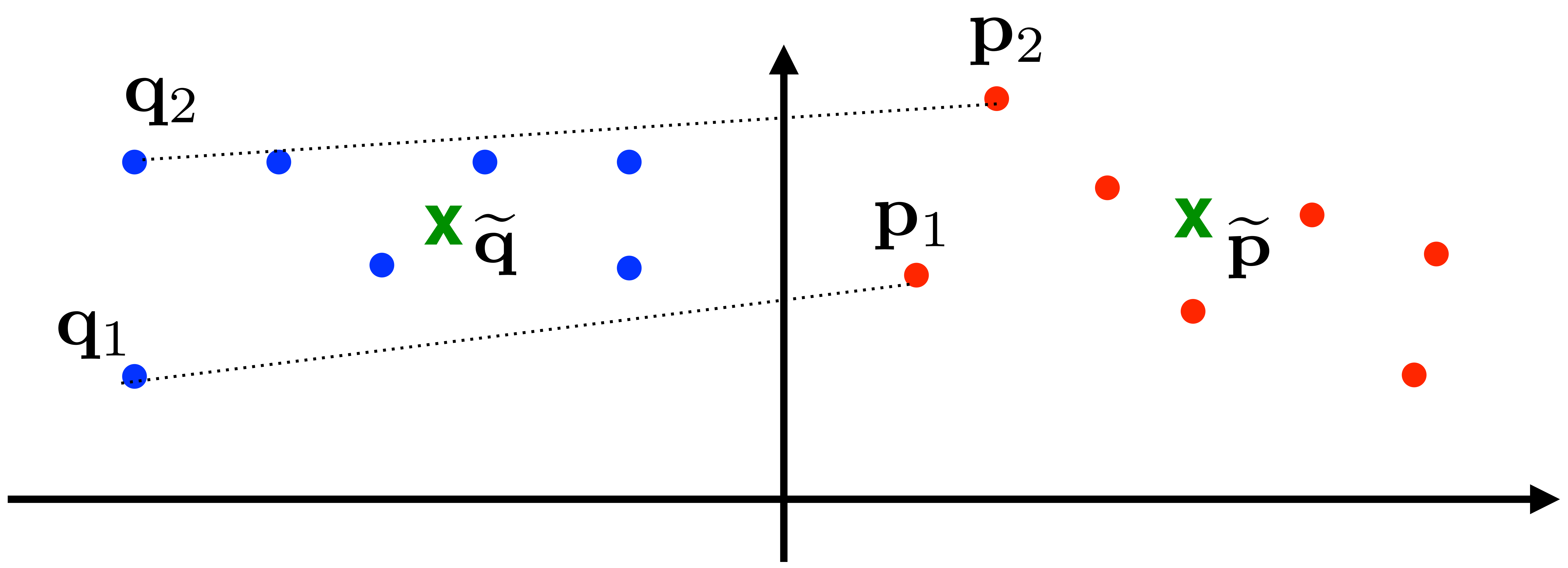
Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$



Absolute orientation problem in SE(2)

