PKR Lab-07 Solution

Task 1. *Consider motion given by a mapping of a general point X to point Y by*

$$\vec{y}_\beta = \mathbf{R}\,\vec{x}_\beta + \vec{o}\,'_\beta\,, \tag{1}$$

where $\vec{x}_\beta$, resp. $\vec{y}_\beta$, are coordinate vectors representing point X, resp. point Y, in a coordinate system with an orthonormal basis β. Matrix $\mathbf{R}$ and vector $\vec{o}\,' = \overrightarrow{OO'}$ are given as follows

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \vec{o}'_\beta = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1. *Write down the matricidal equation determining the coordinates of points on the axis of motion.*
2. *Find all the points on the axis of motion.*

Solution: The axis of motion is the line in $\mathbb{R}^3$ that is left invariant after applying Equation (1). The matrix equation that determines the points on the axis of motion is [1, Equation (9.2)]:

$$(\mathbf{R}-\mathbf{I})^2\vec{x}_\beta = -(\mathbf{R}-\mathbf{I})\vec{o}\,'_\beta$$

Substituting $\mathbf{R}$ and $\vec{o}'_\beta$ to it we obtain

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^2 \vec{x}_\beta = -\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}\vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \tag{2}$$

We can solve this system of linear equations by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Denote $\vec{x}_\beta = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$. From the second row $x_2 - x_3 = 0$ we conclude that $x_2 = x_3$ and we let x_3 to be any real number t. From the first row $x_1 - 2x_2 + x_3 = 1$ we conclude that $x_1 = 2x_2 - x_3 + 1 = t + 1$. Thus, the solutions to (2) are

$$L = \left\{ \begin{bmatrix} t+1 \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

We may verify that the line L is left invariant under the motion given by (1). For this we pick a general point from L and substitute it into (1):

$$\mathbf{R}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + \vec{o}\,'_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We see that L is left invariant (in this case even point-wise, since $\vec{o}\,'_\beta$ is perpendicular to the rotation axis of $\mathbf{R}$). □

Task 2. *Consider unit quaternion*

$$\mathbf{q} = \frac{1}{3}\begin{bmatrix} 0 & -1 & -2 & -2 \end{bmatrix}$$

(a) *For the rotation given by $\mathbf{q}$, find all pairs of $(\theta, \mathbf{v})$ corresponding to its rotation angle $-\pi < \theta \le \pi$ and its rotation axis generated by unit vector $\mathbf{v}$,*

(b) *Find the rotation matrix corresponding to $\mathbf{q}$.*

Solution:

(a) The quaternion is defined by

$$\mathbf{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\mathbf{v} \end{bmatrix}$$

where $(\theta, \mathbf{v})$ define the angle and the normalized axis of rotation. That's why

$$\cos\frac{\theta}{2} = 0 \Rightarrow \sin\frac{\theta}{2} = \pm 1$$

We, e.g., take the pair $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}) = (0, 1)$ which gives

$$\frac{\theta}{2} = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z} \iff \theta = \pi + 4\pi k, k \in \mathbb{Z}$$

By the task we want $-\pi < \theta \leq \pi$, so $\theta = \pi$. We compute the normalized axis of rotation $\mathbf{v}$ by dividing the last 3 coordinates of $\mathbf{q}$ by $\sin\frac{\theta}{2}$:

$$\mathbf{v} = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \end{bmatrix}^\top.$$

If $(\theta, \mathbf{v})$ defines $\mathbf{q}$ for $-\pi < \theta \leq \pi$, then all pairs $(\theta, \mathbf{v})$ with $-\pi < \theta \leq \pi$ that define the rotation given by $\mathbf{q}$ are determined by $\{(\theta, \mathbf{v}), (-\theta, -\mathbf{v})\}$. Since for $\theta = \pi$ the value $-\theta = -\pi$ jumps out of the interval $(-\pi, \pi]$, then we simply add 2π to it, since it doesn't change the rotation matrix (according to the Rodriguez formula [1, Equation 7.22]). Hence, the answer is

$$\left\{ \left(\pi, -\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top\right), \left(\pi, \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top\right) \right\}.$$

(b) The rotation matrix is given by the Rodriguez formula [1, Equation 7.22]:

$$\mathbf{R} = \cos\theta\mathbf{I} + (1 - \cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta\,[\mathbf{v}]_\times$$

$$\mathbf{R} = -\mathbf{I} + 2 \cdot \frac{1}{9} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} + \frac{2}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix}$$

Another way to obtain $\mathbf{R}$ is to use the formula in terms of quaternions [1, Equation 7.67]:

$$\mathbf{R} = \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\ 2(q_2q_3 + q_1q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3q_4 - q_1q_1) \\ 2(q_2q_4 - q_1q_3) & 2(q_3q_4 + q_1q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -7 & 4 & 4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix}$$

□

Task 3. *Consider rotation matrix*

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) Find its (unit) rotation axis and angle $-\pi < \theta \leq \pi$.

