PKR Lab-06 Solution

Task 1. *Consider the rotation matrix with rotation axis generated by vector* $\mathbf{r} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ *that maps vector* $\mathbf{x} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ *to* $\mathbf{y} = \mathbf{R}\mathbf{x} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$.

(a) Find its rotation angle $-\pi < \theta \le \pi$,

(b) Find its rotation matrix $\mathbf{R}$,

(c) Find eigenvalues of $\mathbf{R}$.

Solution: (Method 1). We use the angle-axis parametrization of the rotation:

$$\mathbf{R} = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta[\mathbf{v}]_\times \tag{1}$$

where $\mathbf{v} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$ is the normalized axis of rotation. By the task, $\mathbf{R}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$. Hence, multiplying both sides of Equation (1) by $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ we get

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \mathbf{R}\begin{bmatrix}0\\0\\1\end{bmatrix} = \left(\cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta[\mathbf{v}]_\times\right)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \cos\theta\begin{bmatrix}0\\0\\1\end{bmatrix} + \frac{1-\cos\theta}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{\sin\theta}{\sqrt{3}}\begin{bmatrix}1\\-1\\0\end{bmatrix} \tag{2}$$

From the last equation of Equation (2)

$$0 = \cos\theta + \frac{1-\cos\theta}{3}$$

we can express

$$\cos\theta = -\frac{1}{2}.$$

Substituting it to the second equation in (2) we get

$$0 = \frac{1}{2} - \frac{\sin\theta}{\sqrt{3}}$$

from which we get

$$\sin\theta = \frac{\sqrt{3}}{2}$$

The rotation angle then equals

$$\theta = \operatorname{atan2}(\sin\theta, \cos\theta) = \operatorname{atan2}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{2\pi}{3}.$$

We get the rotation matrix by substituting $\mathbf{v}$ and θ to Equation (1):

$$\mathbf{R} = -\frac{1}{2}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & -1 & 1\\1 & 0 & -1\\-1 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

The eigenvalues of $\mathbf{R}$ are the roots of the characteristic polynomial of $\mathbf{R}$:

$$p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{R}) = \lambda^3 - 1$$

whose roots are the cubic roots of unity $1, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{2}{3}}$.

(Method 2). We now apply a bit different method to find θ. We know that the rotation angle θ is the angle between $\mathbf{p}$ and its image $\mathbf{q}$ under the rotation for some $\mathbf{p} \perp \mathbf{r}$. We claim that

$$\mathbf{p} = [\mathbf{r}]_\perp \mathbf{x}, \quad \mathbf{q} = [\mathbf{r}]_\perp \mathbf{y}, \quad \text{for } [\mathbf{r}]_\perp = \mathbf{I} - \frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2}$$

Obviously, $\mathbf{p} \perp \mathbf{r}$ since

$$\mathbf{r}^\top \mathbf{p} = \mathbf{r}^\top \left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2} \right) \mathbf{x} = \left(\mathbf{r}^\top - \mathbf{r}^\top \right) \mathbf{x} = \mathbf{0}^\top \mathbf{x} = 0$$

Also, $\mathbf{q} = [\mathbf{r}]_\perp \mathbf{y}$ since

$$\mathbf{q} = \mathbf{R}\mathbf{p} = \mathbf{R} \left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2} \right) \mathbf{x} = \left(\mathbf{R} - \mathbf{R}\frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2} \right) \mathbf{x} = \left(\mathbf{R} - \frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2} \right) \mathbf{x} \overset{(1)}{=} \left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^\top}{\|\mathbf{r}\|^2} \right) \mathbf{R}\mathbf{x} = [\mathbf{r}]_\perp \mathbf{y}$$

where $\overset{(1)}{=}$ follows from the fact that $\mathbf{r}^\top \mathbf{R} = \mathbf{r}^\top$. We compute

$$[\mathbf{r}]_\perp = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\mathbf{p} = [\mathbf{r}]_\perp \mathbf{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{q} = \mathbf{R}\mathbf{p} = [\mathbf{r}]_\perp \mathbf{y} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

