



# Electromagnetic Field Theory 2

*(fundamental relations and definitions)*

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# Maxwell('s)-Lorentz('s) Equations

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}$$

Equations of motion  
for fields

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t)$$

Equation of motion  
for particles

$$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)$$

Interaction with materials

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t)$$

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \left( \mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t) \right)$$

*Absolute majority of things happening around us is described by these equations*



# Boundary Conditions

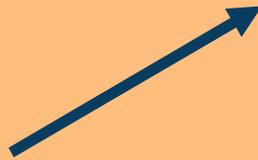
$$\mathbf{n}(\mathbf{r}) \times [\mathbf{E}_1(\mathbf{r}, t) - \mathbf{E}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \times [\mathbf{H}_1(\mathbf{r}, t) - \mathbf{H}_2(\mathbf{r}, t)] = \mathbf{K}(\mathbf{r}, t)$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{B}_1(\mathbf{r}, t) - \mathbf{B}_2(\mathbf{r}, t)] = 0$$

$$\mathbf{n}(\mathbf{r}) \cdot [\mathbf{D}_1(\mathbf{r}, t) - \mathbf{D}_2(\mathbf{r}, t)] = \sigma(\mathbf{r}, t)$$

Normal  
pointing to  
region (1)



# Electromagnetic Potentials

Lorentz('s)  
calibration

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = -\sigma\mu\varphi(\mathbf{r}, t) - \varepsilon\mu \frac{\partial\varphi(\mathbf{r}, t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

*Lorentz('s) calibration depends on particular form of material relations.*



# Wave Equation

Material parameters are assumed independent of frequency

$$\Delta \mathbf{A}(\mathbf{r}, t) - \sigma \mu \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu \mathbf{J}_{\text{source}}(\mathbf{r}, t)$$

Material parameters are assumed independent of coordinates



# Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \mathbf{E} \cdot \mathbf{J}_{\text{source}} dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} + \int_V \sigma |\mathbf{E}|^2 dV + \frac{1}{2} \frac{\partial}{\partial t} \int_V (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) dV$$

Power supplied by sources

Heat losses

*Energy balance in an electromagnetic system*



# Frequency Domain

$$F(\mathbf{r}, t) \in \mathbb{R}$$

$$\hat{F}(\mathbf{r}, \omega) \in \mathbb{C}$$

$$F(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\mathbf{r}, \omega) e^{j\omega t} d\omega$$



$$\hat{F}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} F(\mathbf{r}, t) e^{-j\omega t} dt$$

$$\frac{\partial F(\mathbf{r}, t)}{\partial t} \leftrightarrow j\omega \hat{F}(\mathbf{r}, \omega)$$

$$\frac{\partial F(\mathbf{r}, t)}{\partial r_\xi} \leftrightarrow \frac{\partial \hat{F}(\mathbf{r}, \omega)}{\partial r_\xi}$$

Time derivatives reduce to algebraic multiplication

Spatial derivatives are untouched

*Frequency domain helps us to remove explicit time derivatives*



# Phasors

$$\hat{\mathbf{F}}(\mathbf{r}, -\omega) = \hat{\mathbf{F}}^*(\mathbf{r}, \omega)$$



$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{\pi} \int_0^{\infty} \text{Re}[\hat{\mathbf{F}}(\mathbf{r}, \omega) e^{j\omega t}] d\omega$$

*Reduced frequency domain representation*



# Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r}, \omega) = \hat{\mathbf{J}}(\mathbf{r}, \omega) + j\omega\epsilon\hat{\mathbf{E}}(\mathbf{r}, \omega)$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) = -j\omega\mu\hat{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\nabla \cdot \hat{\mathbf{H}}(\mathbf{r}, \omega) = 0$$

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{r}, \omega) = \frac{\hat{\rho}(\mathbf{r}, \omega)}{\epsilon}$$

*We assume linearity of material relations*



# Wave Equation – Frequency Domain

$$\Delta \hat{\mathbf{A}}(\mathbf{r}, \omega) - j\omega\mu(\sigma + j\omega\varepsilon)\hat{\mathbf{A}}(\mathbf{r}, \omega) = -\mu\hat{\mathbf{J}}_{\text{source}}(\mathbf{r}, \omega)$$

Helmholtz('s) equation



# Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \langle \mathbf{E} \cdot \mathbf{J}_{\text{source}} \rangle dV = \oint_S \langle \mathbf{E} \times \mathbf{H} \rangle \cdot d\mathbf{S} + \int_V \langle \sigma |\mathbf{E}|^2 \rangle dV$$
$$-\frac{1}{2} \int_V \text{Re}[\hat{\mathbf{E}} \cdot \hat{\mathbf{J}}_{\text{source}}^*] dV = \frac{1}{2} \oint_S \text{Re}[\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] \cdot d\mathbf{S} + \frac{1}{2} \int_V \sigma |\hat{\mathbf{E}}|^2 dV$$

Valid for time-harmonic steady state



# Plane Wave

Unitary vector representing the direction of propagation

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega) e^{-jk\mathbf{n}\cdot\mathbf{r}}$$

Electric and magnetic fields are mutually orthogonal

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = \frac{k}{\omega\mu} [\mathbf{n} \times \mathbf{E}_0(\omega)] e^{-jk\mathbf{n}\cdot\mathbf{r}}$$

Electric and magnetic fields are orthogonal to propagation direction

$$\mathbf{n} \cdot \mathbf{E}_0(\omega) = 0$$

$$\mathbf{n} \cdot \mathbf{H}_0(\omega) = 0$$

$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$$

Wave-number

*The simplest wave solution of Maxwell('s) equations*



# Plane Wave Characteristics

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] < 0$$

$$\lambda = \frac{2\pi}{\operatorname{Re}[k]}$$

$$v_f = \frac{\omega}{\operatorname{Re}[k]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\operatorname{Im}[k]}$$

Vacuum



$$k = \frac{\omega}{c_0}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] = 0$$

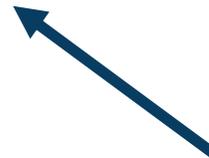
$$\lambda = \frac{c_0}{f}$$

$$v_f = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \Omega$$

$$\delta \rightarrow \infty$$

General isotropic material



# Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{2} \frac{\operatorname{Re}[k]}{\omega \mu} |\mathbf{E}_0(\omega)|^2 e^{2\operatorname{Im}[k] \mathbf{n} \cdot \mathbf{r}} \mathbf{n}$$



# Source Free Maxwell('s) Equations in Free Space

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \sigma(t) * \mathbf{E}(\mathbf{r}, t) + \varepsilon(t) * \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu(t) * \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0$$



$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\varepsilon}(\omega))$$

$$\hat{\mathbf{E}}(\mathbf{k}, \omega) = \frac{\mathbf{k}}{\omega\hat{\varepsilon}(\omega) - j\hat{\sigma}(\omega)} \times \hat{\mathbf{H}}(\mathbf{k}, \omega)$$

$$\hat{\mathbf{H}}(\mathbf{k}, \omega) = -\frac{\mathbf{k}}{\omega\hat{\mu}(\omega)} \times \hat{\mathbf{E}}(\mathbf{k}, \omega)$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}(\mathbf{k}, \omega) = 0$$

$$\mathbf{k} \cdot \hat{\mathbf{H}}(\mathbf{k}, \omega) = 0$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{\mathbf{k}, \omega} \hat{\mathbf{F}}(\mathbf{k}, \omega) e^{j(\mathbf{k} \cdot \mathbf{r} + \omega t)} d\mathbf{k} d\omega$$

