# Computing lexicographic Gröbner basis

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**Algorithm 1:** Multivariate Polynomial Division Algorithm

**Input:** $f, F = (f_1, \ldots, f_s), \geq$ (monomial ordering)
**Output:** $(q_1, \ldots, q_s), r$ such that $f = \sum_{i=1}^{s} q_i f_i + r$, $\mathrm{LT}_{\geq}(r)$ is not divisible by any of $\mathrm{LT}_{\geq}(f_i)$ or $r = 0$

$q_1 \leftarrow \cdots \leftarrow q_s \leftarrow r \leftarrow 0$
$p \leftarrow f$
**while** $p \neq 0$ **do**
  $i \leftarrow 1$
  divisionoccured $\leftarrow FALSE$
  **while** $i \leq s$ **and** divisionoccured $= FALSE$ **do**
    **if** $\mathrm{LT}_{\geq}(f_i)$ divides $\mathrm{LT}_{\geq}(p)$ **then**
      $q_i \leftarrow q_i + \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)}$
      $p \leftarrow p - \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)} f_i$
      divisionoccured $\leftarrow TRUE$
    **else**
      $i \leftarrow i + 1$
  **if** divisionoccured $= FALSE$ **then**
    $r \leftarrow r + \mathrm{LT}_{\geq}(p)$
    $p \leftarrow p - \mathrm{LT}_{\geq}(p)$
**return** $(q_1, \ldots, q_s), r$

**Algorithm 2:** Buchberger's Algorithm

**Input:** $F = (f_1, \ldots, f_s), \geq$ (monomial ordering)
**Output:** Groebner basis $G$ of $F$ w.r.t. $\geq$ monomial ordering

$t \leftarrow s$
$G \leftarrow F$
$B \leftarrow \{(i,j) \mid 1 \leq i < j \leq s\}$
**while** $B \neq \varnothing$ **do**
  Select $(i,j) \in B$
  $r \leftarrow \overline{S_{\geq}(f_i, f_j)}^{(G,\geq)}$
  **if** $r \neq 0$ **then**
    $t \leftarrow t + 1$
    $f_t \leftarrow r$
    $G \leftarrow (f_1, \ldots, f_t)$
    $B \leftarrow B \cup \{(i,t) \mid 1 \leq i \leq t-1\}$
  $B \leftarrow B \setminus \{(i,j)\}$
**return** $G$

**Remark.** *In the implementation of Buchberger's algorithm the notation* $\overline{S_{\geq}(f_i, f_j)}^{(G,\geq)}$ *for the monomial ordering* $\geq$ *is used. It simply denotes the remainder of the division of the S-polynomial of* $f_i$ *and* $f_j$ *w.r.t.* $\geq$

$$S_{\geq}(f_i, f_j) = \frac{\text{LCM}\left(\text{LM}_{\geq}(f_i), \text{LM}_{\geq}(f_j)\right)}{\text{LT}_{\geq}(f_i)} \cdot f_i - \frac{\text{LCM}\left(\text{LM}_{\geq}(f_i), \text{LM}_{\geq}(f_j)\right)}{\text{LT}_{\geq}(f_j)} \cdot f_j$$

*by the ordered tuple of polynomials* $G$ *w.r.t.* $\geq$.

**Task 1.** *Consider the polynomial system* $F = (f_1, f_2) = (xy - 1, x^2 - y)$. *Compute a lexicographic Gröbner basis* $G$ *of* $F$ *w.r.t. the variable ordering* $x > y$ *and retrieve the solutions to* $F$ *from* $G$.