3D Computer Vision

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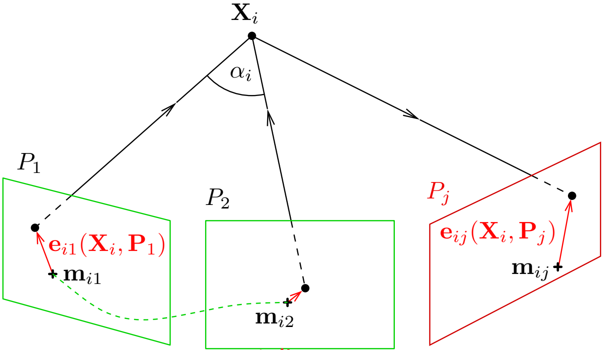
Open Informatics Master's Course

▶Reconstructing Camera System by Gluing Camera Triples

Given: Calibration matrices $\mathbf{K}_j$ and tentative correspondences per camera triples.



Initialization

1. initialize camera cluster $\mathcal{C}$ with a pair P_1, P_2
2. find essential matrix $\mathbf{E}_{12}$ and matches M_{12} by the 5-point algorithm →88
3. construct camera pair
$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$
4. triangulate $\{X_i\}$ per match from M_{12} →106
5. initialize point cloud $\mathcal{X}$ with $\{X_i\}$ satisfying chirality constraint $z_i > 0$ and apical angle constraint $|\alpha_i| > \alpha_T$

• relative cam. orientation 5pt
• decomposition of E
• triangulation

• exterior orientation P3P

Attaching camera $P_j \notin \mathcal{C}$

1. select points $\mathcal{X}_j$ from $\mathcal{X}$ that have matches to P_j
2. estimate $\mathbf{P}_j$ using $\mathcal{X}_j$, RANSAC with the 3-pt alg. (P3P), projection errors $\mathbf{e}_{ij}$ in $\mathcal{X}_j$ →66
3. reconstruct 3D points from all tentative matches from P_j to all P_l, $l \neq k$ that are not in $\mathcal{X}$
4. filter them by the chirality and apical angle constraints and add them to $\mathcal{X}$
5. add P_j to $\mathcal{C}$
6. perform bundle adjustment on $\mathcal{X}$ and $\mathcal{C}$ coming next →139

▶The Projective Reconstruction Theorem

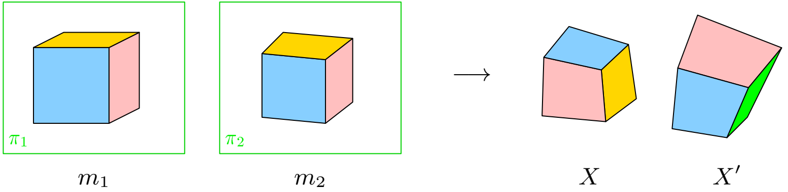
- We can run an analogical procedure when the cameras remain uncalibrated. But:

Observation: Unless $\mathbf{P}_j$ are constrained, then for any number of cameras $j = 1, \ldots, k$

$$\underline{\mathbf{m}}_{ij} \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \underbrace{\mathbf{P}_j \mathbf{H}^{-1}}_{\mathbf{P}'_j} \underbrace{\mathbf{H}\underline{\mathbf{X}}_i}_{\underline{\mathbf{X}}'_i} = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

- when $\mathbf{P}_i$ and $\underline{\mathbf{X}}$ are both determined from correspondences (including calibrations $\mathbf{K}_i$), they are given up to a common 3D homography $\mathbf{H}$

(translation, rotation, scale, shear, pure perspectivity)



- when cameras are internally calibrated ($\mathbf{K}_j$ known) then $\mathbf{H}$ is restricted to a similarity since it must preserve the calibrations $\mathbf{K}_j$ [H&Z, Secs. 10.2, 10.3], [Longuet-Higgins 1981]

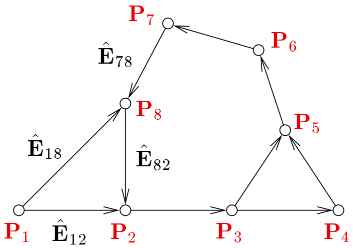
(translation, rotation, scale)

→134 for an indirect proof

▶Reconstructing Camera System from Pairs (Correspondence-Free)

Problem: Given a set of p decomposed pairwise essential matrices $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_\times \hat{\mathbf{R}}_{ij}$ and calibration matrices $\mathbf{K}_i$ reconstruct the camera system $\mathbf{P}_i$, $i = 1, \dots, k$