*Fourier's transform leads to simple algebraic equations*



# Spatial Wave Packet

$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\epsilon}(\omega)) \quad \longrightarrow \quad \omega = \omega(|\mathbf{k}|)$$

This can be electric or magnetic intensity

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_k \hat{\mathbf{F}}_0(\mathbf{k}) e^{j(\mathbf{k}\cdot\mathbf{r} + \omega(|\mathbf{k}|)t)} d\mathbf{k}$$
$$\mathbf{k} \cdot \hat{\mathbf{F}}_0(\mathbf{k}) = 0$$

*General solution to free-space Maxwell's equations*



# Spatial Wave Packet in Vacuum

$$\omega(|\mathbf{k}|) = \pm c_0 |\mathbf{k}|$$

$$\mathbf{k} \cdot \hat{\mathbf{F}}_0^+(\mathbf{k}) = \mathbf{k} \cdot \hat{\mathbf{F}}_0^-(\mathbf{k}) = 0$$

$$\hat{\mathbf{F}}_0^-(\mathbf{k}) = [\hat{\mathbf{F}}_0^+(-\mathbf{k})]^*$$

$$\hat{\mathbf{F}}_0^+(\mathbf{k}) = [\hat{\mathbf{F}}_0^-(-\mathbf{k})]^*$$

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k} \cdot \mathbf{r}} \left[ \hat{\mathbf{F}}_0^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{F}}_0^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\hat{\mathbf{F}}_0^+(\mathbf{k}) = \frac{1}{2} \int_{\mathbf{r}} \left[ \mathbf{F}(\mathbf{r}, 0) + \frac{1}{j c_0 |\mathbf{k}|} \left. \frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \right|_{t=0} \right] e^{-j\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$$

$$\hat{\mathbf{F}}_0^-(\mathbf{k}) = \frac{1}{2} \int_{\mathbf{r}} \left[ \mathbf{F}(\mathbf{r}, 0) - \frac{1}{j c_0 |\mathbf{k}|} \left. \frac{\partial \mathbf{F}(\mathbf{r}, t)}{\partial t} \right|_{t=0} \right] e^{-j\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$$

*The field is uniquely given by initial conditions*



# Spatial Wave Packet in Vacuum

$$\omega(|\mathbf{k}|) = \pm c_0 |\mathbf{k}|$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \left[ \hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\mathbf{H}(\mathbf{r}, t) = -\frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \frac{\mathbf{k}}{Z_0 |\mathbf{k}|} \times \left[ \hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} - \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}^+(\mathbf{k}) = \mathbf{k} \cdot \hat{\mathbf{E}}^-(\mathbf{k}) = 0$$

*Electric and magnetic field are not independent*

# Vacuum Dispersion

1D problem

$$\hat{\mathbf{E}}^+(\mathbf{k}) = \hat{\mathbf{E}}^-(\mathbf{k}) = (2\pi)^2 \hat{\mathbf{E}}^+(k_z) \delta(k_x) \delta(k_y)$$

$$\mathbf{z}_0 \cdot \hat{\mathbf{E}}^+(k_z) = \mathbf{z}_0 \cdot \hat{\mathbf{E}}^-(k_z) = 0$$

Propagation in one direction

no dispersion

$$\mathbf{E}(z, t) = \mathbf{E}^+(z + c_0 t) + \mathbf{E}^-(z - c_0 t)$$

$$\mathbf{H}(z, t) = -\frac{1}{Z_0} \mathbf{z}_0 \times \left[ \mathbf{E}^+(z + c_0 t) - \mathbf{E}^-(z - c_0 t) \right]$$

*1D waves in vacuum propagate without dispersion*

# Vacuum Dispersion

In general this term does not represent translation

$$\left( [x, y, z] \pm c_0 t \right)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} e^{j\mathbf{k}\cdot\mathbf{r}} \left[ \hat{\mathbf{E}}^+(\mathbf{k}) e^{j c_0 t |\mathbf{k}|} + \hat{\mathbf{E}}^-(\mathbf{k}) e^{-j c_0 t |\mathbf{k}|} \right] d\mathbf{k}$$

Waves propagating in all directions

*2D and 3D waves in vacuum always disperse = change shape in time*



# Angular Spectrum Representation

$$\text{Im}[k_z] < 0$$

$$|\mathbf{k}|^2 = k^2 = -j\omega\hat{\mu}(\omega)(\hat{\sigma}(\omega) + j\omega\hat{\epsilon}(\omega)) \quad \longrightarrow \quad k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$

$$\hat{\mathbf{H}}_0(k_x, k_y, \omega) = -\frac{\mathbf{k}}{Z|\mathbf{k}|} \times \hat{\mathbf{E}}_0(k_x, k_y, \omega)$$

$$\hat{\mathbf{E}}_0(k_x, k_y, \omega) = \mathcal{F}_{x,y,t} \{ \mathbf{E}(x, y, 0, t) \}$$

$$\mathbf{E}(x, y, z < 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{j\sqrt{k^2 - k_x^2 - k_y^2} z} dk_x dk_y d\omega$$

$$\mathbf{E}(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{-j\sqrt{k^2 - k_x^2 - k_y^2} z} dk_x dk_y d\omega$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}_0 = 0$$

General solution to free-space Maxwell's equations

# Propagating vs Evanescent Waves

$$k_x^2 + k_y^2 < k^2$$

These waves propagate and  
can carry information to far distances

$$k_x^2 + k_y^2 > k^2$$

These waves exponentially decay in amplitude  
and cannot carry information to far distances

*Field picture losses its resolution with distance from the source plane*



# Paraxial Waves

$$\hat{\mathbf{E}}_0(k_x, k_y, \omega) \quad \Rightarrow \quad k_x^2 + k_y^2 \ll k^2 \quad \Rightarrow \quad \sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{1}{2k}(k_x^2 + k_y^2)$$

$$\mathbf{E}(x, y, z > 0, t) = \frac{1}{(2\pi)^3} \int_{k_x, k_y, \omega} e^{j(k_x x + k_y y - kz + \omega t)} \hat{\mathbf{E}}_0(k_x, k_y, \omega) e^{j\frac{1}{2k}(k_x^2 + k_y^2)z} dk_x dk_y d\omega$$

$$\mathbf{k} \cdot \hat{\mathbf{E}}_0 = 0$$

Propagates almost as a planewave

$$\mathbf{z}_0 \cdot \hat{\mathbf{E}}_0 \approx 0$$

# Gaussian Beam

$$\hat{\mathbf{E}}_{0\perp}(k_x, k_y, \omega) = \mathbf{A}_{0\perp} \pi w_0^2 e^{-\frac{1}{4} w_0^2 (k_x^2 + k_y^2)}$$



$$\mathbf{E}_{\perp}(x, y, z > 0, t) = \frac{1}{2\pi} \int_{\omega} \mathbf{A}_{0\perp} \frac{w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} e^{j \arctan\left(\frac{z}{z_R}\right)} e^{-j \frac{x^2+y^2}{w^2(z)} \left[\frac{z}{z_R}\right]} e^{j\omega \left[t - \frac{z}{c_0}\right]} d\omega$$

