3D Computer Vision

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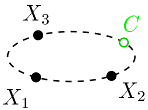
Open Informatics Master's Course

Degenerate (Critical) Configurations for Exterior Orientation



no solution

1. C cocyclic with (X_1, X_2, X_3) camera sees points on a line

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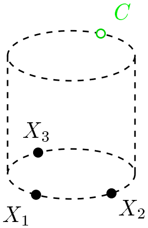

unstable solution

- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C

can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3) camera sees points on a line

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- additional critical configurations depend on the quadratic equations solver [Haralick et al. IJCV 1994]

▶Populating A Little ZOO of Minimal Geometric Problems in CV

problem	**given**	**unknown**	slide
camera resection	6 world–image correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{6}$	$\mathbf{P}$	→62
exterior orientation	$\mathbf{K}$, 3 world–image correspondences $\left\{(X_i,\, m_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{C}$	→66
relative orientation	3 world-world correspondences $\left\{(X_i,\, Y_i)\right\}_{i=1}^{3}$	$\mathbf{R}$, $\mathbf{t}$	→70

- camera resection and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- relative orientation involves no camera (see next)

  it is a recurring problem in 3D vision
- more problems to come

▶The Relative Orientation Problem

Problem: Given point triples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) in a general position in $\mathbf{R}^3$ such that the correspondence $X_i \leftrightarrow Y_i$ is known, determine the relative orientation $(\mathbf{R}, \mathbf{t})$ that maps $\mathbf{X}_i$ to $\mathbf{Y}_i$, i.e.

$$\mathbf{Y}_i = \mathbf{R}\mathbf{X}_i + \mathbf{t}, \quad i = 1, 2, 3\,.$$

Applies to:

- 3D scanners
- merging partial reconstructions from different viewpoints
- generalization of the last step of P3P

Obs: Let the centroid be $\bar{\mathbf{X}} = \frac{1}{3}\sum_i \mathbf{X}_i$ and analogically for $\bar{\mathbf{Y}}$. Then

$$\bar{\mathbf{Y}} = \mathbf{R}\bar{\mathbf{X}} + \mathbf{t}.$$

Therefore

$$\mathbf{Z}_i \stackrel{\text{def}}{=} (\mathbf{Y}_i - \bar{\mathbf{Y}}) = \mathbf{R}(\mathbf{X}_i - \bar{\mathbf{X}}) \stackrel{\text{def}}{=} \mathbf{R}\mathbf{W}_i$$

If all dot products are equal, $\mathbf{Z}_i^\top \mathbf{Z}_j = \mathbf{W}_i^\top \mathbf{W}_j$ for $i, j = 1, 2, 3$, we have

$$\mathbf{R}^* = \begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2 & \mathbf{W}_3\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2 & \mathbf{Z}_3\end{bmatrix}$$

Poor man's solver:

- normalize $\mathbf{W}_i$, $\mathbf{Z}_i$ to unit length, use the above formula, and then find the closest rotation matrix
- but this is equivalent to a non-optimal objective — it ignores errors in vector lengths

An Optimal Algorithm for Relative Orientation

We setup a minimization problem

$$\mathbf{R}^* = \arg\min_{\mathbf{R}} \sum_{i=1}^{3} \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 \quad \text{s.t.} \quad \mathbf{R}^\top\mathbf{R} = \mathbf{I}, \quad \det \mathbf{R} = 1$$

$$\arg\min_{\mathbf{R}} \sum_i \|\mathbf{Z}_i - \mathbf{R}\mathbf{W}_i\|^2 = \arg\min_{\mathbf{R}} \sum_i \left(\|\mathbf{Z}_i\|^2 - 2\mathbf{Z}_i^\top\mathbf{R}\mathbf{W}_i + \|\mathbf{W}_i\|^2\right) = \cdots = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top\mathbf{R}\mathbf{W}_i$$

Obs 1: Let $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$ be the dot-product (Frobenius inner product) over real matrices. Then

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = \operatorname{tr}(\mathbf{A}^\top\mathbf{B}) = \operatorname{vec}(\mathbf{A})^\top \operatorname{vec}(\mathbf{B}) = \mathbf{a} \cdot \mathbf{b}$$

Obs 2: (cyclic property for matrix trace)

$$\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{CAB})$$

Obs 3: ($\mathbf{Z}_i$, $\mathbf{W}_i$ are vectors)

$$\mathbf{Z}_i^\top\mathbf{R}\mathbf{W}_i = \operatorname{tr}(\mathbf{Z}_i^\top\mathbf{R}\mathbf{W}_i) \overset{\text{O2}}{=} \operatorname{tr}(\mathbf{W}_i\mathbf{Z}_i^\top\mathbf{R}) \overset{\text{O1}}{=} (\mathbf{Z}_i\mathbf{W}_i^\top) : \mathbf{R} = \mathbf{R} : (\mathbf{Z}_i\mathbf{W}_i^\top)$$

