3D Computer Vision

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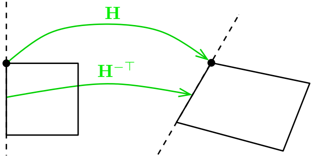
Open Informatics Master's Course

▶Mapping 2D Points and Lines by Homography



$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}}$	(image) point	
$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}}$	(image) line	$\mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top}\underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top}\mathbf{H}^{\top}\mathbf{H}^{-\top}\underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top}\underline{\mathbf{n}} = 0$

Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$

1. extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
2. map by homography, $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
3. if $m'_3 \neq 0$ convert the result $\underline{\mathbf{m}}' = (m'_1, m'_2, m'_3)$ back to Cartesian coordinates (pixels),

$$u' = \frac{m'_1}{m'_3}\,\mathbf{1}, \qquad v' = \frac{m'_2}{m'_3}\,\mathbf{1}$$

- note that, typically, $m'_3 \neq 1$

$m'_3 = 1$ when $\mathbf{H}$ is affine

- an infinite point $\underline{\mathbf{m}} = (u, v, 0)$ maps the same way

Some Homographic Tasters

Rectification of camera rotation: →59 (geometry), →129 (homography estimation)

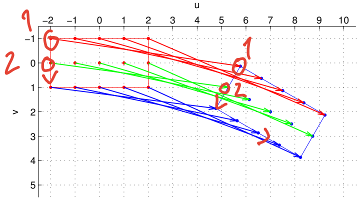
$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^{\top}\mathbf{K}^{-1}$ maps from image plane to facade plane

Homographic Mouse for Visual Odometry: [Mallis 2007]

illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^{\top}}{d}\right)\mathbf{K}^{-1}$ maps from plane to translated plane [H&Z, p. 327]

▶Homography Subgroups: Euclidean Mapping (aka Rigid Motion)



rotation by 30°, then translation by $(7,\ 2)$

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix} = \underline{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}} \in \mathrm{SE}(2)$$

$H\,\underline{x}$

- note: action $H(\mathbf{x}) = \underline{\mathbf{R}}\mathbf{x} + \mathbf{t} \colon \mathbb{R}^2 \to \mathbb{R}^2$, not commutative

EM = The most general homography preserving

1. **lengths**: Let $\mathbf{x}_i' = H(\mathbf{x}_i)$. Then

$$\|\mathbf{x}_2' - \mathbf{x}_1'\| = \|H(\mathbf{x}_2) - H(\mathbf{x}_1)\| \overset{\circledast\ \text{P1; 1pt}}{=} \cdots = \|\mathbf{x}_2 - \mathbf{x}_1\|$$

2. **angles** R — check the dot-product of normalized differences from a point $(\mathbf{x} - \mathbf{z})^\top (\mathbf{y} - \mathbf{z})$ (Cartesian(!))
3. **areas**: $\det \cancel{\mathbf{H}} = 1 \;\Rightarrow\;$ unit Jacobian; follows from 1. and 2.

- eigenvalues $(1,\ e^{-i\phi},\ e^{i\phi})$
- eigenvectors when $\phi \neq k\pi,\ k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

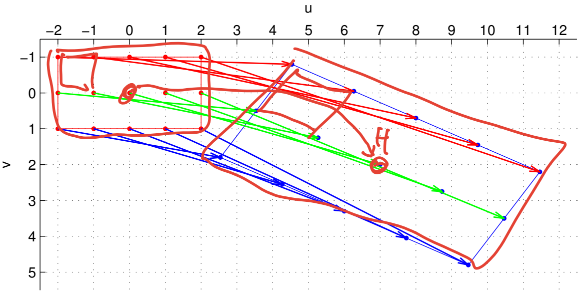
$\mathbf{e}_2, \mathbf{e}_3$ – circular points, i – imaginary unit

4. **circular points**: complex points at infinity $(i, 1, 0)$, $(-i, 1, 0)$ (preserved even by similarity)

- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

▶Homography Subgroups: Affine Mapping (Affinity)

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by 30°
then scaling by $\mathrm{diag}(1,\ 1.5,\ 1)$
then translation by $(7,\ 2)$

Affinity = The most general homography preserving

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints, centers of gravity)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

observe $\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\mathbf{n}}_\infty \quad \Rightarrow \quad \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$

does not preserve

- lengths
- angles
- areas
- circular points

▸Homography Subgroups: General Homography

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \qquad \mathbf{H} \in \mathrm{SL}(3)$$