→81 and →151 on representing $\mathbf{E}$



We construct calibrated camera pairs $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$ see (17)

no hat

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1}\hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1}\hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons i, j correspond to graph nodes — k nodes
- pairs ij correspond to graph edges — p edges

$\hat{\mathbf{P}}_{ij}$ are in different coordinate systems but these are related by similarities $\hat{\mathbf{P}}_{ij}\mathbf{H}_{ij} = \mathbf{P}_{ij}$ — $\mathbf{H}_{ij} \in \mathrm{SIM}(3)$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\in \mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\top & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \overset{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\in \mathbb{R}^{6,4}} \tag{29}$$

- (29) is a system of $24p$ eqs. in $7p + 6k$ unknowns — $7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij})$, $6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each $\hat{\mathbf{P}}_i = (\mathbf{R}_i, \mathbf{t}_i)$ appears on the RHS as many times as is the degree of node $\mathbf{P}_i$ — eg. P_5 3×

Eq. (29) implies $\begin{bmatrix} \mathbf{R}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{R}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i \\ \mathbf{R}_j \end{bmatrix}$ and $\begin{bmatrix} \mathbf{t}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{t}_{ij} + s_{ij}\hat{\mathbf{t}}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{t}_j \end{bmatrix}$

- $\mathbf{R}_{ij}$ and $\mathbf{t}_{ij}$ can be eliminated:

$$\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j, \qquad \hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} = \mathbf{t}_j, \qquad s_{ij} > 0 \tag{30}$$

- note transformations that do not change these equations — assuming no error in $\hat{\mathbf{R}}_{ij}$

  1. $\mathbf{R}_i \mapsto \mathbf{R}_i\mathbf{R}$, 2. $\mathbf{t}_i \mapsto \sigma\,\mathbf{t}_i$ and $s_{ij} \mapsto \sigma s_{ij}$, 3. $\mathbf{t}_i \mapsto \mathbf{t}_i + \mathbf{R}_i\mathbf{t}$

- the global frame is fixed, e.g. by selecting

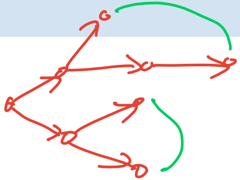
$$\mathbf{R}_1 = \mathbf{I}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \frac{1}{p}\sum_{i,j} s_{ij} = 1 \tag{31}$$

- rotation equations are decoupled from translation equations
- in principle, s_{ij} could correct the sign of $\hat{\mathbf{t}}_{ij}$ from essential matrix decomposition →81
  but $\mathbf{R}_i$ cannot correct the α sign in $\hat{\mathbf{R}}_{ij}$ ⇒ therefore make sure all points are in front of cameras and constrain $s_{ij} > 0$; →83

+ pairwise correspondences are sufficient

− suitable for well-distributed cameras only (dome-like configurations) otherwise intractable or numerically unstable

Finding The Rotation Component in Eq. (30)



1. **Poor Man's Algorithm:**
   a) create a Minimum Spanning Tree of $\mathcal{G}$ from →133
   b) propagate rotations from $\mathbf{R}_1 = \mathbf{I}$ via $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$ from (30)

2. **Rich Man's Algorithm:**
   Consider $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$, $(i,j) \in E(\mathcal{G})$, where $\mathbf{R}$ are a 3×3 rotation matrices
   Errors per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\mathbf{e}_{ij}^c = \hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c, \qquad \text{for all } i, j$$

   Solve

$$\arg\min \sum_{(i,j)\in E(\mathcal{G})} \sum_{c=1}^{3} (\mathbf{e}_{ij}^c)^\top \mathbf{e}_{ij}^c \quad \text{s.t.} \quad (\mathbf{r}_i^k)^\top (\mathbf{r}_j^l) = \begin{cases} 1 & i = j \wedge k = l \\ 0 & i \neq j \wedge k = l \\ 0 & i = j \wedge k \neq l \end{cases}$$

   this is a quadratic programming problem

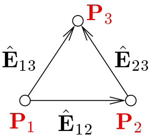
3. **SVD-Lover's Algorithm:**
   Ignore the constraints and project the solution onto rotation matrices see next

Per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \qquad \text{for all } i,\ j \tag{32}$$