Let there be SVD of

$$\sum_i \mathbf{Z}_i\mathbf{W}_i^\top \overset{\text{def}}{=} \mathbf{M} = \mathbf{UDV}^\top$$

Then

$$\mathbf{R} : \mathbf{M} = \mathbf{R} : (\mathbf{UDV}^\top) \overset{\text{O1}}{=} \operatorname{tr}(\mathbf{R}^\top\mathbf{UDV}^\top) \overset{\text{O2}}{=} \operatorname{tr}(\mathbf{V}^\top\mathbf{R}^\top\mathbf{UD}) \overset{\text{O1}}{=} (\mathbf{U}^\top\mathbf{R}\mathbf{V}) : \mathbf{D}$$

cont'd: The Algorithm

We are solving

$$\mathbf{R}^* = \arg\max_{\mathbf{R}} \sum_i \mathbf{Z}_i^\top \mathbf{R} \mathbf{W}_i = \arg\max_{\mathbf{R}} \left(\mathbf{U}^\top \mathbf{R} \mathbf{V}\right) : \mathbf{D}$$

A particular solution is found as follows:

- $\mathbf{U}^\top \mathbf{R} \mathbf{V}$ must be (1) orthogonal, and closest to: (2) diagonal and (3) positive definite $\mathbf{D}$
- Since $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices then the solution to the problem is among $\mathbf{R}^* = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, where $\mathbf{S}$ is diagonal and orthogonal, i.e. one of

$$\pm\,\mathrm{diag}(1,1,1), \quad \pm\,\mathrm{diag}(1,-1,-1), \quad \pm\,\mathrm{diag}(-1,1,-1), \quad \pm\,\mathrm{diag}(-1,-1,1)$$

- $\mathbf{U}^\top \mathbf{V}$ is not necessarily positive definite
- We choose $\mathbf{S}$ so that $(\mathbf{R}^*)^\top \mathbf{R}^* = \mathbf{I}$

Alg:

1. Compute matrix $\mathbf{M} = \sum_i \mathbf{Z}_i \mathbf{W}_i^\top$.
2. Compute SVD $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$.
3. Compute all $\mathbf{R}_k = \mathbf{U}\mathbf{S}_k\mathbf{V}^\top$ that give $\mathbf{R}_k^\top \mathbf{R}_k = \mathbf{I}$.
4. Compute $\mathbf{t}_k = \bar{\mathbf{Y}} - \mathbf{R}_k \bar{\mathbf{X}}$.

- The algorithm can be used for more than 3 points
- Triple pairs can be pre-filtered based on motion invariants (lengths, angles)
- Can be used for the last step of the exterior orientation (P3P) problem →66

Module IV

Computing with a Camera Pair

covered by

[1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1

[2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR* 2006, pp. 630–633

additional references

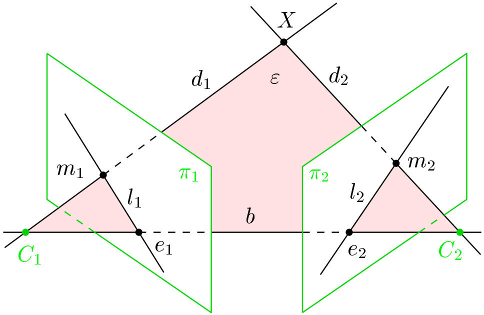
H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293(5828):133–135, 1981.

▶Geometric Model of a Camera Stereo Pair

$$\mathbf{P}_i = \begin{bmatrix}\mathbf{Q}_i & \mathbf{q}_i\end{bmatrix} = \mathbf{K}_i \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} = \mathbf{K}_i \mathbf{R}_i \begin{bmatrix}\mathbf{I} & -\mathbf{C}_i\end{bmatrix} \quad i = 1, 2 \qquad \rightarrow 31$$

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



two-camera setup

Description

- baseline b joins projection centers C_1, C_2

$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$

- epipole $e_i \in \pi_i$ is the image of C_j:

$$\underline{\mathbf{e}}_1 \simeq \mathbf{P}_1 \underline{\mathbf{C}}_2, \quad \underline{\mathbf{e}}_2 \simeq \mathbf{P}_2 \underline{\mathbf{C}}_1$$

