Manipulator Singularities

December 5, 2022

Positional Jacobian and singularities

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{q}(t)) \xrightarrow{\text{chain rule}} \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \frac{\partial f_3}{\partial q_1} & \cdots & \frac{\partial f_3}{\partial q_n} \end{bmatrix}}_{\mathbf{J}(\mathbf{q})} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

A robot configuration $\mathbf{q}^{(0)}$ is **singular** if

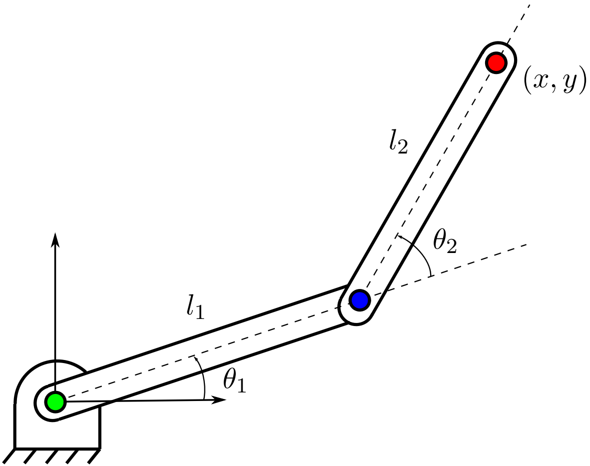
$$\operatorname{rank} \mathbf{J}(\mathbf{q}^{(0)}) < \max_{\mathbf{q} \in \mathbb{R}^n} \operatorname{rank} \mathbf{J}(\mathbf{q})$$

An end-effector pose $\mathbf{x}^{(0)}$ is **singular** if there exists a singular configuration $\mathbf{q}^{(0)}$ such that

$$\mathbf{x}^{(0)} = \mathbf{f}(\mathbf{q}^{(0)})$$

2R manipulator in 2D

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$



Singularities of 2R manipulator in 2D

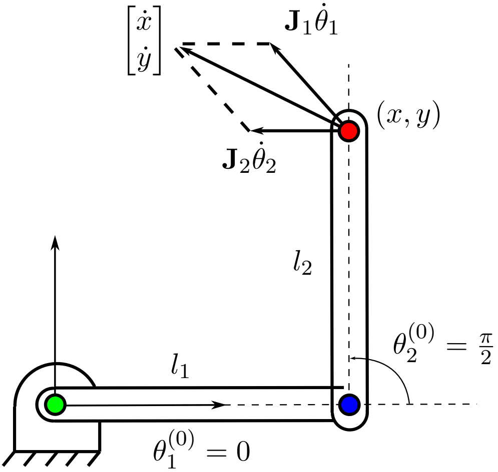
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\mathbf{J}(\boldsymbol{\theta}) = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

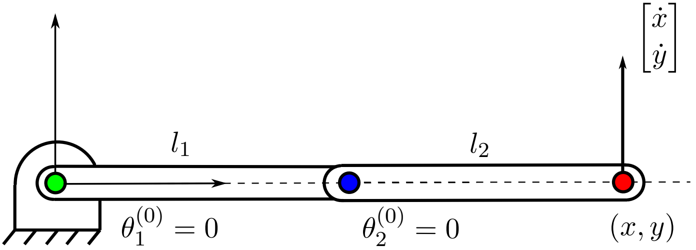
Since $\max_{\boldsymbol{\theta} \in \mathbb{R}^2} \operatorname{rank} \mathbf{J}(\boldsymbol{\theta}) = 2$, then a configuration $\boldsymbol{\theta}^{(0)}$ is singular iff

$$\operatorname{rank} \mathbf{J}(\boldsymbol{\theta}^{(0)}) < 2 \iff \det \mathbf{J}(\boldsymbol{\theta}^{(0)}) = 0 \iff \theta_1^{(0)} \in \mathbb{R}, \theta_2^{(0)} \in \{0, \pi\}$$

2R manipulator in 2D (non-singular configuration)



2R manipulator in 2D (singular configuration)



RPP manipulator in 3D