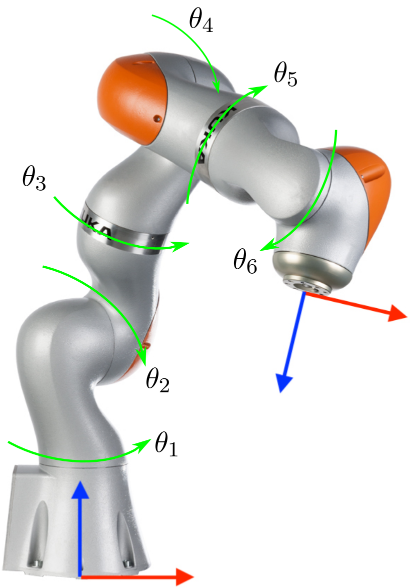
Inverse Kinematic Task (IKT)



Mathematical formulation of IKT

$$\mathbf{M}_e = \mathbf{M}_1^0\mathbf{M}_2^1\mathbf{M}_3^2\mathbf{M}_4^3\mathbf{M}_5^4\mathbf{M}_6^5$$

$$\underbrace{\begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\substack{\text{pose of the} \\ \text{end effector}}} = \prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i + \underbrace{\theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i}_{\text{DH parameters}})$$

$$\underbrace{\prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i + \theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{12 nonzero functions } \mathbf{f}(\boldsymbol{\theta})} = \mathbf{O}$$

We can solve 12 equations $\mathbf{f}(\theta_1, \ldots, \theta_6) = \mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$ either **numerically** or **symbolically**.

Newthon's method (numerical method)

Denote by $\boldsymbol{\theta}^*$ one of the solutions to $\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$ and by $\mathbf{J}$ the Jacobian matrix $\frac{\partial \mathbf{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \in C(\boldsymbol{\theta}, \mathbb{R})^{12\times 6}$.
Two basic operations of the Newthon's method are:

$$\boldsymbol{\theta}_0 = \text{something close to } \boldsymbol{\theta}^* \quad \text{(initialization)}$$

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k - \mathbf{J}^{+}(\boldsymbol{\theta}_k)\mathbf{f}(\boldsymbol{\theta}_k) \quad \text{(improve the previous guess)}$$

Symbolic method

- Equations $\mathbf{f}(\boldsymbol{\theta})$ are polynomial in $\cos\theta_i$ and $\sin\theta_i$
- New variables: $c_i = \cos\theta_i,\ s_i = \sin\theta_i$
- New **polynomial** equations:

$$\begin{matrix}\text{18 equations} \\ \text{12 variables}\end{matrix} \quad \begin{cases} \mathbf{f}(c_1, s_1, \dots, c_6, s_6) = \mathbf{0} \\ c_i^2 + s_i^2 = 1 \quad \forall i = 1, \dots, 6 \end{cases}$$

Groebner bases (symbolic method)

System of polynomial equations

↓ Buchberger algorithm, F4, F5

(based on Polynomial Division)

Groebner basis (lexicographic ordering)

↓ Solving univariate polynomials + Backsubstitution

Solutions

Monomials and terms

1. **Monomial** is a product of variables, e.g., x^3y^5 or the constant 1
2. **Term** is a product of a nonzero constant with a monomial, e.g., $3x^2yzw^4$
3. Consider 3 variables x, y, z and order them, e.g.,

$$x > y > z$$

Every monomial can be written now as $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\mathbf{x} = (x, y, z)$ and $\boldsymbol{\alpha} \in \mathbb{Z}^3_{\geq 0}$, e.g.,

$$y^2x^3z = \mathbf{x}^{\boldsymbol{\alpha}} \quad \text{with} \quad \boldsymbol{\alpha} = (3, 2, 1)$$

4. We say that term $a\,\mathbf{x}^{\boldsymbol{\alpha}}$ **divides** term $b\,\mathbf{x}^{\boldsymbol{\beta}}$ if

$$\beta_i - \alpha_i \geq 0 \quad \forall i = 1, \ldots, n$$

Lexicographic monomial ordering

1. We define the relation
$$\mathbf{x}^{\boldsymbol{\alpha}} \geq_{\text{lex}} \mathbf{x}^{\boldsymbol{\beta}}$$
if the left-most nonzero element of $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is positive or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.
2. For example,
$$\mathbf{x}^{\boldsymbol{\alpha}} = x^3y^3z \geq_{\text{lex}} x^3y^2z^{10} = \mathbf{x}^{\boldsymbol{\beta}}$$
since
$$\boldsymbol{\alpha} - \boldsymbol{\beta} = (3,3,1) - (3,2,10) = (0, 1, -9)$$
3. We can extend this relation to terms by saying that
$$a\,\mathbf{x}^{\boldsymbol{\alpha}} \geq_{\text{lex}} b\,\mathbf{x}^{\boldsymbol{\beta}} \iff \mathbf{x}^{\boldsymbol{\alpha}} \geq_{\text{lex}} \mathbf{x}^{\boldsymbol{\beta}}$$

Multivariate Polynomial Division Algorithm

Algorithm 1: Multivariate Polynomial Division Algorithm

Input: $f, F = (f_1, \dots, f_s), \geq$ (monomial ordering)
Output: $(q_1, \dots, q_s), r$ such that $f = \sum_{i=1}^{s} q_i f_i + r$, $\mathrm{LT}_{\geq}(r)$ is not divisible by any of $\mathrm{LT}_{\geq}(f_i)$ or $r = 0$

$q_1 \leftarrow \dots \leftarrow q_s \leftarrow r \leftarrow 0$
$p \leftarrow f$
while $p \neq 0$ **do**
  $i \leftarrow 1$
  divisionoccured $\leftarrow FALSE$
  while $i \leq s$ **and** divisionoccured $= FALSE$ **do**
    if $\mathrm{LT}_{\geq}(f_i)$ divides $\mathrm{LT}_{\geq}(p)$ **then**
      $q_i \leftarrow q_i + \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)}$
      $p \leftarrow p - \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)} f_i$
      divisionoccured $\leftarrow TRUE$
    else
      $i \leftarrow i + 1$
  if divisionoccured $= FALSE$ **then**
    $r \leftarrow r + \mathrm{LT}_{\geq}(p)$
    $p \leftarrow p - \mathrm{LT}_{\geq}(p)$
return $(q_1, \dots, q_s), r$