- $l_i \in \pi_i$ is the image of optical ray d_j, $j \neq i$ and also the epipolar plane

$$\varepsilon = (C_2, X, C_1)$$

- l_j is the epipolar line ('epipolar') in image π_j induced by m_i in image π_i

Epipolar constraint relates $\underline{\mathbf{m}}_1$ and $\underline{\mathbf{m}}_2$: corresponding d_2, b, d_1 are coplanar

a necessary condition →87

Epipolar Geometry Example: Forward Motion

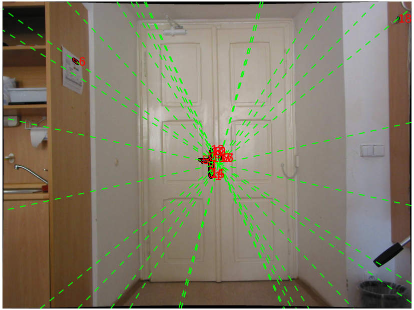


image 1

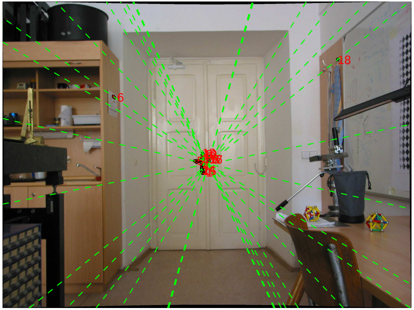
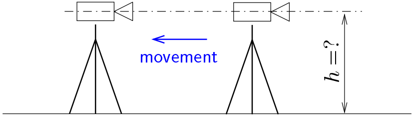


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

Epipole is the image of the other camera's center.
How high was the camera above the floor?



▶Cross Products and Maps by Skew-Symmetric 3×3 Matrices

- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_\times \mathbf{m}$, where $[\mathbf{b}]_\times$ is a 3×3 skew-symmetric matrix

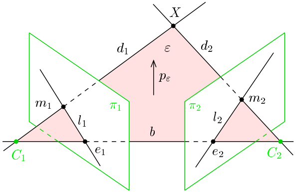
$$[\mathbf{b}]_\times = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \qquad \text{assuming} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

1. $[\mathbf{b}]_\times^\top = -[\mathbf{b}]_\times$ — the general antisymmetry property
2. $\mathbf{A}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ for all $\mathbf{x}$ — skew-sym mtx generalizes cross products
3. $[\mathbf{b}]_\times^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_\times$
4. $\|[\mathbf{b}]_\times\|_F = \sqrt{2}\,\|\mathbf{b}\|$ — Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\mathrm{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
5. $\mathrm{rank}\,[\mathbf{b}]_\times = 2$ iff $\|\mathbf{b}\| > 0$ — check minors of $[\mathbf{b}]_\times$
6. $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$
7. eigenvalues of $[\mathbf{b}]_\times$ are $(0, \lambda, -\lambda)$
8. for any 3×3 regular $\mathbf{B}$: $\mathbf{B}^\top [\mathbf{B}\mathbf{z}]_\times \mathbf{B} = \det \mathbf{B}\, [\mathbf{z}]_\times$ — follows from the factoring on →39
9. in particular: if $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$ then $[\mathbf{R}\mathbf{b}]_\times = \mathbf{R}[\mathbf{b}]_\times \mathbf{R}^\top$

- note that if $\mathbf{R}_b$ is rotation about $\mathbf{b}$ then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
- note $[\mathbf{b}]_\times$ is not a homography; it is not a rotation matrix — it is the logarithm of a rotation mtx

▶Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = \begin{bmatrix} \mathbf{Q}_i & \mathbf{q}_i \end{bmatrix} = \mathbf{K}_i \begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \end{bmatrix}, \; i = 1, 2$$

defs:

$\mathbf{R}_{21}$ – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^\top$

$\mathbf{t}_{21}$ – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21}\mathbf{t}_1 = -\mathbf{R}_2\mathbf{b}$ →74

$\mathbf{b}$ – baseline vector (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q} = -\mathbf{R}^\top\mathbf{t}$ →33 and 35

$$0 = \underbrace{\mathbf{d}_2^\top \; \mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1}\underline{\mathbf{m}}_2)^\top}_{\text{optical ray}} \; \underbrace{\mathbf{Q}_1^\top \underline{\mathbf{l}}_1}_{\text{optical plane}} = \underline{\mathbf{m}}_2^\top \; \underbrace{\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top (\underline{\mathbf{e}}_1 \times \underline{\mathbf{m}}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \underline{\mathbf{m}}_2^\top \; \underbrace{\left(\mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times\right)}_{\text{fundamental matrix } \mathbf{F}} \; \underline{\mathbf{m}}_1$$

