3D Computer Vision

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Open Informatics Master's Course

▸Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. Consider the following spatial line (world frame)

   $\mathbf{d} \in \mathbb{R}^3$ line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\,\mathbf{d}$$

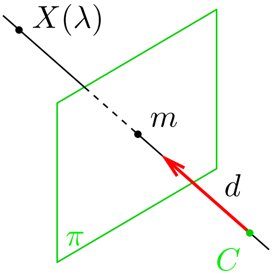
2. The projection of the (finite) point $X(\lambda)$ is

$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{X}(\lambda)\\ 1\end{bmatrix} = \mathbf{Q}(\cancel{\mathbf{C}} + \lambda\mathbf{d}) + \cancel{\mathbf{q}} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

QC = -q

$$= \lambda\,\underbrace{\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}}_{P}\underset{X}{\begin{bmatrix}\mathbf{d}\\ 0\end{bmatrix}}$$

... which is also the image of a point at infinity in $\mathbb{P}^3$



- optical ray line corresponding to image point m is the set

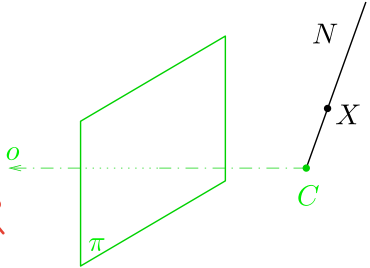
$$\mathbf{X}(\mu) = \mathbf{C} + \mu\,\mathbf{Q}^{-1}\underline{\mathbf{m}}, \qquad \mu \in \mathbb{R} \qquad (\mu = 1/\lambda)$$

- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in world coordinate frame

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π

1. points X on a given line N parallel to π project to a point at infinity $(u, v, 0)$ in π:
$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \overset{\lambda \in \mathbb{R}}{\simeq} \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$
2. therefore the set of points X is parallel to π iff
$$0 = \mathbf{q}_3^\top \mathbf{X} + q_{34} \cancel{\neq 0} \qquad q_3 \in \mathbb{R}^3,\ q_{34} \in \mathbb{R}$$
3. this is a plane equation with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)
$$\boxed{\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3}$$
if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]



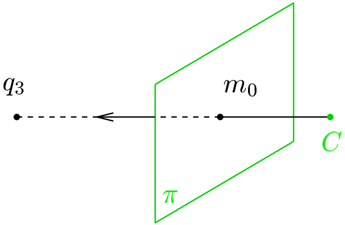
- the axis is expressed in world coordinate frame

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to the principal point m_0
3. then

$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

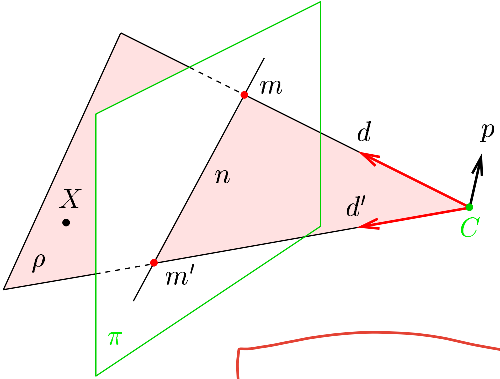


principal point: $\boxed{\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3}$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p containing the projection center C and a given image line n.



optical ray given by m $\quad \mathbf{d} \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$((Q^{-1})^{-1})^\top = Q^\top$

$$\mathbf{p} \simeq \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^\top(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^\top\underline{\mathbf{n}}$$

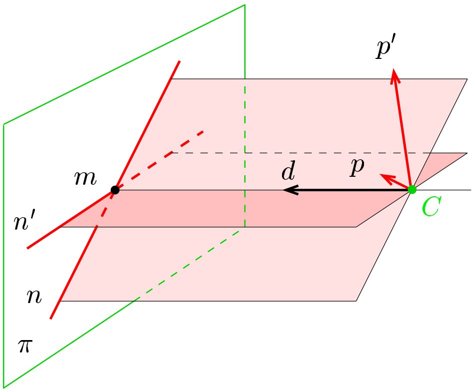
- note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^\top(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top \underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\to 30} = \underline{\mathbf{n}}^\top\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^\top\underline{\mathbf{n}})^\top\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \boxed{\underline{\boldsymbol{\rho}} \simeq \mathbf{P}^\top\underline{\mathbf{n}}}$

$\rho_1\,x + \rho_2\,y + \rho_3\,z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\mathbf{m}}$$

▶Summary: Projection Center; Optical Ray, Axis, Plane

General (finite) camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P}), \quad \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ projection center (world coords.) →35

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ optical ray direction (world coords.) →36

$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$ outward optical axis (world coords.) →37

$\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$ principal point (in image plane) →38