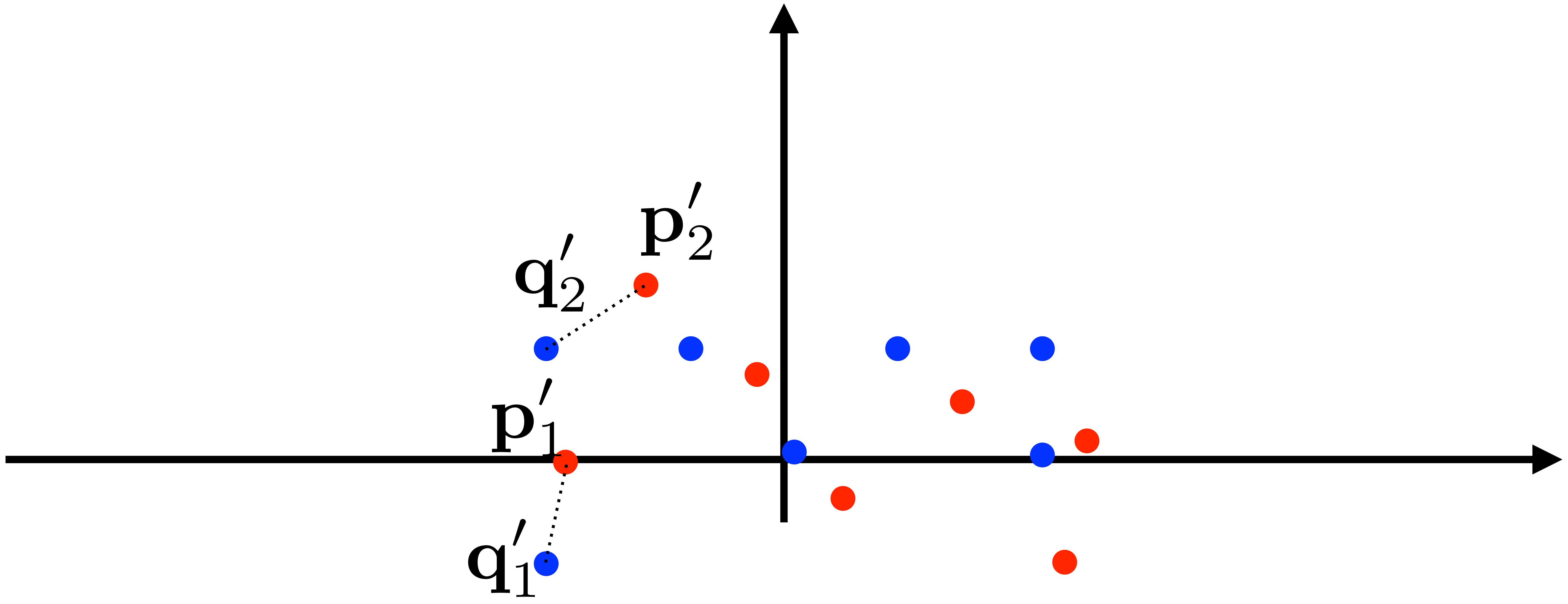
$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Substitution: $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of $\mathbf{t}$

Depends only on θ

Solution: $$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2$$



Absolute orientation problem in SE(2)

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

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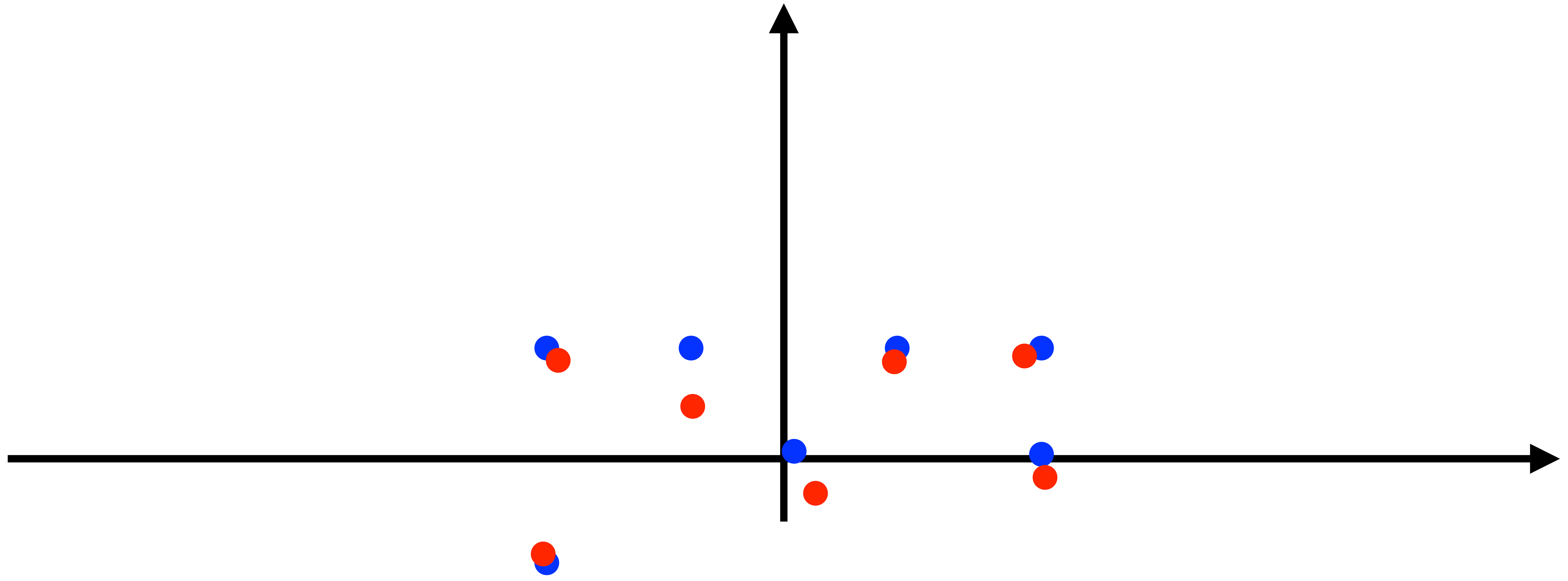