Epipolar constraint $\underline{\mathbf{m}}_2^\top \mathbf{F} \, \underline{\mathbf{m}}_1 = 0$ is a point-line incidence constraint

- point $\underline{\mathbf{m}}_2$ is incident on epipolar line $\underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$
- point $\underline{\mathbf{m}}_1$ is incident on epipolar line $\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2$

- $\mathbf{F}\underline{\mathbf{e}}_1 = \mathbf{F}^\top\underline{\mathbf{e}}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\underline{\mathbf{e}}_1 \simeq \mathbf{Q}_1\mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1\mathbf{C}_2 - \mathbf{Q}_1\mathbf{C}_1 = \mathbf{K}_1\mathbf{R}_1\mathbf{b} = -\mathbf{K}_1\mathbf{R}_1\mathbf{R}_2^\top\mathbf{t}_{21} = -\mathbf{K}_1\mathbf{R}_{21}^\top\mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [\underline{\mathbf{e}}_1]_\times = \mathbf{Q}_2^{-\top}\mathbf{Q}_1^\top [-\mathbf{K}_1\mathbf{R}_{21}^\top\mathbf{t}_{21}]_\times \overset{\circledast 1}{=} \cdots \simeq \mathbf{K}_2^{-\top}[-\mathbf{t}_{21}]_\times \mathbf{R}_{21}\mathbf{K}_1^{-1}$$ fundamental

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2\mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} \overset{\to 76/9}{=} \mathbf{R}_{21} \underbrace{[\mathbf{R}_1\mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21}[-\mathbf{R}_{21}^\top\mathbf{t}_{21}]_\times$$ essential

▶The Structure and the Key Properties of the Fundamental Matrix

$$\mathbf{F} = (\underbrace{\mathbf{Q}_2\mathbf{Q}_1^{-1}}_{\text{epipolar homography } \mathbf{H}_e})^{-\top}[\mathbf{e}_1]_\times = \underbrace{\mathbf{K}_2^{-\top}\mathbf{R}_{21}\mathbf{K}_1^{\top}}_{\mathbf{H}_e^{-\top}}[\overbrace{\mathbf{e}_1}^{\text{left epipole}}]_\times \overset{\to 76}{\simeq} [\overbrace{\mathbf{H}_e\mathbf{e}_1}^{\text{right epipole}}]_\times \mathbf{H}_e = \mathbf{K}_2^{-\top}\underbrace{[-\mathbf{t}_{21}]_\times \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}}\mathbf{K}_1^{-1}$$

1. $\mathbf{E}$ captures relative camera pose only [Longuet-Higgins 1981]

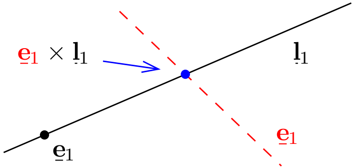
(the change of the world coordinate system does not change $\mathbf{E}$)

$$\begin{bmatrix}\mathbf{R}'_i & \mathbf{t}'_i\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i & \mathbf{t}_i\end{bmatrix} \cdot \begin{bmatrix}\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_i\mathbf{R} & \mathbf{R}_i\mathbf{t} + \mathbf{t}_i\end{bmatrix},$$

then

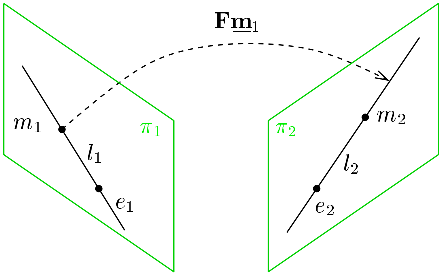
$$\mathbf{R}'_{21} = \mathbf{R}'_2\mathbf{R}'^{\top}_1 = \dots = \mathbf{R}_{21} \qquad \mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21}\mathbf{t}'_1 = \dots = \mathbf{t}_{21}$$

2. the translation length $\mathbf{t}_{21}$ is lost since $\mathbf{E}$ is homogeneous
3. $\mathbf{F}$ maps points to lines and it is not a homography
4. $\mathbf{H}_e$ maps epipoles to epipoles, $\mathbf{H}_e^{-\top}$ epipolar lines to epipolar lines: $\underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1$



another epipolar line map: $\underline{\mathbf{l}}_2 \simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times \underline{\mathbf{l}}_1 = \mathbf{F}(\underline{\mathbf{e}}_1 \times \underline{\mathbf{l}}_1)$