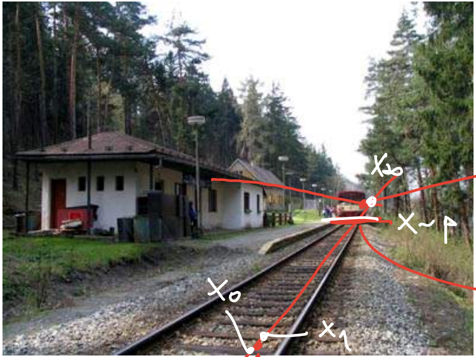
$\underline{\boldsymbol{\rho}} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ optical plane (world coords.) →39

$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0 \\ 0 & f/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$ camera (calibration) matrix (f, u_0, v_0 in pixels) →31

$\mathbf{R}$ camera rotation matrix (cam coords.) →30

$\mathbf{t}$ camera translation vector (cam coords.) →30

What Can We Do with An 'Uncalibrated' Perspective Camera?



P_∞
vanishing point
úběžník

P

How far is the engine from a given point on the tracks?

X_0

distance between sleepers (ties) 0.806m but we cannot count them, the image resolution is too low

We will review some life-saving theory. . .
. . . and build a bit of geometric intuition. . .

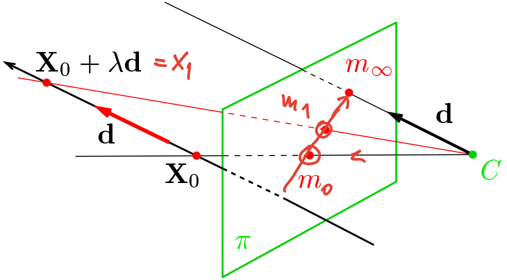
In fact

- 'uncalibrated' = the image contains a 'calibrating object' that suffices for the task at hand

▶Vanishing Point

Vanishing point (V.P.): The limit m_∞ of the projection of a point $\mathbf{X}(\lambda)$ that moves along a space line $\mathbf{X}(\lambda) = \mathbf{X}_0 + \lambda\mathbf{d}$ infinitely in one direction.

the image of the point at infinity on the line



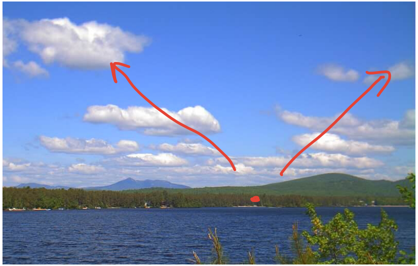
$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda\to\pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots \simeq \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

- the V.P. of a spatial line with directional vector $\mathbf{d}$ is $\underline{\mathbf{m}}_\infty \simeq \mathbf{Q}\,\mathbf{d}$
- V.P. is independent on line position $\mathbf{X}_0$, it depends on its directional vector only
- all parallel (world) lines share the same (image) V.P., including the optical ray defined by m_∞

Some Vanishing Point "Applications"

where is the sun?



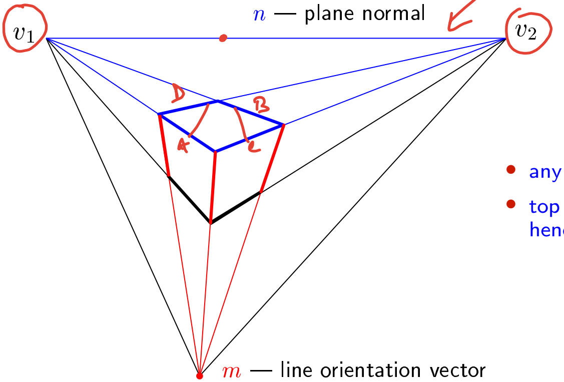
what is the wind direction?
(must have video)

fly above the lane,
at constant altitude!

►Vanishing Line

Vanishing line (V.L.): The set of vanishing points of all lines in a plane the image of the line at infinity in the plane and in all parallel planes (!)

horizon = vanishing l.



- any box with parallel edges
- top (blue) and bottom (black) box planes are parallel, hence they share V.L. n

- V.L. n corresponds to spatial plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

  because this is the normal vector of a parallel optical plane (!) →39

- a spatial plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top}\mathbf{p}$.