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$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

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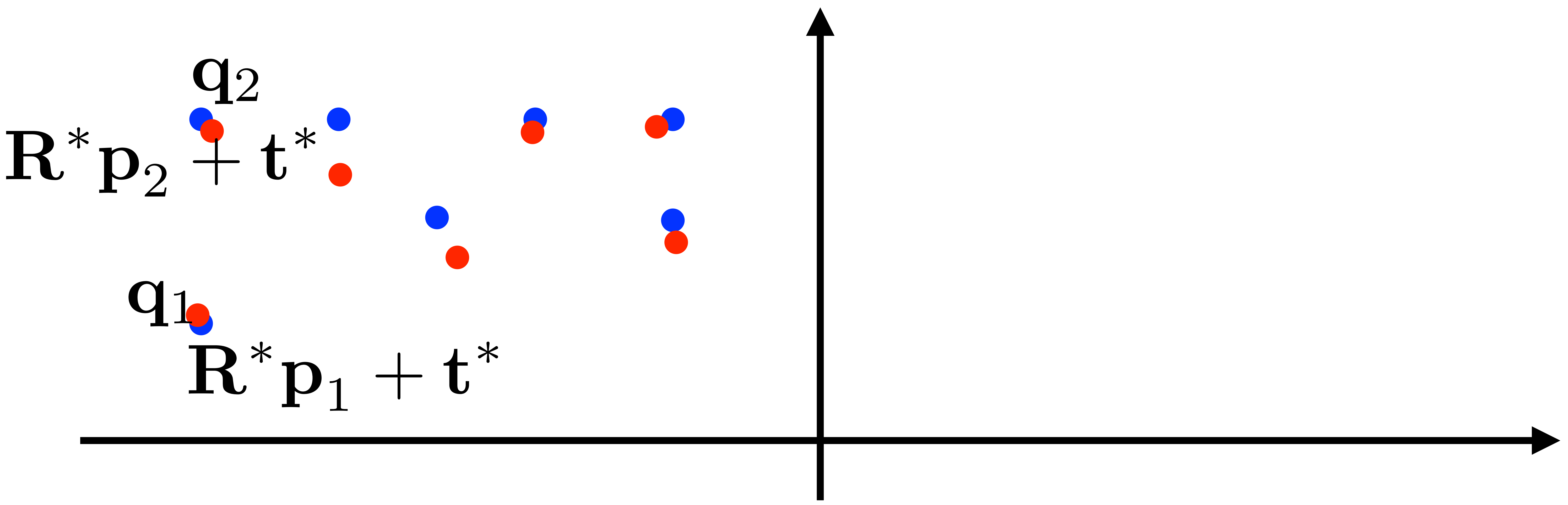
Can be always zero by appropriate choice of $\mathbf{t}$

Solution: $\theta^\star = \arg\min_{\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 \qquad \mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star}\tilde{\mathbf{p}}$



Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} - \begin{bmatrix} q'_x \\ q'_y \end{bmatrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{matrix} \cos(\theta)p'_x - \sin(\theta)p'_y - q'_x \\ \sin(\theta)p'_x + \cos(\theta)p'_y - q'_y \end{matrix} \right\|^2$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} - \begin{bmatrix} q_x' \\ q_y' \end{bmatrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{matrix} \cos(\theta)p_x' - \sin(\theta)p_y' - q_x' \\ \sin(\theta)p_x' + \cos(\theta)p_y' - q_y' \end{matrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p_x' \cos(\theta) - p_y' \sin(\theta) - q_x'\right)^2 + \left(p_x' \sin(\theta) + p_y' \cos(\theta) - q_y'\right)^2$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x\right)^2 + \left(p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y\right)^2$$

