Inverse Kinematics of 6R Manipulator by Gröbner Basis Computation

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Mathematical Formulation of IKT

$$\mathbf{M}_e = \mathbf{M}_1^0\mathbf{M}_2^1\mathbf{M}_3^2\mathbf{M}_4^3\mathbf{M}_5^4\mathbf{M}_6^5$$

$$\underbrace{\begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{pose of the end effector}} = \prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i + \underbrace{\theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i}_{\text{DH parameters}})$$

$$\underbrace{\prod_{i=1}^{6} \mathbf{M}_i^{i-1}(\theta_i + \theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{12 nonzero functions } \mathbf{f}(\boldsymbol{\theta})} = \mathbf{O}$$

Symbolic formulation

Every matrix $\mathbf{M}_i^{i-1}$ can be decomposed as

$$\begin{bmatrix} \cos(\theta_i + \theta_{i_{\text{offset}}}) & -\sin(\theta_i + \theta_{i_{\text{offset}}}) & 0 & 0 \\ \sin(\theta_i + \theta_{i_{\text{offset}}}) & \cos(\theta_i + \theta_{i_{\text{offset}}}) & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New variables:

$$c_i = \cos(\theta_i + \theta_{i_{\text{offset}}}), \quad s_i = \sin(\theta_i + \theta_{i_{\text{offset}}})$$

New polynomial equations:

$$\begin{cases} \prod_{i=1}^{6} \mathbf{M}_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O} \\ {c_i}^2 + {s_i}^2 = 1, \quad i = 1, \ldots, 6 \end{cases}$$

Compute θ_i from c_i and s_i:

$$\theta_i = \operatorname{atan2}(s_i, c_i) - \theta_{i_{\text{offset}}}$$

Simplified equations

The inverse matrix $\left(\mathbf{M}_i^{i-1}\right)^{-1}$ has the form:

$$\begin{bmatrix} 1 & 0 & 0 & -a_i \\ 0 & \cos\alpha_i & \sin\alpha_i & 0 \\ 0 & -\sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_i & s_i & 0 & 0 \\ -s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & -d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is polynomial in c_i and s_i, the former polynomial equations of IKT of degree 6 in c_i and s_i can be simplified to the following ones of degree 3:

$$\prod_{i=4}^{6} \mathbf{M}_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \left(\prod_{i=1}^{3} \mathbf{M}_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) \right)^{-1} \begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O},$$

$${c_i}^2 + {s_i}^2 = 1, \quad i = 1, \ldots, 6$$

Solvability of equations

1. Groebner Basis computation is done in exact arithmetics over the rational numbers and therefore the rational input must be provided.
2. At the same time, the input must satisfy all identities on sines, cosines and rotations, otherwise the equations would have no solution.

That's why,

1. the parameters d_i, a_i, $\mathbf{t}_e$ must be given using `Fraction` numbers
2. $\cos\alpha_i$ and $\sin\alpha_i$ must be given as a tuple of `Fraction` numbers such that the sum of their squares is exactly 1
3. the rotation matrix $\mathbf{R}_e$ must be given using `Fraction` numbers such that $\mathbf{R}_e^\top \mathbf{R}_e = \mathbf{I}$ and $\det \mathbf{R}_e = 1$ hold exactly.

Rational approximation of a floating point number

Input: floating point number n, positive tolerance tol
Output: fraction number f such that $|f - n| < tol$
Steps:

1. If $tol \geq 1$: a rational approximation f of n is:

$$f = \frac{\lfloor n \rfloor}{1}$$

2. If $tol < 1$: represent tol in scientific notation and take the exponent e, i.e.

$$tol = m \times 10^e, \quad m \in [1, 10),\ e \in \mathbb{Z}_{<0}$$

and take a rational approximation f of n given by:

$$f = \frac{\lfloor n \cdot 10^{-e} \rfloor}{10^{-e}}$$

Rational approximation of a floating point number

Example

Let the floating point number be given by

$$n = 10.123456789$$

and the tolerance of the rational approximation be given by another floating point number

$$tol = 0.000{\color{red}2}5932 = 2.5932 \times 10^{-5} \Rightarrow e = -5$$

The fraction which approximates n is

$$f = \frac{\lfloor n \cdot 10^{-e} \rfloor}{10^{-e}} = \frac{\lfloor 10.123456789 \cdot 10^{5} \rfloor}{10^{5}} = \frac{1012345}{10^{5}}$$

$$|f - n| = |10.12345 - 10.123456789| = 0.000006789 < tol$$

Rational sine and cosine

Rational parametrization of the unit circle:

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \quad t \in \mathbb{Q}$$

Trigonometric meaning of t:

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{2t^2}{2t} = t$$

Rational sine and cosine

Input: floating point number $\theta \in (-\pi, \pi]$, positive tolerance tol
Output: rational approximations c, s of $\cos\theta, \sin\theta$ such that

$$c^2 + s^2 = 1 \text{ (exactly)}$$

Steps:

1. If $\cos\theta + 1 < tol$, then return

$$c = -1, \quad s = 0$$

2. Compute a rational approximation t of $\tan\frac{\theta}{2}$
3. Compute cosine and sine corresponding to t:

$$c = \frac{1 - t^2}{1 + t^2}, \quad s = \frac{2t}{1 + t^2}$$

Rational sine and cosine

Example

Let the angle and the tolerance be given by

$$\theta = 1.2345, \quad tol = 0.0023$$

The rational approximation of $\tan\frac{\theta}{2}$ is given by

$$t = \texttt{rat\_approx}\left(\tan\frac{\theta}{2}, tol\right) = \texttt{rat\_approx}\left(0.709766, 0.0023\right) =$$

$$= \frac{\lfloor 0.709766 \cdot 10^3 \rfloor}{10^3} = \frac{709}{1000}$$

$$c = \frac{1-t^2}{1+t^2} = \frac{497319}{1502681} \approx \cos\theta, \quad s = \frac{2t}{1+t^2} = \frac{1418000}{1502681} \approx \sin\theta$$

Rational 3×3 rotation matrix

Parametrization of exact 3×3 rotation matrices:

$$\mathbf{R} = \frac{1}{\sum_{i=1}^{4} q_i^2} \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2 q_3 - q_1 q_4) & 2(q_2 q_4 + q_1 q_3) \\ 2(q_2 q_3 + q_1 q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3 q_4 - q_1 q_2) \\ 2(q_2 q_4 - q_1 q_3) & 2(q_3 q_4 + q_1 q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix},$$

$$q_1, q_2, q_3, q_4 \in \mathbb{Q}$$

In the next slide we will use the function `q2r` which is defined by the above formula.

Rational 3×3 rotation matrix

Algorithm 1: Rational 3×3 rotation matrix

Input: vector of float numbers $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^\top$ with $\|\mathbf{q}\| \approx 1$, positive tolerance *tol*

Output: 3×3 matrix $\mathbf{R}$ with fraction numbers such that

$$\mathbf{R}^\top \mathbf{R} = \mathbf{I}, \ \det \mathbf{R} = 1 \ \text{(exactly)} \quad \& \quad \|\mathbf{R} - \mathtt{q2r}(\mathbf{q})\|_\mathrm{F} < tol$$

$tol_q \leftarrow tol$
while *TRUE* **do**
    $\mathbf{q}_{rat} \leftarrow \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$
    for ($k \leftarrow 1; k \leq 4; k \leftarrow k+1$)
        $\mathbf{q}_{rat}[k] \leftarrow \mathtt{rat\_approx}(q_k, tol_q)$
    $\mathbf{R} \leftarrow \mathtt{q2r}(\mathbf{q}_{rat})$
    if $\|\mathbf{R} - \mathtt{q2r}(\mathbf{q})\|_\mathrm{F} < tol$ **then**
        return $\mathbf{R}$
    else
        $tol_q \leftarrow \frac{tol_q}{10}$

Rational 3×3 rotation matrix

Example

Let the quaternion and the tolerance be given by

$$\mathbf{q} = \begin{bmatrix} 0.748 & 0.654 & 0.108 & 0.012 \end{bmatrix}, \quad tol = 0.0011$$

The output of the above algorithm gives the rational rotation matrix

$$\mathbf{R} = \begin{bmatrix} \frac{243853}{249757} & \frac{30828}{249757} & \frac{44316}{249757} \\ \frac{39804}{249757} & \frac{35827}{249757} & -\frac{243948}{249757} \\ -\frac{36468}{249757} & \frac{245244}{249757} & \frac{30067}{249757} \end{bmatrix}$$

which comes from the (non-unit) rational quaternion

$$\mathbf{q}_{rat} = \begin{bmatrix} \frac{187}{250} & \frac{327}{500} & \frac{27}{250} & \frac{3}{250} \end{bmatrix} \approx \mathbf{q}$$

Reduced lexicographic Gröbner basis of IKT equations

IKT equations:

$$\prod_{i=4}^{6} \mathbf{M}_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \left(\prod_{i=1}^{3} \mathbf{M}_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i)\right)^{-1} \begin{bmatrix} \mathbf{R}_e & \mathbf{t}_e \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O},$$

$${c_i}^2 + {s_i}^2 = 1, \quad i = 1, \ldots, 6$$

For the following ordering of variables

$$c_1 > s_1 > \cdots > c_6 > s_6$$

the reduced lexicographic Gröbner basis of IKT equations looks like:

$${s_6}^{16} + a_{1,15} \cdot {s_6}^{15} + a_{1,14} \cdot {s_6}^{14} + \cdots + a_{1,1} \cdot s_6 + a_{1,0}$$

$$c_6 + a_{2,15} \cdot {s_6}^{15} + a_{2,14} \cdot {s_6}^{14} + \cdots + a_{2,1} \cdot s_6 + a_{2,0}$$

$$\vdots$$

$$c_1 + a_{12,15} \cdot {s_6}^{15} + a_{12,14} \cdot {s_6}^{14} + \cdots + a_{12,1} \cdot s_6 + a_{12,0}$$