Gaussian profile  
in amplitude

Paraxial  
corrections

Planewave-like  
propagation

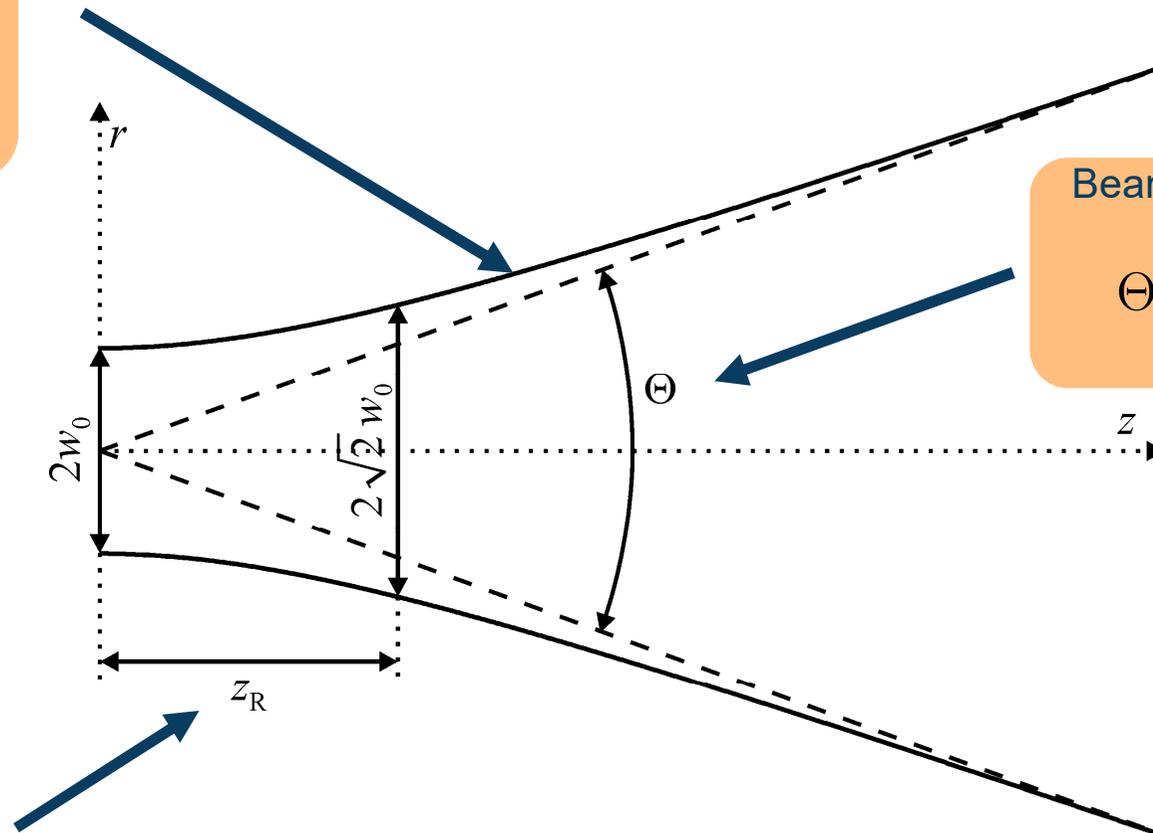
*Approximates radiation of sources large in comparison to wavelength*



# Gaussian Beam

Half-width of the beam

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$



Beam divergence

$$\Theta = \frac{2\lambda}{\pi w_0}$$

Rayleigh's distance

$$z_R = \frac{1}{2} k w_0^2 = \frac{\pi w_0^2}{\lambda}$$


# Gaussian Beam – Time-Harmonic Case

$$\langle \mathbf{S} \rangle = \frac{1}{2} \operatorname{Re} \left[ \hat{\mathbf{E}}(x, y, z, \omega) \times \hat{\mathbf{H}}^*(x, y, z, \omega) \right] = \mathbf{z}_0 S_0 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}}$$

86.5 % of power flows through the beam width

Power density at origin

$$w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2}$$



# Material Dispersion

Causality requirement

$$\varepsilon(\tau) = 0, \tau < 0$$

Stability requirement

$$\varepsilon(\tau) \rightarrow 0, \tau \rightarrow \infty$$

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \mathbf{E}(\mathbf{r}, t - \tau) d\tau$$

$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mu(\tau) \mathbf{H}(\mathbf{r}, t - \tau) d\tau$$

$$\mathbf{J}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \sigma(\tau) \mathbf{E}(\mathbf{r}, t - \tau) d\tau$$



$$\hat{\mathbf{D}}(\mathbf{r}, \omega) = \hat{\varepsilon}(\omega) \hat{\mathbf{E}}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{B}}(\mathbf{r}, \omega) = \hat{\mu}(\omega) \hat{\mathbf{H}}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\sigma}(\omega) \hat{\mathbf{E}}(\mathbf{r}, \omega)$$

*Even single planewave undergoes time dispersion when materials are present*



# Lorentz's Dispersion Model

Losses                      Resonance                      Coupling strength

$$\frac{\partial^2 \mathbf{P}(t)}{\partial t^2} + \Gamma \frac{\partial \mathbf{P}(t)}{\partial t} + \omega_0^2 \mathbf{P}(t) = \varepsilon_0 \omega_p^2 \mathbf{E}(t)$$

↓

$$\varepsilon(\omega) = \varepsilon_0 \left( 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\Gamma} \right)$$

↓

$$\varepsilon(\omega) = \varepsilon_0 \left( 1 + \sum_i \frac{\omega_{p,i}^2}{\omega_{0,i}^2 - \omega^2 + j\omega\Gamma_i} \right)$$

Dispersion model able to describe vast amount of natural materials



# Drude's Dispersion Model

Special case of  
Lorentz's dispersion

$$\omega_0 = 0$$

$$\omega_p^2 = \frac{\sigma_0 \Gamma}{\epsilon_0}$$

Permittivity model

$$\epsilon(\omega) = \epsilon_0 \left( 1 - \frac{\omega_p^2}{\omega(\omega - j\Gamma)} \right)$$

Conductivity model

$$\sigma(\omega) = \frac{\sigma_0}{1 + j\frac{\omega}{\Gamma}}$$

Collisionless plasma

$$\frac{\Gamma}{\omega} \ll 1 \Rightarrow \epsilon(\omega) \approx \epsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right)$$

Dispersion model describing neutral plasma



# Appleton's Dispersion Model

$$\frac{\partial^2 \mathbf{P}(t)}{\partial t^2} + \omega_c \frac{\partial \mathbf{P}(t)}{\partial t} \times \mathbf{z}_0 = \varepsilon_0 \omega_p^2 \mathbf{E}(t)$$

Cyclotron frequency

Direction of magnetization

Plasma frequency

$\hat{\varepsilon} \neq \hat{\varepsilon}^T$   
Propagation in opposite directions is not the same

$$\hat{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & \frac{-j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 0 \\ \frac{j\omega_c \omega_p^2}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{bmatrix}$$