Derivative: $$\sum_{\mathbf{p}',\mathbf{q}'} 2(p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x) \cdot (-p'_x \sin(\theta) - p'_y \cos(\theta))$$

$$+ \quad 2(p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y) \cdot (p'_x \cos(\theta) - p'_y \sin(\theta)) = 0$$

Simplify: $$\sum_{\mathbf{p}',\mathbf{q}'} p'^2_x \cdot \left(-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)\right)$$

$$+ \; p'^2_y \cdot \left(\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta)\right)$$

$$+ \; p'_x p'_y \cdot \left(-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta)\right)$$

$$+ \; p'_x \cdot \left(q'_x \sin(\theta) - q'_y \cos(\theta)\right)$$

$$+ \; p'_y \cdot \left(q'_x \cos(\theta) + q'_y \sin(\theta)\right) \quad = 0$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x\right)^2 + \left(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y\right)^2$$

Derivative:

$$\sum_{\mathbf{p}',\mathbf{q}'} 2(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x)\cdot(-p'_x\sin(\theta) - p'_y\cos(\theta))$$

$$+ \quad 2(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y)\cdot(p'_x\cos(\theta) - p'_y\sin(\theta)) = 0$$

Simplify:

$$\sum_{\mathbf{p}',\mathbf{q}'} \cancel{p'^2_x\cdot\left(-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)\right)}$$

$$+ \quad \cancel{p'^2_y\cdot\left(\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta)\right)}$$

$$+ \quad \cancel{p'_xp'_y\cdot\left(-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta)\right)}$$

$$+ \quad p'_x\cdot\left(q'_x\sin(\theta) - q'_y\cos(\theta)\right)$$

$$+ \quad p'_y\cdot\left(q'_x\cos(\theta) + q'_y\sin(\theta)\right) = 0$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x\right)^2 + \left(p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y\right)^2$$

Derivative:

$$\sum_{\mathbf{p}',\mathbf{q}'} p'_x \cdot \left(q'_x \sin(\theta) - q'_y \cos(\theta)\right) + p'_y \cdot \left(q'_x \cos(\theta) + q'_y \sin(\theta)\right) = 0$$

Solve:

$$\sum_{\mathbf{p}',\mathbf{q}'} p'_x \cdot \left(q'_x \tan(\theta) - q'_y\right) + p'_y \cdot \left(q'_x + q'_y \tan(\theta)\right) = 0$$

$$\sum_{\mathbf{p}',\mathbf{q}'} \tan(\theta) \cdot (p'_x q'_x + p'_y q'_y) + (p'_y q'_x - p'_x q'_y) = 0$$

$$\theta^\star = \arctan\left(\frac{\sum_{\mathbf{p}',\mathbf{q}'} p'_x q'_y - p'_y q'_x}{\sum_{\mathbf{p}',\mathbf{q}'} p'_x q'_x + p'_y q'_y}\right) = \arctan\left(\frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}}\right)$$

$\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'^\top_i$ … covariance matrix

Absolute orientation problem in **SE(2)**

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Depends only on θ

Can be always zero by appropriate choice of $\mathbf{t}$

Substitution: $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Solution: $\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'^{\top}_i$ … covariance matrix

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 = \arctan\left(\frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}}\right)$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star}\tilde{\mathbf{p}}$$

Absolute orientation problem in **SE(3)**

$$\mathbf{z}^{v} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Depends only on θ (first term)

Can be always zero by appropriate choice of $\mathbf{t}$ (second term)

Substitution: $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Solution: $\mathbf{H} = \sum_i \mathbf{p'}_i \mathbf{q'}_i^\top$ … covariance matrix with SVD decomposition $\mathbf{H} = \mathbf{USV}^\top$

$$\mathbf{R}^\star = \arg\min_{\mathbf{R}} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|^2 = \mathbf{VU}^\top$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star}\tilde{\mathbf{p}}$$

Python:

```python
H = P @ Q.T
U, S, V = np.linalg.svd(H, full_matrices=True)
```