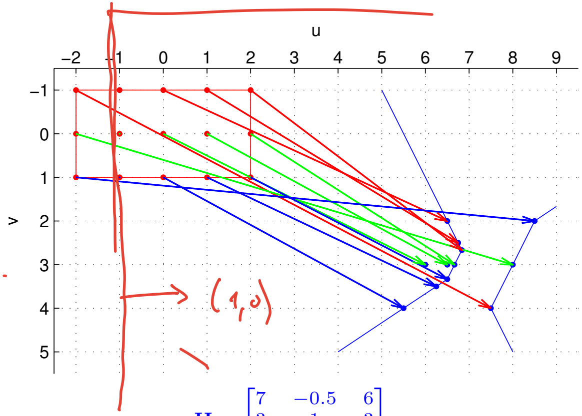
$\det \mathbf{H} \neq 0$

preserves only

- incidence and concurrency
- collinearity
- cross-ratio (ratio of ratios) on the line →46

does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$



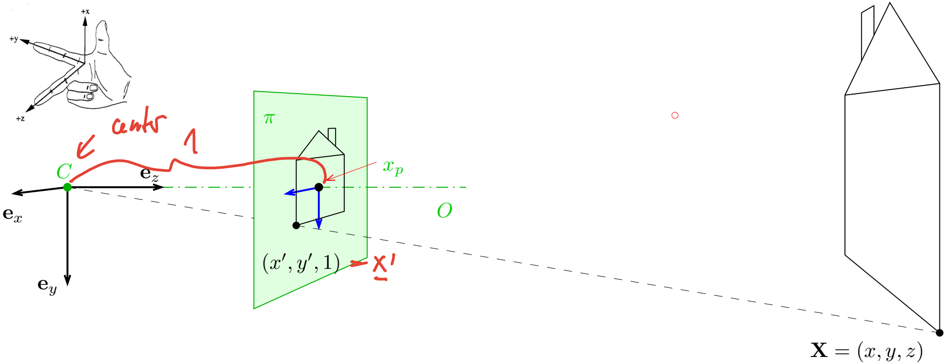
$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

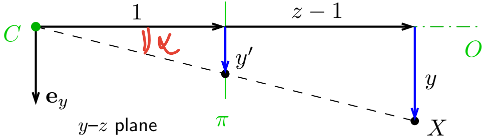
(where in the picture is the line n?)

$ax + by + c = 0$

▶Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking ‘down the street’
2. right-handed canonical coordinate system (x, y, z) with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$
3. origin = center of projection C
4. image plane π at unit distance from C
5. optical axis O is perpendicular to π
6. principal point x_p: intersection of O and π
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

$$\tan\alpha = \frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}, \qquad x' = \underline{\frac{x}{z}}$$

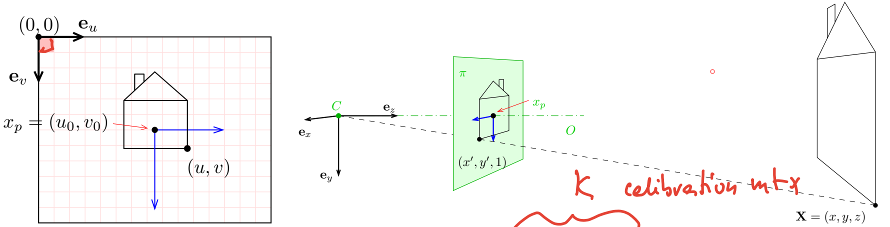
▸Natural and Canonical Image Coordinate Systems

projected point **in canonical camera** $(z \neq 0)$

$$\underbrace{(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right)}_{} = \underbrace{\frac{1}{z}}_{\lambda \neq 0}(x, y, z) \simeq (x, y, z) \equiv \underline{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}} \cdot \underline{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}} = \underline{\mathbf{P}_0\, \underline{\mathbf{X}}} = \underline{m}$$

projected point **in scanned image**

scale by f and translate origin to image corner



projection mtx

$$\boxed{\begin{aligned} u &= f\,\frac{x}{z} + u_0 \\ v &= f\,\frac{y}{z} + v_0 \end{aligned}}$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \overbrace{\begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{K}\ \text{calibration mtx}} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \underbrace{\mathbf{K}\mathbf{P}_0}_{}\,\underline{\mathbf{X}} = \underline{\mathbf{P}}\,\underline{\mathbf{X}}$$

- ‘calibration’ matrix $\mathbf{K}$ transforms canonical $\mathbf{P}_0$ to standard perspective camera $\mathbf{P}$

focal length [px]

►Computing with Perspective Camera Projection Matrix

Projection from world to image in standard camera $\mathbf{P}$:

$$\underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx + u_0 z \\ fy + v_0 z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}}_{\text{(a)}} \simeq \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underline{\mathbf{m}}$$

cross-check: $\dfrac{m_1}{m_3} = \dfrac{f\,x}{z} + u_0 = u, \qquad \dfrac{m_2}{m_3} = \dfrac{f\,y}{z} + v_0 = v$ when $m_3 \neq 0$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