Dispersion model describing magnetized neutral plasma



# Propagation in Appleton's Dispersion Model

$$\hat{\mathbf{E}} = \mathbf{E}_0 e^{jk_z z}$$
$$\mathbf{k} \cdot \mathbf{E}_0 = 0$$

Planewave propagation  
along magnetization

Fundamental modes  
are circularly polarized  
waves

$$\frac{k_z^2}{k_0^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)}$$

$$\hat{E}_x = \mp j \hat{E}_y$$

Dispersion model describing magnetized neutral plasma

# Radiation

Microscopic  
charge velocity

$$\frac{\partial \mathbf{v}(t)}{\partial t} \neq 0$$

Macroscopic  
current density

$$\frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} \neq 0$$



$$\int_{-\infty}^{\infty} \oint_S (\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)) \cdot d\mathbf{S} dt \neq 0$$

Any surface  
circumscribing  
the sources

Only accelerating charges can radiate



# Time-Harmonic Electric Dipole

$$\hat{P}(\mathbf{r}, \omega) = \mathbf{z}_0 p_z(\omega) \delta(x) \delta(y) \delta(z)$$

$\rho(\mathbf{r}, \omega) \approx 0$

$$\hat{\mathbf{A}}(\mathbf{r}, \omega) = j Z_0 k^2 (\mathbf{r}_0 \cos \theta - \theta_0 \sin \theta) p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = c_0 k^3 \varphi_0 \sin \theta \left( -1 + \frac{j}{kr} \right) p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[ 2\mathbf{r}_0 \cos \theta \left( \frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left( -1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

Elementary source of radiation



# Time-Harmonic Electric Dipole - Field Zones

Static zone

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[ 2\mathbf{r}_0 \cos \theta \left( \frac{j}{kr} + \frac{1}{k^2 r^2} \right) + \theta_0 \left( -1 + \frac{j}{kr} + \frac{1}{k^2 r^2} \right) \sin \theta \right] p_z(\omega) \frac{e^{-jkr}}{4\pi kr}$$

Induction zone

Radiation zone

Static, quasi-static and fully dynamic terms all appear in the formula



# Time-Harmonic Electric Dipole - Radiation Zone

$$\hat{P}(\mathbf{r}, \omega) = \mathbf{z}_0 p_z(\omega) \delta(x) \delta(y) \delta(z)$$



$$\hat{\mathbf{E}}_\infty(\mathbf{r}, \omega) \approx -Z_0 c_0 k^3 \theta_0 p_z(\omega) \frac{e^{-jkr}}{4\pi kr} \sin \theta$$

$$\hat{\mathbf{H}}_\infty(\mathbf{r}, \omega) \approx \frac{1}{Z_0} \mathbf{r}_0 \times \hat{\mathbf{E}}_\infty(\mathbf{r}, \omega)$$

$$\langle \mathbf{S}_\infty \rangle = \frac{1}{2} \text{Re}[\hat{\mathbf{E}}_\infty \times \hat{\mathbf{H}}_\infty^*] = \frac{1}{2Z_0} |\hat{\mathbf{E}}_\infty(\mathbf{r}, \omega)|^2 \mathbf{r}_0$$

Radiated power [W]

$$P_{\text{rad}} = \frac{c_0^2 Z_0 k^4}{12\pi} |p_z(\omega)|^2$$

$$\mathbf{r}_0 \cdot \hat{\mathbf{E}}_\infty \approx 0$$

Farfield has a planewave-like geometry



# Time-Harmonic Electric Dipole – General Case

$$\hat{\mathbf{P}}(\mathbf{r}, \omega) = \hat{\mathbf{p}}(\omega) \delta(\mathbf{r} - \mathbf{r}')$$



$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = c_0 k^3 \left( \frac{\mathbf{R}}{R} \times \hat{\mathbf{p}} \right) \left( 1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{4\pi kR}$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = Z_0 c_0 k^3 \left[ -\frac{\mathbf{R}}{R} \times \left( \frac{\mathbf{R}}{R} \times \hat{\mathbf{p}} \right) + \left( 3 \frac{\mathbf{R}}{R} \left[ \hat{\mathbf{p}} \cdot \frac{\mathbf{R}}{R} \right] - \hat{\mathbf{p}} \right) \left( \frac{1}{k^2 R^2} + \frac{j}{kR} \right) \right] \frac{e^{-jkR}}{4\pi kR}$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

Elementary source of radiation



# General Radiator

$$\hat{\mathbf{J}}(\mathbf{r}, \omega), \mathbf{J}(\mathbf{r}, t)$$



$$\hat{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) \frac{e^{-jk_0 R}}{R} dV'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{R}{c_0}\right)}{R} dV'$$

$$R = |\mathbf{r} - \mathbf{r}'|$$

time  
retardation

Superposition of dipole fields



# Field in Radiation Zone – General Case FD

$$kR \gg 1 \quad \wedge \quad r \gg r'$$

$$\hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{-jk_0 r}}{r} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) e^{jk_0 \mathbf{r}_0 \cdot \mathbf{r}'} dV'$$

$$\hat{\mathbf{H}}_{\infty}(\mathbf{r}, \omega) \approx -\frac{j\omega}{Z_0} \mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega)$$

$$\hat{\mathbf{E}}_{\infty} \approx -Z_0 (\mathbf{r}_0 \times \hat{\mathbf{H}}_{\infty})$$

$$\hat{\mathbf{E}}_{\infty}(\mathbf{r}, \omega) \approx j\omega \mathbf{r}_0 \times (\mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega))$$

$$\langle \mathbf{S}_{\infty} \rangle = \frac{1}{2Z_0} \omega^2 \left| \mathbf{r}_0 \times \hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega) \right|^2 \mathbf{r}_0$$

Farfield has a planewave-like geometry



# Field in Radiation Zone – General Case TD

$$\mathbf{A}_\infty(\mathbf{r}, t) \approx \frac{\mu}{4\pi r} \int_{V'} \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c_0} + \frac{\mathbf{r}_0 \cdot \mathbf{r}'}{c_0}\right) dV'$$

$$\mathbf{E}_\infty \approx -Z_0(\mathbf{r}_0 \times \mathbf{H}_\infty) \quad \mathbf{H}_\infty(\mathbf{r}, t) \approx -\frac{1}{Z_0} \mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t)$$

$$\mathbf{E}_\infty(\mathbf{r}, t) \approx \mathbf{r}_0 \times (\mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t))$$

$$\mathbf{S}_\infty \approx \frac{1}{Z_0} \left| \mathbf{r}_0 \times \dot{\mathbf{A}}_\infty(\mathbf{r}, t) \right|^2 \mathbf{r}_0$$

Farfield has a planewave-like geometry



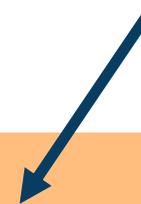
# Radiation Zone = Rays

$$\hat{\mathbf{A}}_{\infty}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{-jk_0 r}}{r} \int_{V'} \hat{\mathbf{J}}(\mathbf{r}', \omega) e^{jk_0 \mathbf{r}_0 \cdot \mathbf{r}'} dV'$$