- fix c and denote $\mathbf{r}^c = \left[\mathbf{r}_1^c, \mathbf{r}_2^c, \ldots, \mathbf{r}_k^c\right]^\top$ c-th columns of all rotation matrices stacked; $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (32) becomes $\mathbf{D}\,\mathbf{r}^c = \mathbf{0}$ — $\mathbf{D} \in \mathbb{R}^{3p,3k}$
- $3p$ equations for $3k$ unknowns $\to p \geq k$ — in a 1-connected graph we have to fix $\mathbf{r}_1^c = [1, 0, 0]$

Ex: $(k = p = 3)$



$\to$

$$\begin{aligned}\hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c &= \mathbf{0}\\ \hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c &= \mathbf{0}\\ \hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c &= \mathbf{0}\end{aligned}$$

$\to$

$$\mathbf{D}\,\mathbf{r}^c = \begin{bmatrix}\hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I}\\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{r}_1^c\\ \mathbf{r}_2^c\\ \mathbf{r}_3^c\end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} r^1 & r^2 & r^3 \\ r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \\ r_3^1 & r_3^2 & r_3^3 \end{bmatrix}$$

- must hold for any c

Idea: [Martinec & Pajdla CVPR 2007]

1. find the space of all $\mathbf{r}^c \in \mathbb{R}^{3k}$ that solve (32) — $\mathbf{D}$ is sparse, use `[V,E] = eigs(D'*D,3,0);` (Matlab)
2. choose 3 unit orthogonal vectors in this space — 3 smallest eigen~~vectors~~ values
3. find closest rotation matrices per cam. using SVD — because $\|\mathbf{r}^c\| = 1$ is necessary but insufficient; $\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$, where $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ ↑ up to det

- global world rotation is arbitrary

From (30) and (31):

$0 < d \le 3$ – rank of camera center set, p – #pairs, k – #cameras

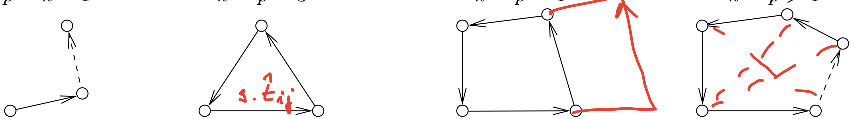
$$\hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \sum_{i,j} s_{ij} = p, \qquad s_{ij} > 0, \qquad \mathbf{t}_i \in \mathbb{R}^d$$

- in rank d: $d \cdot p + d + 1$ indep. eqns for $d \cdot k + p$ unknowns $\to p \ge \frac{d(k-1)-1}{d-1} \stackrel{\text{def}}{=} Q(d,k)$

Ex: Chains and circuits

construction of $\mathbf{t}_i$ from sticks of known orientation $\hat{\mathbf{t}}_{ij}$ and unknown length s_{ij}?

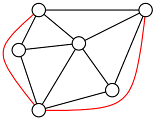
$p = k - 1$ | $k = p = 3$ | $k = p = 4$ | $k = p > 4$



$k \le 2$ for any d | $3 \ge d \ge 2$: non-collinear ok | $3 \ge d \ge 3$: non-planar ok | $3 \ge d \ge k - 1$: impossible

- equations insufficient for chains, trees, or when $d = 1$ — collinear cameras
- 3-connectivity implies sufficient equations for $d = 3$ — cams. in general pos. in 3D
  - s-connected graph has $p \ge \lceil \frac{sk}{2} \rceil$ edges for $s \ge 2$, hence $p \ge \lceil \frac{3k}{2} \rceil \ge Q(3,k) = \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any k when $d = 2$ — coplanar cams
  - since $p \ge \lceil 2k \rceil \ge Q(2,k) = 2k - 3$
  - maximal planar tringulated graphs have $p = 3k - 6$ and give a solution for $k \ge 3$

maximal planar triangulated graph example:



Linear equations in (30) and (31) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \qquad \mathbf{t} = \left[\mathbf{t}_1^\top, \mathbf{t}_2^\top, \ldots, \mathbf{t}_k^\top, s_{12}, \ldots, s_{ij}, \ldots\right]^\top$$

assuming measurement errors $\mathbf{D}\mathbf{t} = \epsilon$ and $d = 3$, we have

$$\mathbf{t} \in \mathbb{R}^{3k+p}, \quad \mathbf{D} \in \mathbb{R}^{3p,3k+p} \quad \text{sparse}$$

and

$$\mathbf{t}^* = \arg\min_{\mathbf{t},\, s_{ij}>0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D}\, \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

▶Bundle Adjustment

Goal: Use a good (and expensive) error model and improve the initial estimates of all parameters

Given:

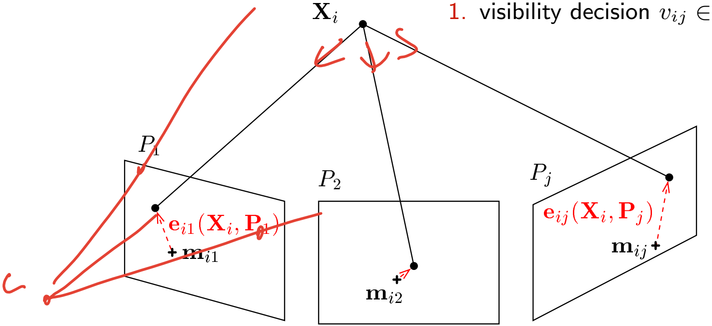
1. set of 3D points $\{\mathbf{X}_i\}_{i=1}^p$
2. set of cameras $\{\mathbf{P}_j\}_{j=1}^c$
3. fixed tentative projections $\mathbf{m}_{ij}$

Required:

1. corrected 3D points $\{\mathbf{X}'_i\}_{i=1}^p$
2. corrected cameras $\{\mathbf{P}'_j\}_{j=1}^c$

Latent:

1. visibility decision $v_{ij} \in \{0, 1\}$ per $\mathbf{m}_{ij}$



- for simplicity, $\mathbf{X}$, $\mathbf{m}$ are considered Cartesian (not homogeneous)
- we have projection error $\mathbf{e}_{ij}(\mathbf{X}_i, \mathbf{P}_j) = \mathbf{x}_i - \mathbf{m}_i$ per image feature, where $\underline{\mathbf{x}}_i = \mathbf{P}_j \underline{\mathbf{X}}_i$
- for simplicity, we will work with scalar error $e_{ij} = \|\mathbf{e}_{ij}\|$

Robust Objective Function for Bundle Adjustment

The data model is constructed by marginalization over v_{ij}, as in the Robust Matching Model →116

$$p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \prod_{\text{pts:}i=1}^{p} \prod_{\text{cams:}j=1}^{c} \Big((1-P_0)p_1(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j) + P_0\, p_0(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j)\Big)$$

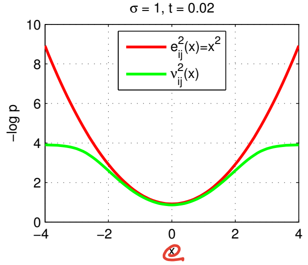
marginalized negative log-density is (→117)

$$-\log p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \sum_i \sum_j \underbrace{-\log\Big(e^{-\frac{e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)}{2\sigma_1^2}} + t\Big)}_{\rho(e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)) = \nu_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)} \stackrel{\text{def}}{=} \sum_i \sum_j \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)$$

- we can use LM, e_{ij} is the exact projection error function (not Sampson error)
- ν_{ij} is a 'robust' error fcn.; it is non-robust ($\nu_{ij} = e_{ij}$) when $t = 0$
- $\rho(\cdot)$ is a 'robustification function' often found in M-estimation
- the $\mathbf{L}_{ij}$ in Levenberg-Marquardt changes to vector

$$(\mathbf{L}_{ij})_l = \frac{\partial \nu_{ij}}{\partial \theta_l} = \underbrace{\frac{1}{1 + t\, e^{e_{ij}^2(\theta)/(2\sigma_1^2)}}}_{\text{small for } e_{ij} \gg \sigma_1} \cdot \frac{1}{\nu_{ij}(\theta)} \cdot \frac{1}{4\sigma_1^2} \cdot \frac{\partial e_{ij}^2(\theta)}{\partial \theta_l} \qquad (33)$$

but the LM method stays the same as before →108–109



- outliers (wrong v_{ij}): almost no impact on $\mathbf{d}_s$ in normal equations because the red term in (33) scales contributions to both sums down for the particular ij

$$-\sum_{i,j} \mathbf{L}_{ij}^\top \nu_{ij}(\theta^s) = \Big(\sum_{i,j}^{k} \mathbf{L}_{ij}^\top \mathbf{L}_{ij}\Big) \mathbf{d}_s$$

▶Sparsity in Bundle Adjustment

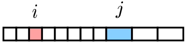
We have $q = 3p + 11k$ parameters: $\boldsymbol{\theta} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p; \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k)$ — points, cameras

We will use a multi-index $r = 1, \ldots, z$, $z = p \cdot k$. Then — r correspond to point-cam pairs (i, j)