▶Cross Ratio

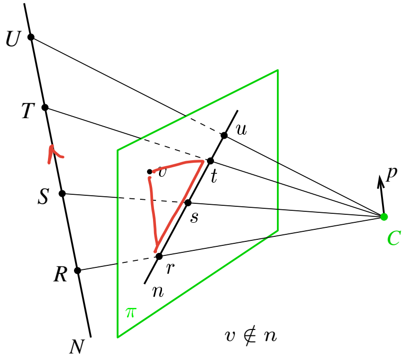
$|\cdot|$ - signed distance

Four distinct collinear spatial points R, S, T, U define cross-ratio

$$\boxed{[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{SR}|}\frac{|\overrightarrow{US}|}{|\overrightarrow{TU}|}}$$

$R \quad S \quad T \quad U$

a mnemonic (∞)



- $|\overrightarrow{RT}|$ – signed distance from R to T in the arrow direction
- each point X is once in numerator and once in denominator
- if X is 1st in a numerator term, it is 2nd in a denominator term
- there are six cross-ratios from four points:

$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU],\ \cdots$$

Obs: $$[RSTU] = \frac{|\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}}\ \underline{\mathbf{r}}\ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{u}}\ \underline{\mathbf{s}}\ \underline{\mathbf{v}}|}{|\underline{\mathbf{t}}\ \underline{\mathbf{u}}\ \underline{\mathbf{v}}|}, \qquad |\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}| = \det[\underline{\mathbf{r}}\ \underline{\mathbf{t}}\ \underline{\mathbf{v}}] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \quad \text{mixed product} \tag{1}$$

Corollaries:

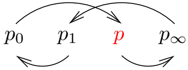
1. • cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ — proof: plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
2. • cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will "see" the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points R, S, T, U may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

▶1D Projective Coordinates

The 1-D projective coordinate of a point P is defined by the following cross-ratio:

$$[P] \overset{\text{def}}{=} [P_0\, P_1\, P\, P_\infty] = [p_0\, p_1\, p\, p_\infty] = \frac{|\overrightarrow{p_0\, p}|}{|\overrightarrow{p_1\, p_0}|}\,\frac{|\overrightarrow{p_\infty\, p_1}|}{|\overrightarrow{p\, p_\infty}|} = [p]$$



naming convention:

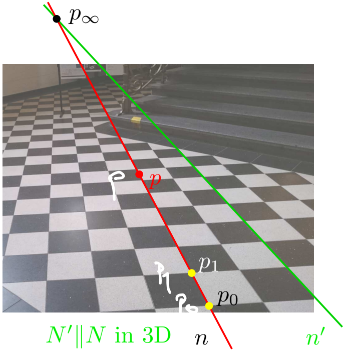
P_0 – the origin	$[P_0] = 0$
P_1 – the unit point	$[P_1] = 1$
P_∞ – the supporting point	$[P_\infty] = \pm\infty$

$$[P] = [p]$$

$[P]$ is equal to Euclidean coordinate along N

$[p]$ is its measurement in the image plane

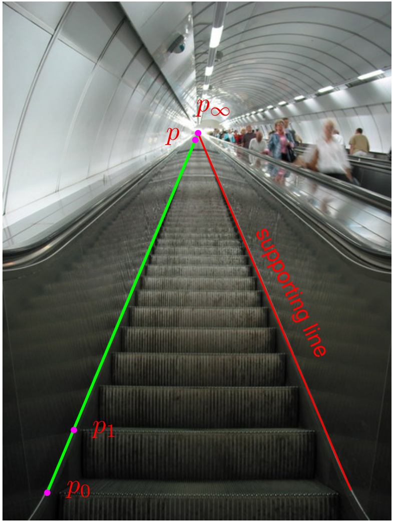
if the sign is not of interest, any cross-ratio containing $|p_0\, p|$ does the job



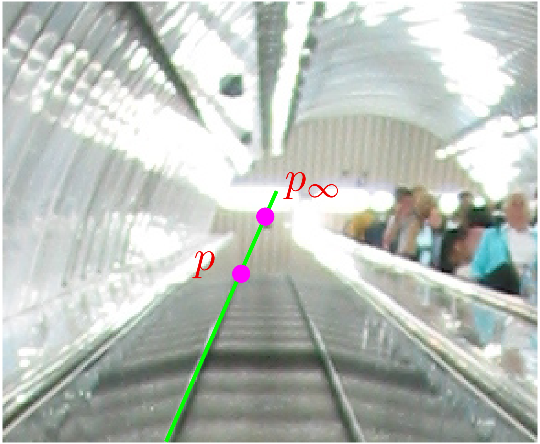
Applications

- Given the image of a 3D line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined →48
- Finding V.P. of a line through a regular object →49