4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \left\{ \mathbf{J}(x, y, z, t) \right\}$$



$$\mathbf{A}_{\infty}(\mathbf{r}, t) \approx \mathcal{F}_{\omega}^{-1} \left\{ \frac{\mu_0}{4\pi} \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \frac{e^{-jk_0 r}}{r} \right\}$$

Radiation diagram is formed by Fourier('s) transform of sources



# Angular Spectrum Representation (Sources)

4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \{ \mathbf{J}(x, y, z, t) \}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\text{Im}[k_z] < 0$$

$$\hat{\mathbf{G}} = \frac{\hat{\mathbf{J}}(k_x, k_y, \mp k_z, \omega)}{2k_z} e^{\mp jk_z z}$$

Valid only outside the source region

$$\mathbf{H} \left( x, y, \begin{matrix} z > \max(z') \\ z < \min(z') \end{matrix}, t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left\{ [k_x, k_y, \mp k_z] \times \hat{\mathbf{G}} \right\}$$

$$\mathbf{E} \left( x, y, \begin{matrix} z > \max(z') \\ z < \min(z') \end{matrix}, t \right) = \mathcal{F}_{k_x, k_y, \omega}^{-1} \left\{ \frac{Z_0}{k} [k_x, k_y, \mp k_z] \times \left( [k_x, k_y, \mp k_z] \times \hat{\mathbf{G}} \right) \right\}$$

*General solution to free-space Maxwell's equations*



# Angular Spectrum in Radiation Zone

$$\mathbf{F}(x, y, z > 0, \omega) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ \hat{\mathbf{L}}(k_x, k_y) e^{-jk_z z} \right\}$$

$$k_0 r \rightarrow \infty \quad \Downarrow \quad \begin{aligned} k_z &= \sqrt{k_0^2 - k_x^2 - k_y^2} \\ \text{Im}[k_z] &< 0 \end{aligned} \quad \mathbf{r}_0 = \frac{[x, y, z]}{r}$$

$$\mathbf{F}_\infty(\mathbf{r}, \omega) = \frac{1}{(2\pi)^2} \int_{k^2 > k_x^2 + k_y^2} \hat{\mathbf{L}}(k_x, k_y) e^{jk_0 r \left( \frac{k_x}{k} r_{0x} + \frac{k_y}{k} r_{0y} - \frac{k_z}{k} r_{0z} \right)} dk_x dk_y$$

$$k_0 r \rightarrow \infty \quad \Downarrow \quad \text{Stationary phase method}$$

$$\mathbf{F}_\infty(\mathbf{r}, \omega) = \frac{jk_0 r_{0z}}{2\pi} \hat{\mathbf{L}}(-k_0 r_{0x}, -k_0 r_{0y}) \frac{e^{-jk_0 r}}{r}$$

Farfield is made of propagating planewaves



# Angular Spectrum in Radiation Zone

4D Fourier('s) transform

$$\hat{\mathbf{J}}(k_x, k_y, k_z, \omega) = \mathcal{F}_{x,y,z,t} \left\{ \mathbf{J}(x, y, z, t) \right\}$$

$$\mathbf{H}_\infty(\mathbf{r}, t) \approx \mathcal{F}_\omega^{-1} \left\{ -\frac{jk_0}{4\pi} \mathbf{r}_0 \times \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \frac{e^{-jk_0 r}}{r} \right\}$$

$$\mathbf{E}_\infty(\mathbf{r}, t) \approx \mathcal{F}_\omega^{-1} \left\{ \frac{jk_0 Z_0}{4\pi} \mathbf{r}_0 \times \left[ \mathbf{r}_0 \times \hat{\mathbf{J}}(-k_0 \mathbf{r}_0, \omega) \right] \frac{e^{-jk_0 r}}{r} \right\}$$

$$\mathbf{r}_0 = \frac{[x, y, z]}{r}$$

Farfield is made of propagating planewaves



# Planar Material Boundary

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

Incident wave

Reflected (1 → 1) / Transmitted (2 → 1) wave

$$\mathbf{H}(z < 0) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ \mathbf{H}_1^+(k_x, k_y, \omega) e^{-jk_{z1}z} + \mathbf{H}_1^-(k_x, k_y, \omega) e^{jk_{z1}z} \right\}$$

$$\mathbf{E}(z < 0) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ Z_1 \frac{[k_x, k_y, -k_{z1}] \times \mathbf{H}_1^+(k_x, k_y, \omega)}{k_1} e^{-jk_{z1}z} + Z_1 \frac{[k_x, k_y, k_{z1}] \times \mathbf{H}_1^-(k_x, k_y, \omega)}{k_1} e^{jk_{z1}z} \right\}$$

$$\mathbf{H}(z > 0) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ \mathbf{H}_2^+(k_x, k_y, \omega) e^{-jk_{z2}z} + \mathbf{H}_2^-(k_x, k_y, \omega) e^{jk_{z2}z} \right\}$$

$$\mathbf{E}(z > 0) = \mathcal{F}_{k_x, k_y}^{-1} \left\{ Z_2 \frac{[k_x, k_y, -k_{z2}] \times \mathbf{H}_2^+(k_x, k_y, \omega)}{k_2} e^{-jk_{z2}z} + Z_2 \frac{[k_x, k_y, k_{z2}] \times \mathbf{H}_2^-(k_x, k_y, \omega)}{k_2} e^{jk_{z2}z} \right\}$$

Boundary is at  $z = 0$

Reflected (2 → 2) / Transmitted (1 → 2) wave

Incident wave

Field is composed of incident, reflected and transmitted waves



# Planar Material Boundary – Boundary Conditions

$$\left[ k_x, k_y, \mp k_{z1} \right] \cdot \mathbf{H}_1^\pm = 0$$

$$\left[ k_x, k_y, \mp k_{z2} \right] \cdot \mathbf{H}_2^\pm = 0$$

$$\mathbf{z}_0 \times \mathbf{H}_1^+ + \mathbf{z}_0 \times \mathbf{H}_1^- = \mathbf{z}_0 \times \mathbf{H}_2^+ + \mathbf{z}_0 \times \mathbf{H}_2^-$$

$$Z_1 \frac{\mathbf{z}_0 \times \left( \left[ k_x, k_y, -k_{z1} \right] \times \mathbf{H}_1^+ \right)}{k_1} + Z_1 \frac{\mathbf{z}_0 \times \left( \left[ k_x, k_y, k_{z1} \right] \times \mathbf{H}_1^- \right)}{k_1} =$$

$$Z_2 \frac{\mathbf{z}_0 \times \left( \left[ k_x, k_y, -k_{z2} \right] \times \mathbf{H}_2^+ \right)}{k_2} + Z_2 \frac{\mathbf{z}_0 \times \left( \left[ k_x, k_y, k_{z2} \right] \times \mathbf{H}_2^- \right)}{k_2}$$

$k_x, k_y$  are equal on both sides

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\text{Im} \left[ k_z \right] < 0$$