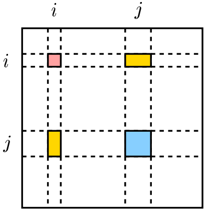
$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{r=1}^{z} \nu_r^2(\boldsymbol{\theta}), \qquad \boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s, \qquad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\boldsymbol{\theta}^s) = \left(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag}(\mathbf{L}_r^\top \mathbf{L}_r) \right) \mathbf{d}_s$$

The block-form of $\mathbf{L}_r$ in Levenberg-Marquardt (→108) is zero except in columns i and j:

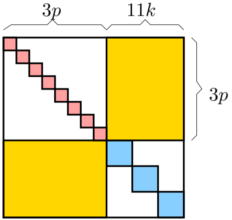
r-th error term is $\nu_r^2 = \rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j))$

$\mathbf{L}_r =$ 

$r = (i, j)$ blocks:
- ■: $\mathbf{X}_i$, 1×3
- ■: $\mathbf{P}_j$, 1×11

$\mathbf{L}_r^\top \mathbf{L}_r =$ 

blocks:
- ■: $\mathbf{X}_i - \mathbf{X}_i$, 3×3
- ■: $\mathbf{X}_i - \mathbf{P}_j$, 3×11
- ■: $\mathbf{P}_j - \mathbf{P}_j$, 11×11

$\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r =$ 

- "points-first-then-cameras" parameterization scheme

▶Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{x} \text{ such that} \quad \mathbf{b} \stackrel{\text{def}}{=} -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \Big(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag}\big(\mathbf{L}_r^\top \mathbf{L}_r\big)\Big)\mathbf{x} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{x}$$

- $\mathbf{A}$ is very large — approx. $3 \cdot 10^4 \times 3 \cdot 10^4$ for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$ is sparse and symmetric, $\mathbf{A}^{-1}$ is dense — direct matrix inversion is prohibitive

> Choleski: symmetric positive definite matrix $\mathbf{A}$ can be decomposed to $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is lower triangular. If $\mathbf{A}$ is sparse then $\mathbf{L}$ is sparse, too.

1. decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ — `L = chol(A);` transforms the problem to $\mathbf{L}\underbrace{\mathbf{L}^\top\mathbf{x}}_{\mathbf{c}} = \mathbf{b}$
2. solve for $\mathbf{x}$ in two passes:

$$\mathbf{L}\,\mathbf{c} = \mathbf{b} \qquad \mathbf{c}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{b}_i - \sum_{j<i} \mathbf{L}_{ij}\mathbf{c}_j\Big) \qquad \text{forward substitution, } i = 1, \dots, q \text{ (params)}$$

$$\mathbf{L}^\top\mathbf{x} = \mathbf{c} \qquad \mathbf{x}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{c}_i - \sum_{j>i} \mathbf{L}_{ji}\mathbf{x}_j\Big) \qquad \text{back-substitution}$$

- Choleski decomposition is fast (does not touch zero blocks)
  non-zero elements are $9p + 121k + 66pk \approx 3.4 \cdot 10^6$; ca. $250\times$ fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse $\mathbf{A}$ and diagonal pivoting for semi-definite $\mathbf{A}$ — see above; [Triggs et al. 1999]
- λ controls the definiteness

Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%    L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%    for sparse square symmetric positive definite matrix A,
%    especially efficient for arrowhead sparse matrices.

% (c) 2010 Radim Sara (sara@cmp.felk.cvut.cz)

 [p,q] = size(A);
 if p ~= q, error 'Matrix A is not square'; end

 L = sparse(q,q);
 F = ones(q,1);
 for i=1:q
  F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
  for j = F(i):i-1
   k = max(F(i),F(j));
   a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
   L(i,j) = a/L(j,j);
  end
  a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
  if a < 0, error 'Matrix A is not positive definite'; end
  L(i,i) = sqrt(a);
 end
end
```

▶Gauge Freedom (kalibrační invariance)

1. The external frame is not fixed: See Projective Reconstruction Theorem →132

$$\underline{\mathbf{m}}_{ij} \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.

- $\mathbf{P}$ is 12 numbers for 11 parameters
- we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
- but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints
2. fixing the scale (e.g. $s_{12} = 1$)

3a. either imposing constraints on projective entities
- cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
- points, e.g. $(\underline{\mathbf{X}}_i)_4 = 1$ or $\|\underline{\mathbf{X}}_i\|^2 = 1$ — the 2nd: can represent points at infinity

3b. or using minimal representations
- points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
- rotation matrices can be represented by skew-symmetric matrices →149

Thank You