Summary

- **Static environment + known correspondences** is required assumption
- Given 3D-3D (or 2D-2D) correspondences, **globally optimal alignment in L2 has closed-form solution** (i.e. least-squares solution constrained on SE(3) manifold)
- **Applications:**
  - Lidar-Lidar or Lidar-Robot Calibration
  - Localization from (un)known correspondences
  - Computer graphics for alignment of 3D models
- **Next:** Localization from unknown correspondences ICP

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \widetilde{\mathbf{q}}\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}_i' + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}_i' - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

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$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}_i' + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}_i' - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \mathbf{t}') =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i'\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i')\mathbf{t}'}_{=0} + \|\mathbf{t}'\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i)\mathbf{t}'}_{=0} + \|\mathbf{t}'\|_2^2 =$$

$$= \arg\min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \|\mathbf{t}'\|_2^2$$

we can reach second term zero by $\mathbf{t} = \widetilde{\mathbf{q}} - \mathbf{R}\widetilde{\mathbf{p}} = \mathbf{t}^*$

Proof [Arun-TPAMI-87]

$$= \arg\min_{\mathbf{R}\in SO(3),\mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \|\mathbf{t}'\|_2^2$$

we can reach second term zero by $\mathbf{t} = \tilde{\mathbf{q}} - \mathbf{R}\tilde{\mathbf{p}} = \mathbf{t}^*$

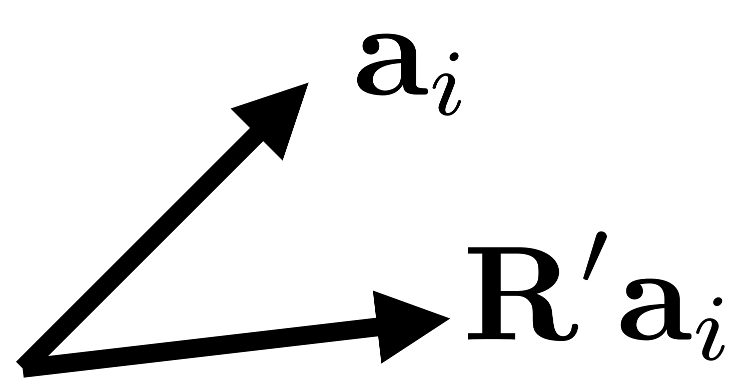
$$\arg\min_{\mathbf{R}\in SO(3)} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 = \arg\max_{\mathbf{R}\in SO(3)} \sum_i \mathbf{q}_i'^{\top}\mathbf{R}\mathbf{p}'_i =$$

$$= \arg\max_{\mathbf{R}\in SO(3)} \sum_i \underbrace{\mathbf{q}_i'^{\top}}_{\mathbf{a}_i} \underbrace{\mathbf{R}\mathbf{p}'_i}_{\mathbf{b}_i} = \arg\max_{\mathbf{R}\in SO(3)} \text{ trace } \mathbf{R}\,\underbrace{\mathbf{P}\mathbf{Q}^{\top}}_{\mathbf{H}} = \mathbf{V}\mathbf{U}^{\top}$$

$$\arg\max_{\mathbf{R}',\mathbf{R}^*\in SO(3)} \text{ trace } \mathbf{R}'\mathbf{R}^*\mathbf{U}\mathbf{S}\mathbf{V}^{\top}$$ … expand into two rotations

$$\arg\max_{\mathbf{R}'\in SO(3)} \text{trace } \mathbf{R}'\,\underbrace{\mathbf{V}\mathbf{U}^{\top}}_{\mathbf{R}^*}\,\underbrace{\mathbf{U}\mathbf{S}\mathbf{V}^{\top}}_{\mathbf{H}} = \arg\max_{\mathbf{R}'\in SO(3)} \text{trace } \mathbf{R}'\,\underbrace{(\mathbf{V}\sqrt{\mathbf{S}})}_{\mathbf{A}}\,\underbrace{(\sqrt{\mathbf{S}}\mathbf{V})^{\top}}_{\mathbf{A}^{\top}} =$$

$$= \arg\max_{\mathbf{R}'\in SO(3)} \sum_i \mathbf{a}_i^{\top}\mathbf{R}'\mathbf{a}_i = \mathbf{E}$$



$$\text{trace } \mathbf{B}\mathbf{A}^{\top} = \sum_i \mathbf{a}_i^{\top}\mathbf{b}_i$$