Relations valid for both propagative and evanescent waves

# Perpendicular Incidence – Matrix Form

$$k_x = k_y = 0$$

Transmission matrix  
(multilayer cascade)

$$[S] = \frac{1}{T_{22}} \begin{bmatrix} -T_{21} & 1 \\ \det T & T_{12} \end{bmatrix}$$

$$[T] = \frac{1}{S_{12}} \begin{bmatrix} -\det S & S_{22} \\ -S_{11} & 1 \end{bmatrix}$$

$$\begin{bmatrix} E_2^+ \\ E_2^- \end{bmatrix} = [T] \begin{bmatrix} E_1^+ \\ E_1^- \end{bmatrix}$$

$$[T] = \frac{1}{2Z_1} \begin{bmatrix} Z_1 + Z_2 & Z_1 - Z_2 \\ Z_1 - Z_2 & Z_1 + Z_2 \end{bmatrix}$$

$$\begin{bmatrix} E_1^- \\ E_2^+ \end{bmatrix} = [S] \begin{bmatrix} E_1^+ \\ E_2^- \end{bmatrix}$$

$$[S] = \frac{1}{Z_2 + Z_1} \begin{bmatrix} Z_2 - Z_1 & 2Z_1 \\ 2Z_2 & Z_1 - Z_2 \end{bmatrix}$$

Scattering matrix  
(experiments)

Matrices with the use across the electrical engineering

# Perpendicular Incidence – Interesting Cases

Wavelength inside the slab

$$d = \frac{\lambda_2}{2} n$$



$$T = \begin{bmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix}, S = \begin{bmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{bmatrix}$$

Transparent dielectric layer

Bragg's mirror  
(dielectric mirror)

$$k_0 (n_1 d_1 + n_2 d_2) = N\pi$$

Alternating dielectric layers

Technically important special cases



# Oblique Incidence – TM / TE Case

$$k_x = k_x$$
$$k_y = 0$$

Snell's law of refraction

$$k_1 \sin \alpha_{\text{inc}} = k_2 \sin \alpha_{\text{trans}}$$

$$k_1 \sin \alpha_{\text{inc}} = k_1 \sin \alpha_{\text{refl}}$$

The angle of incidence equals to the angle of reflection

Snell's law is a property of k-vectors



# Oblique Incidence – TM Case

$$\begin{aligned}k_y &= 0 \\ H_x &= 0 \\ H_z &= 0\end{aligned}$$

$$\begin{aligned}R_{1 \rightarrow 1}^{\text{TM}} &= \frac{E_{1x}^-}{E_{1x}^+} = \frac{\frac{k_{z2}}{k_2} Z_2 - \frac{k_{z1}}{k_1} Z_1}{\frac{k_{z2}}{k_2} Z_2 + \frac{k_{z1}}{k_1} Z_1} \\ T_{1 \rightarrow 2}^{\text{TM}} &= \frac{E_{2x}^+}{E_{1x}^+} = \frac{2Z_2 \frac{k_{z2}}{k_2}}{\frac{k_{z2}}{k_2} Z_2 + \frac{k_{z1}}{k_1} Z_1}\end{aligned}$$

Generalization of reflection and transmission to oblique incidence



# Oblique Incidence – TM Case

$$k_y = 0$$
$$H_x = 0$$
$$H_z = 0$$

Brewster's angle

$$R_{1 \rightarrow 1}^{\text{TM}} = 0$$



$$\frac{k_x}{k_1} = \sqrt{\frac{\varepsilon_2 (\mu_2 \varepsilon_1 - \mu_1 \varepsilon_2)}{\mu_1 (\varepsilon_1^2 - \varepsilon_2^2)}}$$

Vanishing reflection on a boundary

Simplification for pure dielectrics

$$\frac{k_x}{k_1} = \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right)^{-\frac{1}{2}}$$

Can be used for polarizing unpolarized light beams



# Oblique Incidence – TE Case

$$\begin{aligned}k_y &= 0 \\ E_x &= 0 \\ E_z &= 0\end{aligned}$$

$$\begin{aligned}R_{1 \rightarrow 1}^{\text{TE}} &= \frac{E_{1y}^-}{E_{1y}^+} = \frac{\frac{k_{z1}}{k_1} Z_2 - \frac{k_{z2}}{k_2} Z_1}{\frac{k_{z1}}{k_1} Z_2 + \frac{k_{z2}}{k_2} Z_1} \\ T_{1 \rightarrow 2}^{\text{TE}} &= \frac{E_{2y}^+}{E_{1y}^+} = \frac{2Z_2 \frac{k_{z1}}{k_1}}{\frac{k_{z1}}{k_1} Z_2 + \frac{k_{z2}}{k_2} Z_1}\end{aligned}$$

Generalization of reflection and transmission to oblique incidence



# Oblique Incidence – TE Case

$$\begin{aligned}k_y &= 0 \\ E_x &= 0 \\ E_z &= 0\end{aligned}$$

Brewster's angle

$$R_{1 \rightarrow 1}^{\text{TE}} = 0$$



$$\frac{k_x}{k_1} = \sqrt{\frac{\mu_2 (\varepsilon_2 \mu_1 - \varepsilon_1 \mu_2)}{\varepsilon_1 (\mu_1^2 - \mu_2^2)}}$$

Vanishing reflection on a boundary

Simplification for pure magnetics

$$\frac{k_x}{k_1} = \left(1 + \frac{\mu_1}{\mu_2}\right)^{-\frac{1}{2}}$$

Unrealistic scenario for natural materials



# Oblique Incidence – Total Reflection

$$\frac{k_x}{k_1} > \frac{k_2}{k_1} = \frac{n_2}{n_1} < 1$$



$$\left| R_{1 \rightarrow 1}^{\text{TM}} \right| = \left| R_{1 \rightarrow 1}^{\text{TE}} \right| = 1$$

Valid for both, the TM and the TE case



# Guided TEM Wave

Wave propagation identical to a planewave

$$k^2 = -j\omega\mu(\sigma + j\omega\varepsilon)$$

Geometry of a planewave

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \mathbf{E}_\perp(x, y, \omega) e^{-jkz}$$

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = \mathbf{H}_\perp(x, y, \omega) e^{-jkz}$$

$$\hat{\mathbf{H}} = \frac{k}{\omega\mu} (\mathbf{z}_0 \times \hat{\mathbf{E}})$$

$$\begin{aligned} \Delta_\perp \mathbf{E}_\perp &= 0 \\ \Delta_\perp \mathbf{H}_\perp &= 0 \end{aligned}$$

$$\mathbf{n} \times \mathbf{E}_\perp = 0$$

Boundary condition  
on the conductor

Generalization of a planewave

# Circuit Parameters of the TEM Wave

$$\hat{U}(z, \omega) = \hat{U}_0(\omega) e^{-jkz}$$

$$\hat{I}(z, \omega) = \hat{I}_0(\omega) e^{-jkz}$$

Enclosing conductor

$$\hat{I}_0(\omega) = \oint_l \mathbf{H}_\perp \cdot d\mathbf{l} = \frac{k}{\omega\mu} \cdot \frac{Q_{\text{pul}}}{\varepsilon}$$

$$\hat{U}_0(\omega) = -\int_A^B \mathbf{E}_\perp \cdot d\mathbf{l} = \frac{\omega\mu}{k} \cdot \frac{\Phi_{\text{pul}}}{\mu}$$

$$Z_{\text{TRL}} = \frac{\hat{U}_0(\omega)}{\hat{I}_0(\omega)} = \frac{\omega\mu}{k} \cdot \frac{\varepsilon}{C_{\text{pul}}} = \frac{\omega\mu}{k} \cdot \frac{L_{\text{pul}}}{\mu} = \sqrt{\frac{L_{\text{pul}}}{C_{\text{pul}}}}$$

$$v_{\text{phase}} = \frac{1}{\sqrt{C_{\text{pul}} L_{\text{pul}}}} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Between conductors