Application: Counting Steps



- Namesti Miru underground station in Prague

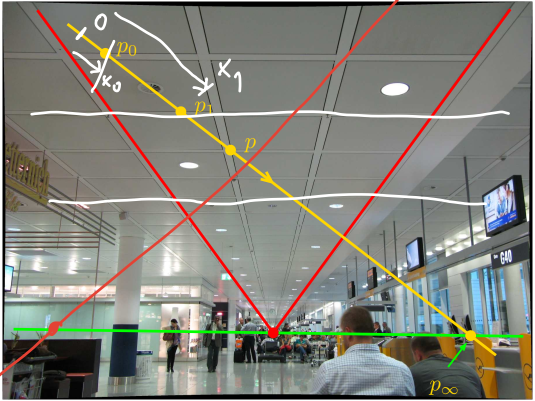


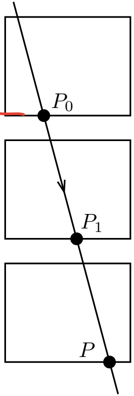
detail around the vanishing point

Result: $[P] = 214$ steps (correct answer is 216 steps)

4Mpx camera

Application: Finding the Horizon from Repetitions





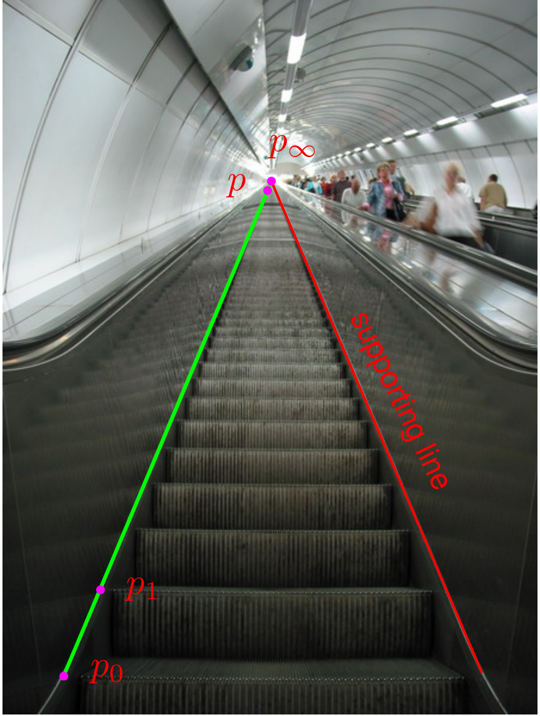
in 3D: $|P_0P| = 2|P_0P_1|$ then [H&Z, p. 218]

$$[P_0P_1PP_\infty] = \frac{|P_0P|}{|P_1P_0|} = 2 \quad\Rightarrow\quad x_\infty = \frac{x_0\,(2\,x - x_1) - x\,x_1}{x + x_0 - 2\,x_1}$$

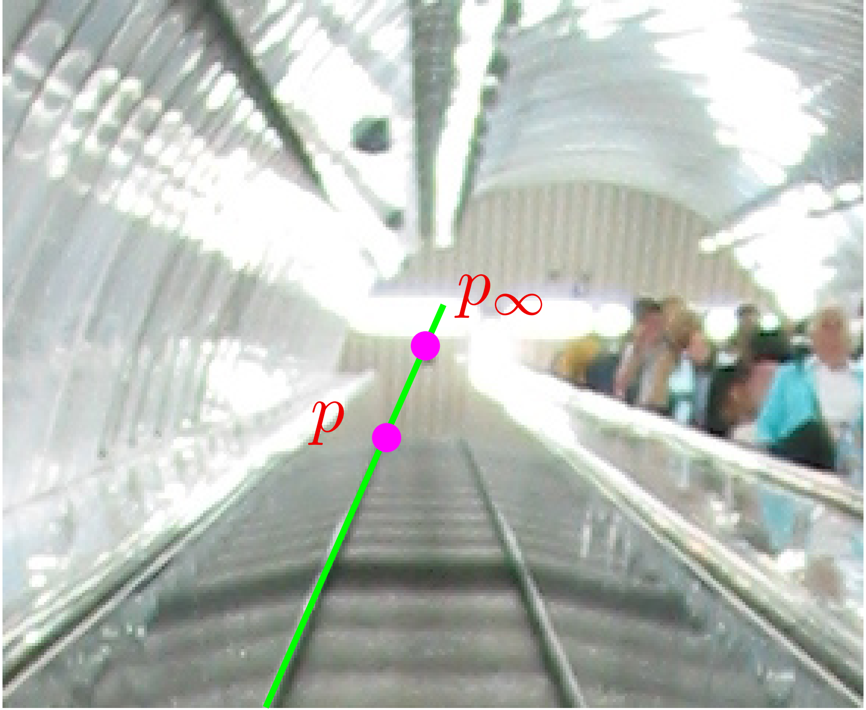
- x – 1D coordinate along the yellow line, positive in the arrow direction
- could be applied to counting steps (→48) if there was no supporting line

⊛ P1; 1pt: How high is the camera above the floor?

Thank You



p_∞
p
supporting line
p_1
p_0



p_∞
p