(u_0, v_0) – principal point in pixels

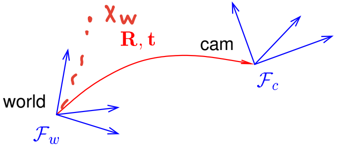
Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)

   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π — i.e. points with $z = 0$

▶Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:



$$\mathbf{X}_c = \underbrace{\mathbf{R}\,\mathbf{X}_w + \mathbf{t}}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame $\mathcal{F}_c$

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame $\mathcal{F}_c$

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0\begin{bmatrix}\mathbf{X}_c\\ 1\end{bmatrix} = \mathbf{K}\mathbf{P}_0\begin{bmatrix}\mathbf{R}\mathbf{X}_w + \mathbf{t}\\ 1\end{bmatrix} = \mathbf{K}\underbrace{\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}}_{\mathbf{P}_0}\underbrace{\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ \mathbf{0}^\top & 1\end{bmatrix}}_{\mathbf{T}\ \in SE(3)}\begin{bmatrix}\mathbf{X}_w\\ 1\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ (a 3×4 mtx) discards the last row of $\mathbf{T}$

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\boxed{\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\mathbf{C}$ – camera position in the world reference frame $\mathcal{F}_w$ — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

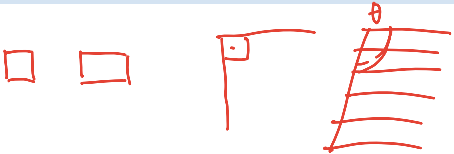
$\mathbf{r}_3^\top$ – optical axis in the world reference frame $\mathcal{F}_w$ — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}\underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

▶Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- skew angle θ of the digitization raster
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0 \\ 0 & f/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

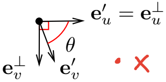
units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Give the parameters f, a, θ, u_0, v_0 a precise meaning by decomposing $\mathbf{K}$ to simple maps; deadline LD+2 wk

Hints:

1. image projects to orthogonal system $F^{\perp}$, then it maps by skew to F', then by scale $a\,f$, f to F'', then by translation by u_0, v_0 to F'''
2. Skew: Do not confuse it with the shear mapping. Express point $\mathbf{x}$ as

$$\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u^{\perp}\mathbf{e}_u^{\perp} + v^{\perp}\mathbf{e}_v^{\perp}$$



$\mathbf{e}_{:}$ are unit-length basis vectors; consider their four pairwise dot-products.

3. $\mathbf{K}$ maps from $F^{\perp}$ to F''' as

$$w'''\,[u''', v''', 1]^{\top} = \mathbf{K}[u^{\perp}, v^{\perp}, 1]^{\top}$$

▶Summary: Projection Matrix of a General Finite Perspective Camera

$$\underline{\mathbf{m}} \underset{\simeq}{\simeq} \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \underbrace{\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}} \simeq \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

$xP \simeq P$

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

std. $\frac{1}{P_{34}} \cdot P$

finite camera: $\det \mathbf{K} \neq 0$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite perspective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left 3×3 submatrix $\mathbf{Q}$ non-singular.

random finite camera: `Q = rand(3,3); while det(Q)==0, Q = rand(3,3); end, P = [Q, rand(3,1)];`

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$	full rank	(if finite perspective camera; see [H&Z, Sec. 6.3] for cameras at infinity)
$\mathbf{K} \in \mathbb{R}^{3,3}$	upper triangular with positive diagonal elements	
$\mathbf{R} \in \mathbb{R}^{3,3}$	rotation mtx: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$	

P=

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \begin{bmatrix}\mathbf{KR} & \mathbf{Kt}\end{bmatrix}$ also →35
2. RQ decomposition of $\mathbf{Q} = \mathbf{KR}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \underline{\mathbf{Q}}\,\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}} \qquad \mathbf{QR}_{32} = \begin{bmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & 0 & \cdot\end{bmatrix},\ (\mathbf{QR}_{32})\mathbf{R}_{31} = \begin{bmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ 0 & 0 & \cdot\end{bmatrix},\ \mathbf{QR}_{32}\mathbf{R}_{31}\mathbf{R}_{21} = \begin{bmatrix}\cdot & \cdot & \cdot\\ 0 & \cdot & \cdot\\ 0 & 0 & \cdot\end{bmatrix}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g.