Per unit length

Velocity of phase propagation



# The Telegraph Equations

$$\frac{\partial U(z,t)}{\partial z} = -L_{\text{pul}} \frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C_{\text{pul}} \frac{\partial U(z,t)}{\partial t}$$

*Circuit analog of Maxwell's equations*



# Guided TE and TM Waves

Wave propagation differs from a planewave

$$k_z^2 = k^2 - k_\perp^2$$

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = \left[ \mathbf{E}_\perp(\mathbf{r}_\perp, \omega) + z_0 E_z(\mathbf{r}_\perp, \omega) \right] e^{-jk_z z}$$

$$\hat{\mathbf{H}}(\mathbf{r}, \omega) = \left[ \mathbf{H}_\perp(\mathbf{r}_\perp, \omega) + z_0 H_z(\mathbf{r}_\perp, \omega) \right] e^{-jk_z z}$$

$$\mathbf{E}_\perp = -\frac{1}{k_\perp^2} \left( jk_z \nabla_\perp E_z - j\omega\mu z_0 \times \nabla_\perp H_z \right)$$

$$\mathbf{H}_\perp = -\frac{1}{k_\perp^2} \left( jk_z \nabla_\perp H_z + (\sigma + j\omega\varepsilon) z_0 \times \nabla_\perp E_z \right)$$

$$\Delta_\perp E_z + k_\perp^2 E_z = 0$$

$$\Delta_\perp H_z + k_\perp^2 H_z = 0$$

TEM mode must be completed with TE and TM modes to form a complete set

# PEC Waveguides – pure TE, TM modes

Impedances differ from those of a plane wave

Boundary condition on the conductor

$$\mathbf{n} \times \hat{\mathbf{E}} = 0$$

$$Z^{\text{TE/TM}} \mathbf{H}_{\perp} = \mathbf{z}_0 \times \mathbf{E}_{\perp}$$

$$Z^{\text{TE}} = \frac{\omega\mu}{k_z}, \quad Z^{\text{TM}} = \frac{jk_z}{\sigma + j\omega\epsilon}$$



- Wavenumbers  $k_{\perp} > 0$  form a **discrete set**
- Modes are **orthogonal** in waveguide cross-section
- Modes form a **complete set** in waveguide cross-section

*TEM mode must be completed with TE and TM modes to form a complete set*

# PEC Waveguides – modal orthogonality

$$\int_S \mathbf{E}_{\perp\alpha} \cdot \mathbf{E}_{\perp\beta}^* dS = C \delta_{\alpha\beta}$$

cross-section of  
the waveguide



$$\int_S (\mathbf{E}_{\perp\alpha} \times \mathbf{H}_{\perp\beta}^*) \cdot \mathbf{z}_0 dS = \frac{1}{Z_{\beta}^*} \int_S \mathbf{E}_{\perp\alpha} \cdot \mathbf{E}_{\perp\beta}^* dS$$

$$\int_S \mathbf{H}_{\perp\alpha} \cdot \mathbf{H}_{\perp\beta}^* dS = \frac{1}{Z_{\alpha} Z_{\beta}^*} \int_S \mathbf{E}_{\perp\alpha} \cdot \mathbf{E}_{\perp\beta}^* dS$$

*Waveguide modes form an orthogonal set*



# PEC Waveguides – modal decomposition

positive direction

$$\hat{\mathbf{E}}^+(\mathbf{r}, \omega) = \sum_{\alpha} C_{\alpha}^+ \left[ \mathbf{E}_{\perp\alpha}(\mathbf{r}_{\perp}, \omega) + \mathbf{z}_0 E_{z\alpha}(\mathbf{r}_{\perp}, \omega) \right] e^{-jk_{z\alpha}z}$$

$$\hat{\mathbf{H}}^+(\mathbf{r}, \omega) = \sum_{\alpha} C_{\alpha}^+ \left[ \mathbf{H}_{\perp\alpha}(\mathbf{r}_{\perp}, \omega) + \mathbf{z}_0 H_{z\alpha}(\mathbf{r}_{\perp}, \omega) \right] e^{-jk_{z\alpha}z}$$

negative direction

$$\hat{\mathbf{E}}^-(\mathbf{r}, \omega) = \sum_{\alpha} C_{\alpha}^- \left[ \mathbf{E}_{\perp\alpha}(\mathbf{r}_{\perp}, \omega) - \mathbf{z}_0 E_{z\alpha}(\mathbf{r}_{\perp}, \omega) \right] e^{jk_{z\alpha}z}$$

$$\hat{\mathbf{H}}^-(\mathbf{r}, \omega) = \sum_{\alpha} C_{\alpha}^- \left[ -\mathbf{H}_{\perp\alpha}(\mathbf{r}_{\perp}, \omega) + \mathbf{z}_0 H_{z\alpha}(\mathbf{r}_{\perp}, \omega) \right] e^{jk_{z\alpha}z}$$

*Any field within a waveguide can be composed of its modes*



# PEC Waveguides – Field Sources

Known field at arbitrary cross-section of the waveguide

$$C_{\beta}^{\pm} = \frac{1}{2} \frac{\int_S \left[ \mathbf{E}_{\perp\beta}^* (\mathbf{r}_{\perp}, \omega) \cdot \hat{\mathbf{E}} (\mathbf{r}, \omega) \pm |Z_{\beta}|^2 \mathbf{H}_{\perp\beta}^* (\mathbf{r}_{\perp}, \omega) \cdot \hat{\mathbf{H}} (\mathbf{r}, \omega) \right] dS}{\int_S \mathbf{E}_{\perp\beta}^* (\mathbf{r}_{\perp}, \omega) \cdot \mathbf{E}_{\perp\beta} (\mathbf{r}_{\perp}, \omega) dS} e^{\pm jk_{z\beta} z}$$

transversal field of the waveguide mode

*Tangential fields within the cross-section fully define the field everywhere*



# PEC Waveguides – Field Sources

Valid to the right (+) or to the left (-) of the source region

Full field of the waveguide mode

Source current density existing within the waveguide

$$C_{\beta}^{\pm} = -\frac{Z_{\beta}}{2} \frac{\int_V \hat{\mathbf{E}}_{\beta}^{\mp}(\mathbf{r}, \omega) \cdot \hat{\mathbf{J}}_s(\mathbf{r}, \omega) dV}{\int_S \mathbf{E}_{\perp\beta}(\mathbf{r}_{\perp}, \omega) \cdot \mathbf{E}_{\perp\beta}(\mathbf{r}_{\perp}, \omega) dS}$$

transversal field of the waveguide mode

*This is how waveguide modes are excited*



# Dielectric Waveguides – mixed TE + TM modes

Boundary condition  
on dielectric interface

$$\mathbf{n} \times [\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2] = 0$$

$$\mathbf{n} \times [\hat{\mathbf{H}}_1 - \hat{\mathbf{H}}_2] = 0$$



- **Finite** number of guided modes
- **Continuum** of radiating modes
- Only **combination** of guided and radiating modes **forms a complete set** in the waveguide cross-section