(b) Find all unit quaternions corresponding to $\mathbf{R}$.

Solution:

(a) The rotation axis is given by the eigenvector corresponding to the eigenvalue $\lambda = 1$:

$$\mathbf{R}\mathbf{v} = \mathbf{v} \iff (\mathbf{R} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

We solve the linear homogeneous system of equations:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We obtain 1 zero row in the row echelon form of $\mathbf{R}-\mathbf{I}$ indicating that $\dim\ker(\mathbf{R}-\mathbf{I})=1$. Let's denote $\mathbf{v}=\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}$. Then we let v_3 to be any real number t. From the second equation $-v_2+v_3=0$ we obtain $v_2=v_3=t$. From the first $-v_1+v_2=0$ we obtain $v_1=v_2=t$. Thus, all the solutions to this linear system may be described by

$$\left\{\begin{bmatrix}t\\t\\t\end{bmatrix}\;\middle|\; t\in\mathbb{R}\right\}$$

Out of this set we take one of unit norm, e.g.,

$$\mathbf{v}=-\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

The rotation angle can be determined from the Rodriguez formula:

$$\mathbf{R}=\cos\theta\mathbf{I}+(1-\cos\theta)\mathbf{v}\mathbf{v}^\top+\sin\theta\left[\mathbf{v}\right]_\times$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\cos\theta\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}+\frac{1}{3}(1-\cos\theta)\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}+\frac{1}{\sqrt{3}}\sin\theta\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix}$$

Out of these 9 equations we pick 2 given by the elements $(1,1)$ and $(1,2)$:

$$(1,1):\quad 0=\cos\theta+\frac{1}{3}(1-\cos\theta)\iff\cos\theta=-\frac{1}{2}$$

$$(1,2):\quad 1=\frac{1}{3}(1-\cos\theta)+\frac{1}{\sqrt{3}}\sin\theta\iff\sin\theta=\frac{\sqrt{3}}{2}$$

from which we deduce that $\theta=\frac{2\pi}{3}$. Another way to compute the rotation axis and angle is to apply the formula [1, Equations 7.40, 7.41] for non-symmetric rotations:

$$\theta=\arccos\left(\frac{1}{2}\operatorname{trace}\mathbf{R}-1\right)=\frac{2\pi}{3},\quad \mathbf{v}=\frac{1}{2\sin\theta}\begin{bmatrix}r_{32}-r_{23}\\r_{13}-r_{31}\\r_{21}-r_{12}\end{bmatrix}=-\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Notice that for a symmetric rotation (i.e., a rotation by π) $\mathbf{v}$ is undefined, so this formula is not applicable. As was noted before, all the angle-axis (for $-\pi<\theta\leq\pi$) solutions to $\mathbf{R}$ are given by $\{(\theta,\mathbf{v}),(-\theta,-\mathbf{v})\}$.

(b) One way is to apply the formula

$$\mathbf{q}_{1,2}=\pm\begin{bmatrix}\cos\frac{\theta}{2}\\\sin\frac{\theta}{2}\mathbf{v}\end{bmatrix}=\pm\begin{bmatrix}\cos\frac{\pi}{3}\\\sin\frac{\pi}{3}\mathbf{v}\end{bmatrix}=\pm\begin{bmatrix}\frac{1}{2}\\-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\end{bmatrix}=\pm\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$$

The other way is to use [1, Equation 7.74]:

$$\mathbf{q}_{1,2}=\pm\frac{1}{2\sqrt{\operatorname{trace}\mathbf{R}+1}}\begin{bmatrix}\operatorname{trace}\mathbf{R}+1\\r_{32}-r_{23}\\r_{13}-r_{31}\\r_{21}-r_{12}\end{bmatrix}=\pm\frac{1}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$$

Again, this formula works only for non-symmetric rotations.

□

References

[1] Tomas Pajdla, *Elements of geometry for robotics*, `https://cw.fel.cvut.cz/b221/_media/courses/pkr/pro-lecture-2021.pdf`.