The cosine of an angle between $\mathbf{p}$ and $\mathbf{q}$ can be computed as

$$\cos\theta = \cos\angle(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^\top \mathbf{q}}{\|\mathbf{p}\| \, \|\mathbf{q}\|} = -\frac{1}{2} \Rightarrow \theta = \arccos\left(-\frac{1}{2}\right) = \pi - \arccos\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

We now must determine the correct direction of the rotation axis: its direction is given by the right hand rule, where 4 fingers go from $\mathbf{p}$ to $\mathbf{q}$ in the direction of the smallest angle (namely, $\theta = \arccos\left(\frac{\mathbf{p}^\top \mathbf{q}}{\|\mathbf{p}\|\|\mathbf{q}\|}\right)$) and the thumb shows the direction of $\mathbf{v}$, since only in that case [1, Equation (7.14)] holds true (where we consider $\vec{v}$ as $\mathbf{v}$). Algebraically, $\mathbf{v} = a\mathbf{r}$ must be chosen in such a way that

1. $\|\mathbf{v}\| = 1$,
2. $\mathbf{v}^\top (\mathbf{p} \times \mathbf{q}) > 0$

The cross product $\mathbf{p} \times \mathbf{q}$ equals

$$\mathbf{p} \times \mathbf{q} = \begin{bmatrix} p_2 q_3 - p_3 q_2 \\ p_3 q_1 - p_1 q_3 \\ p_1 q_2 - p_2 q_1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then, obviously, $a = \frac{1}{\sqrt{3}}$, since then $\|\mathbf{v}\| = 1$ and

$$\mathbf{v}^\top (\mathbf{p} \times \mathbf{q}) = \frac{1}{3\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} > 0,$$

i.e. $\mathbf{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. The rotation matrix then, according to Equation (1), equals

$$\mathbf{R} = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta [\mathbf{v}]_\times = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Remark. *Notice that in Method 1 we first pick unit* $\mathbf{v} = a\mathbf{r}$ *(doesn't matter in which direction) and then find* θ *accordingly to* (1)*. If we first picked* $\mathbf{v} = -\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$*, then we would obtain* $\theta = -\frac{2\pi}{3}$*. While in Method 2 we first pick* $\theta \in [0, \pi]$ *(since* arccos *returns values from this interval) and then choose* $\mathbf{v} = a\mathbf{r}$ *according to the right hand rule. We note that the map* $(\theta, \mathbf{v}) \mapsto \mathbf{R}$ *is* 2*-to-*1 *except for the identity rotation: if* $(\theta, \mathbf{v})$ *generate* $\mathbf{R}$*, then* $(-\theta, -\mathbf{v})$ *generate it as well and there no other angle-axis pairs that generate the same* $\mathbf{R}$*.*

□

Task 2. *Find all* 2×2 *rotation matrices* $\mathbf{R}$ *such that*

$$\mathbf{R}\mathbf{R} = \mathbf{R}^\top$$

Solution: By multiplying by $\mathbf{R}$ both sides we obtain

$$\mathbf{R}^3 = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}^3 = \mathbf{I} \iff \begin{bmatrix}\cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \iff \cos 3\theta = 1, \sin 3\theta = 0$$

which means that

$$3\theta = 2k\pi, k \in \mathbb{Z} \iff \theta = \frac{2k\pi}{3}, k \in \mathbb{Z}$$

All rotation matrices $\mathbf{R}$ can be obtained by taking $k = 0, 1, 2$, i.e.

$$\mathbf{R}_1 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$

□

References

[1] Tomas Pajdla, *Elements of geometry for robotics*, `https://cw.fel.cvut.cz/b221/_media/courses/pkr/pro-lecture-2021.pdf`.