Logical reasoning and programming, lab session 7

(October 31, 2022)

7.1 Decide whether for any formula φ holds:

(a) $\varphi \equiv \forall\varphi$,
(b) $\varphi \equiv \exists\varphi$,
(c) $\models \varphi$ iff $\models \forall\varphi$,
(d) $\models \varphi$ iff $\models \exists\varphi$,

where $\forall\varphi$ ($\exists\varphi$) is the universal (existential) closure of φ. If not, does at least one implication hold?

7.2 Show that for any set of formulae Γ and a formula φ holds if $\Gamma \models \varphi$, then $\forall\Gamma \models \varphi$, where $\forall\Gamma = \{\, \forall\psi \colon \psi \in \Gamma \,\}$. Does the opposite direction hold?

7.3 Does it hold $\Gamma \models \varphi$ iff $\forall\Gamma \models \forall\varphi$?

7.4 Find a set of formulae Γ and a formula φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$.

7.5 Unify the following pairs of formulae:

(a) $\{p(X, Y) \doteq p(Y, f(Z))\}$,
(b) $\{p(a, Y, f(Y)) \doteq p(Z, Z, U)\}$,
(c) $\{p(X, g(X)) \doteq p(Y, Y)\}$,
(d) $\{p(X, g(X), Y) \doteq p(Z, U, g(U))\}$,
(e) $\{p(g(X), Y) \doteq p(Y, Y), p(Y, Y) \doteq p(U, f(W))\}$.

Note: You can check your results in SWISH using `unify_with_occurs_check/2`.

7.6 What is the size of the maximal term that is produced when you try to unify

$$\{f(g(X_1, X_1), g(X_2, X_2), \ldots, g(X_{n-1}, X_{n-1})) \doteq f(X_2, X_3, \ldots, X_n\}.$$

7.7 We say that a binary predicate q is the transitive closure of a binary predicate p, if $q(s, t)$ iff there is a sequence of terms $s = t_1, t_2, \ldots, t_{n-1}, t_n = t$ such that $p(t_i, t_{i+1})$, for $1 \leq i < n$. Is the formula

$$\forall X \forall Z (q(X, Z) \leftrightarrow (p(X, Z) \vee \exists Y (p(X, Y) \wedge q(Y, Z))))$$

a correct definition of q?

7.8 The compactness theorem in First-Order Logic says that a set of sentences has a model iff every finite subset of it has a model. Use this theorem to show that the transitive closure is not definable in FOL.

Hint: Assume for a contradiction that φ is a formula that expresses that q is the transitive closure of p. Let $\psi^n(a, b) = \neg(\exists X_1 \ldots \exists X_{n-1}(p(a, X_1) \wedge p(X_1, X_2) \wedge \cdots \wedge p(X_{n-1}, b))$ (Hence $\psi^1(a, b)$ means $\neg p(a, b)$ and $\psi^2(a, b)$ means $\neg(\exists X_1(p(a, X_1) \wedge p(X_1, b))))$. What can you say about the satisfiability of $\Gamma = \{\varphi\} \cup \{q(a, b)\} \cup \{\psi^1(a, b), \psi^2(a, b), \ldots\}$?

7.9 Show that the resolution rule is correct.

7.10 Derive the empty clause $\Box$ using the resolution calculus from:

(a) $\{\{\neg p(X), \neg p(f(X))\}, \{p(f(X)), p(X)\}, \{\neg p(X), p(f(X))\}\}$

(b) $\{\{\neg p(X, a), \neg p(X, Y), \neg p(Y, X)\}, \{p(X, f(X)), p(X, a)\}, \{p(f(X), X), p(X, a)\}\}$

7.11 Prove using the resolution calculus that from

$$\forall X \forall Y (p(X, Y) \rightarrow p(Y, X))$$
$$\forall X \forall Y \forall Z((p(X, Y) \wedge p(Y, Z)) \rightarrow p(X, Z))$$
$$\forall X \exists Y p(X, Y)$$

follows $\forall X p(X, X)$.

7.12 Check PyRes; simple resolution-based theorem provers for first-order logic. You can find proofs for the previous examples using them. For example, use

```
pyres-fof.py -tifb -HPickGiven5 -nlargest
```

There are various heuristics (`FIFO`, `SymbolCount`, `PickGiven5`, and `PickGiven2`) and literal selections (`first`, `smallest`, `largest`, `leastvars`, and `eqleastvars`) available. Use `-p` to see a proof.

7.13 List all the possible applications of the factoring rule on the clause

$$\{p(X, f(Y), Z), p(T, T, g(a)), p(f(b), S, g(W)), \neg s(Z, T), \neg s(c, d)\}.$$

If it is possible to use the factoring rule several times, then produce even these results.