- proof by point/line 'transmutation' (left)
- point $\underline{\mathbf{e}}_1$ does not lie on line $\underline{\mathbf{e}}_1$ (dashed): $\underline{\mathbf{e}}_1^\top\underline{\mathbf{e}}_1 \neq 0$
- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ is not a homography, unlike $\mathbf{H}_e^{-\top}$ but it does the same job for epipolar line mapping
- no need to decompose $\mathbf{F}$ to obtain $\mathbf{H}_e$



$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F}\, \underline{\mathbf{m}}_1$$

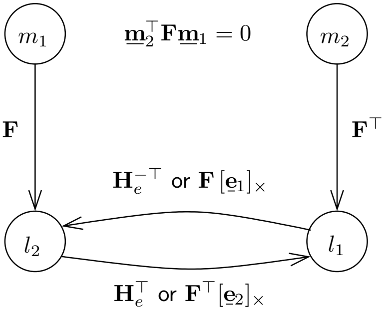
$$\underline{\mathbf{e}}_1 \simeq \operatorname{null}(\mathbf{F}), \qquad \underline{\mathbf{e}}_2 \simeq \operatorname{null}(\mathbf{F}^\top)$$

$$\underline{\mathbf{e}}_1 \simeq \mathbf{H}_e^{-1}\underline{\mathbf{e}}_2 \qquad \underline{\mathbf{e}}_2 \simeq \mathbf{H}_e\underline{\mathbf{e}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top\underline{\mathbf{m}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{F}\underline{\mathbf{m}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{H}_e^\top\underline{\mathbf{l}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{H}_e^{-\top}\underline{\mathbf{l}}_1$$

$$\underline{\mathbf{l}}_1 \simeq \mathbf{F}^\top[\underline{\mathbf{e}}_2]_\times\underline{\mathbf{l}}_2 \qquad \underline{\mathbf{l}}_2 \simeq \mathbf{F}[\underline{\mathbf{e}}_1]_\times\underline{\mathbf{l}}_1$$

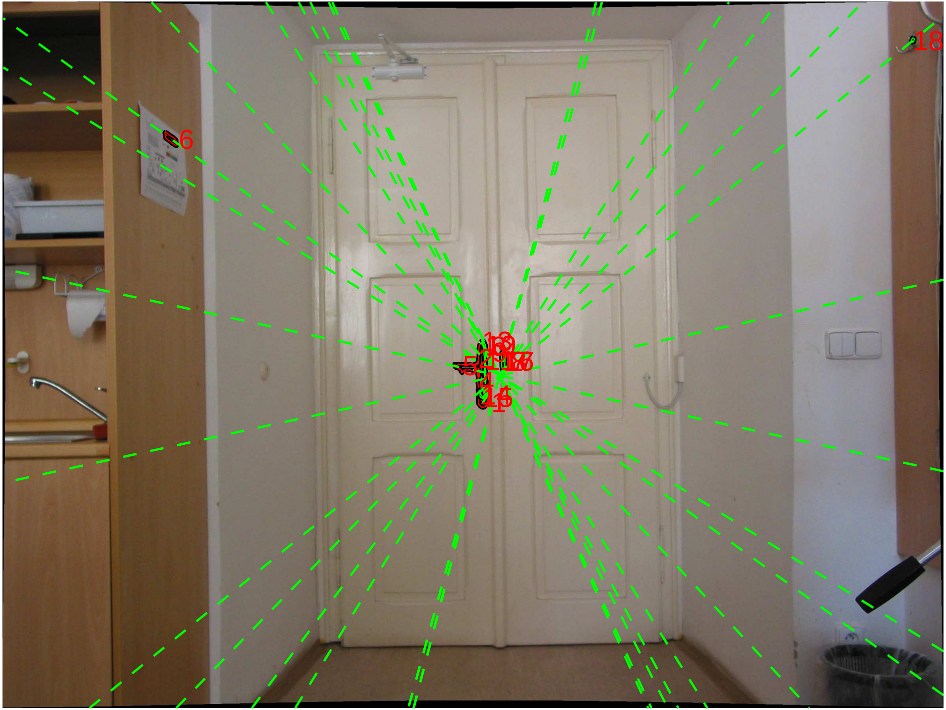


- $\mathbf{F}[\underline{\mathbf{e}}_1]_\times$ maps epipolar lines to epipolar lines but it is not a homography
- $\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1}$ is the epipolar homography→78
  $\mathbf{H}_e^{-\top}$ maps epipolar lines to epipolar lines, where

$$\mathbf{H}_e = \mathbf{Q}_2\mathbf{Q}_1^{-1} = \mathbf{K}_2\mathbf{R}_{21}\mathbf{K}_1^{-1}$$

you have seen this →59

Thank You



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