(see the next slide for derivation details)

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix} \text{ gives } \begin{matrix} c^2 + s^2 = 1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33}\end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{KR} = \mathbf{KT}^{-1}\mathbf{TR}$, where $\mathbf{T} = \operatorname{diag}(-1,-1,1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive 'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array [q#1,#2 &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q . R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}}\right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\, q_{3,2} + q_{1,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2}\, q_{3,2} + q_{1,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\, q_{3,2} + q_{2,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2}\, q_{3,2} + q_{2,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

►Center of Projection (Optical Center)

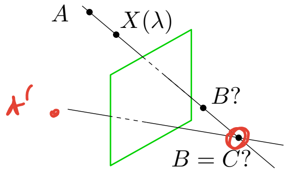
Observation: finite $\mathbf{P}$ has a non-trivial right null-space rank 3 but 4 columns

Theorem

Let $\mathbf{P}$ be a camera and let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\boxed{\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}}$. Then $\underline{\mathbf{B}}$ is equivalent to the projection center $\underline{\mathbf{C}}$ (homogeneous, in world coordinate frame).

Proof.

1. Let AB be a spatial line (B given from $\mathbf{P}\underline{\mathbf{B}} = \mathbf{0}$, $A \neq B$). Then



$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R} \qquad \text{(world frame)}$$

2. It projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + (1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}} \simeq \mathbf{P}\,\underline{\mathbf{A}}$$

- the entire line projects to a single point ⇒ it must pass through the projection center of $\mathbf{P}$
- this holds for any choice of $A \neq B$ ⇒ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▸Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. Consider the following spatial line (world frame)

   $\mathbf{d} \in \mathbb{R}^3$ line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

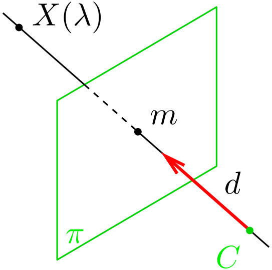
$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\,\mathbf{d}$$

2. The projection of the (finite) point $X(\lambda)$ is

$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X}(\lambda) \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$

…which is also the image of a point at infinity in $\mathbb{P}^3$

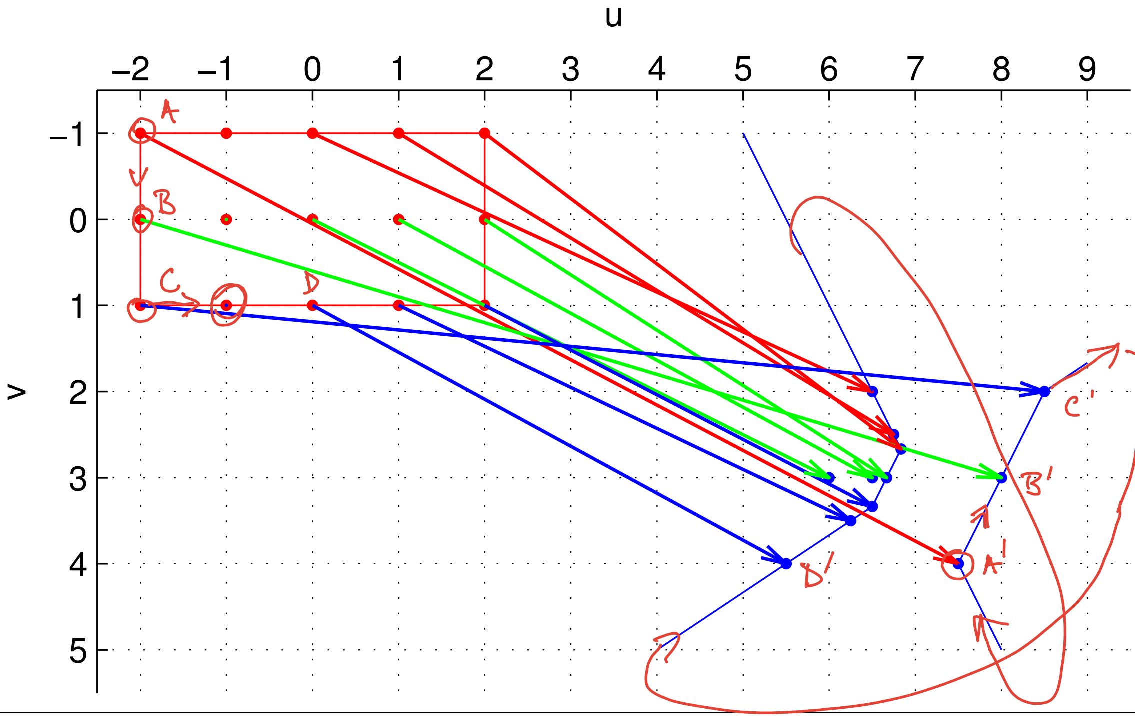


- optical ray line corresponding to image point m is the set

$$\mathbf{X}(\mu) = \mathbf{C} + \mu\,\mathbf{Q}^{-1}\underline{\mathbf{m}}, \qquad \mu \in \mathbb{R} \qquad (\mu = 1/\lambda)$$

- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in world coordinate frame

Thank You


u
−2
−1
0
1
2
3
4
5
6
7
8
9
v
−1
0
1
2
3
4
5
A
B
C
D
A'
B'
C'
D'