*General field is not guided by a dielectric waveguide*

# Cavity Resonators

Master equation

$$\Delta \hat{\mathbf{E}}_n + \frac{\omega_n^2}{c_0^2} \epsilon_r \mu_r \hat{\mathbf{E}}_n = 0$$

Boundary condition  
on the conductor

$$\mathbf{n} \times \hat{\mathbf{E}}_n = 0$$

Nontrivial solution only  
exists in a lossless cavity

Notice that field value is  
defined on a closed surface

- Eigenfrequencies  $\omega_n > 0$  form a **discrete set**
- Modes are **orthogonal** in the volume of the cavity
- Modes form a **complete set** in the volume of the cavity

*Field in a lossless cavity forms an exception to the uniqueness theorem*

# Closed Waveguide as a Cavity Resonator

Dispersion relation  $k_{z\alpha} = \frac{p\pi}{L}$

$$\frac{\omega_\alpha^2}{c_0^2} \varepsilon_r \mu_r = k_{z\alpha}^2 + k_\perp^2$$

$$\hat{\mathbf{E}}_\alpha(\mathbf{r}, \omega) = \mathbf{E}_{\alpha\perp}(\mathbf{r}_\perp, \omega) \sin(k_{z\alpha} z) + j z_0 E_{z\alpha}(\mathbf{r}_\perp, \omega) \cos(k_{z\alpha} z)$$

$$\hat{\mathbf{H}}_\alpha(\mathbf{r}, \omega) = j \mathbf{H}_{\perp\alpha}(\mathbf{r}_\perp, \omega) \cos(k_{z\alpha} z) + z_0 H_{z\alpha}(\mathbf{r}_\perp, \omega) \sin(k_{z\alpha} z)$$

Electric and magnetic fields  
are 90° out of phase

*Waveguide modes already solve the wave equation*

# Excitation of a PEC Cavity

Vector potential is expanded  
into cavity modes

$$\mathbf{A}(\mathbf{r}, \omega) = \sum_{\alpha} C_{\alpha}(\omega) \mathbf{A}_{\alpha}(\mathbf{r})$$

Modes of lossless cavity

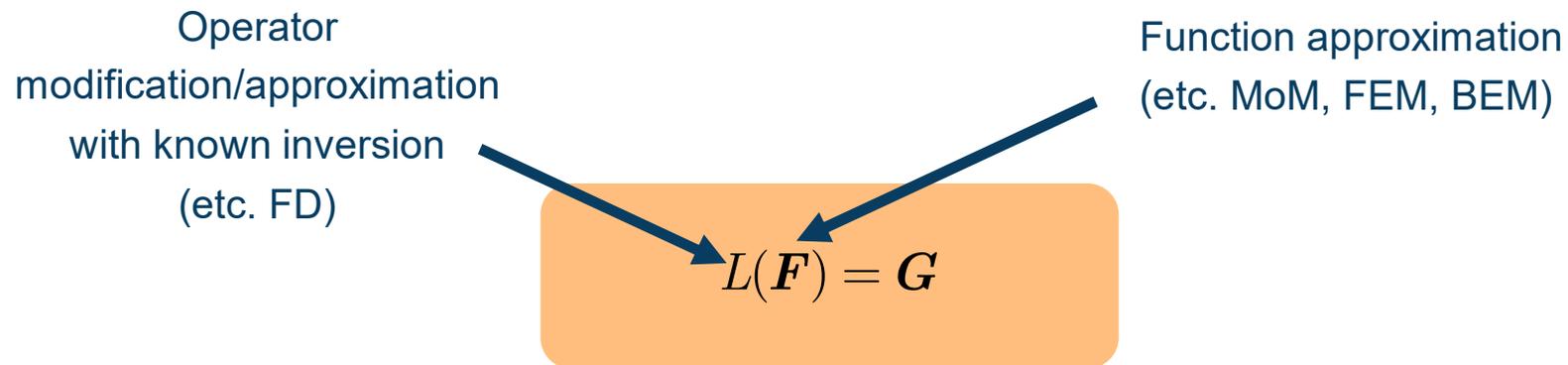
$$\Delta \mathbf{A}_{\alpha}(\mathbf{r}) + \lambda_{\alpha}^2 \mathbf{A}_{\alpha}(\mathbf{r}) = 0$$
$$\mathbf{n}(\mathbf{r}) \times \mathbf{E}_{\alpha}(\mathbf{r}) = 0$$

$$C_{\alpha}(\omega) = -\frac{\mu(\omega)}{k^2(\omega) - \lambda_{\alpha}^2} \cdot \frac{\int_V \mathbf{A}_{\alpha}^*(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}, \omega) dV}{\int_V \mathbf{A}_{\alpha}^*(\mathbf{r}) \cdot \mathbf{A}_{\alpha}(\mathbf{r}) dV}$$

*Modes of a lossless cavity are used as a basis*



# Computational Electromagnetics (CEM)



*Typical real-life problems cannot be solved with pen and paper.*



# Finite Differences in Frequency Domain

Approximating derivatives with differences:

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2}$$

Differential equations  $\xrightarrow{\text{Discretization } x, y, z \rightarrow i, j, k}$  System of linear equations

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0$$

*Common method to find waveguide or resonator modes.*



# Finite Differences in Time Domain

Additional time discretization:  $t \rightarrow q$

Schemes

Implicit:

Computes full matrices for discretization in time and space.

Explicit:

Treats time as an evolution variable.

System of linear equations:

$$g_{i',j',k'}^{q'} = \sum_{i,j,k,q} a_{i,j,k}^q f_{i,j,k}^q$$

Matrix multiplication:

$$g_{i',j',k'}^{q'} = \sum_{i,j,k,q < q'} a_{i,j,k}^q f_{i,j,k}^q$$

+ Boundary and initial conditions

$$\Delta \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

*Versatile method that can even handle non-linearities.*



# Finite Element Method

Galerkin method applied to a differential equation.

$$\sum_n a_n \langle \psi_m, \mathbf{L}(\psi_n) \rangle = \langle \psi_m, \mathbf{G} \rangle$$

Testing within scalar product

$$\mathbf{F} = \sum_n a_n \psi_n$$

Expansion coefficients

Basis functions (satisfy boundary conditions)

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0$$

Powerful method to solve bounded problems described by a differential equation.



# Method of Moments

Galerkin method applied to an integral equation.

$$\sum_n a_n \langle \psi_m, \mathbf{L}(\psi_n) \rangle = \langle \psi_m, \mathbf{G} \rangle$$

Testing within scalar product

$$\mathbf{F} = \sum_n a_n \psi_n$$

Expansion coefficients

Basis functions (satisfy boundary conditions)

$$\frac{j}{\omega \epsilon_0 \chi} \mathbf{J}(\mathbf{r}) = -\frac{j\omega\mu_0}{4\pi} \left( \mathbf{1} + \frac{1}{k_0^2} \nabla \nabla \cdot \right) \int_V \left[ \mathbf{J}_i(\mathbf{r}') + \mathbf{J}(\mathbf{r}') \right] \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dV$$

Method for unbounded problems, which discretizes only current supporting regions.





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